Math 280A Homework Problems Fall 2018

Problems are from Resnick, S. A Probability Path, Birkhauser, or from the lecture notes. The problems from the lecture notes are restated here. In the lecture note problems listed in the assignments, you should look up the corresponding problem in the lecture notes where more context is given including extra standing assumptions for the problem. This context and standing assumptions are not always extracted out when I construct the homework sheets.

1.1 Homework 1. Due Friday, October 5, 2018
• Read over Lecture notes Chapter 1.
• Lecture note Exercises: 1.1, 1.2, and 1.3.

1.2 Homework 2. Due Friday, October 12, 2018
• Lecture note Exercises: 4.2, 4.3, 4.7, 4.8, 4.9, 4.10, 4.11, 4.12.
• Look at Resnick, p. 20-27: 9, 12, 17, 23.
• Hand in Resnick, p. 20-27: 5, 18, 40*.
*Notes on Resnick’s #40: (i) \(B((0,1])\) should be \(B([0,1))\) in the statement of this problem, (ii) \(k\) is an integer, (iii) \(r \geq 2\).

1.3 Homework 3. Due Friday, October 19, 2018
• Look at Resnick, p. 20-27: 9, 12, 17, 23.
• Hand in Resnick, p. 20-27: 5, 18, 40*.

1.4 Homework 4. Due a, October 26, 2018
• Look at lecture note exercises: 5.10, 5.19.
• Hand in lecture note exercises: 5.13, 5.14, 5.15, 5.16, 5.17, 5.18.

1.5 Homework 5. Due Friday, November 2, 2018
• Look at Resnick, § 2.6, p. 63–70: 3, 14.
• Look at lecture note exercises: 6.1.
• Hand in Resnick, § 2.6, p. 63–70: 6, 11.

1.6 Homework 6. Due Friday, November 9, 2018
• Hand in Lecture note Exercises: 6.5, 6.6, 9.1, 9.2, 9.6.
• Look at Lecture note Exercises: 9.3, 9.4.
• Look at Resnick, p. 85–90: 3, 7, 8, 12, 17, 21.
• Hand in from Resnick, p. 85–90: 4, 6*, 9, 15, 18**.
* In #6, the random variable \(X\) is understood to take values in the extended real numbers.
** In #18, I would write the left side in terms of an expectation.

1.7 Homework 7. Due Wednesday, November 21, 2018
• Hand in Lecture note Exercises: 9.7, 10.7, 10.29.
• Look at Lecture note Exercises: 9.8, 9.10, 9.11, 10.9.
• Hand in from Resnick, p. 155–166: 6b, 7, 38.
• Look at Resnick, p. 155–166: 13, 26, 37.
1.8 Homework 8. Due Friday, November 30, 2018

- Hand in Lecture note Exercises: 10.3, 10.4, 10.8, 10.14, 3.19, 10.16
- Look at Lecture note Exercises: 10.5, 10.18, 10.19, 10.20
- **Hand in** from Resnick, § 5.10: 29, 36 [In # 36, please assume all random variables are real valued.]

1.9 Homework 9. Due Friday, December 7, 2018

- Look at Lecture note Exercise 12.2, 12.3, 12.5, 12.6
- **Hand in** Lecture note Exercises: 10.32, 12.4, 15.1

For this last homework set you are to work **alone** and only use the text book or the lecture notes as references. If you have questions about these problems, please ask them in class so that everyone gets the same information.
Math 280B Homework Problems Winter 2019

Problems are from Resnick, S. A Probability Path, Birkhauser, or from the lecture notes. The problems from the lecture notes are restated here. In the lecture note problems listed in the assignments, you should look up the corresponding problem in the lecture notes where more context is given including extra standing assumptions for the problem. This context and standing assumptions are not always extracted out when I construct the homework sheets.

2.1 Homework 1. Due Monday, January 14, 2019

- Look at Lecture note Exercise. 3.5, 13.3, 13.4
- Look at from Resnick § 5.10: 8, 18, 19, 22
- Hand in Lecture note Exercises: 3.1, 3.2, 3.3, 3.6, 3.7
- Hand in from Resnick § 5.10: 9

2.2 Homework 2. Due Wednesday, January 23, 2019

- Look at Lecture note Exercise 3.8, 3.9, 3.13, 3.15, 3.17
- Look at from Resnick § 4.6: 3, 5
- Hand in Lecture note Exercises: 3.10, 3.11, 3.12, 3.14, 3.16, 3.18, 3.21
- Hand in from Resnick § 4.6: 6, 19

2.3 Homework 3. Due Monday, January 28, 2019

You should work alone on this homework set! Please ask questions about these problems in class.

- Look at Lecture note Exercise: Read Proposition 12.30, 3.19
- Look at from Resnick § 4.6: 28, 29.
- Hand in Lecture note Exercises: 3.60, 3.20

* Exercise 3.19 has been replaced by Exercise 3.60 in the problems to be handed in as Exercise 3.19 was already given last quarter. Sorry about that.

2.4 Homework 4. Due Monday, February 4, 2019

- Look at Lecture note Exercise: 3.27, 3.28, 3.29
- Look at from Resnick § 5.10: #39
- Hand in Lecture note Exercise 3.4, 3.22, 3.23, 3.24, 3.25, 3.26

2.5 Homework 5. Due Monday, February 11, 2019

- Look at Lecture note Exercise: 3.30, 3.33, 3.37
- Hand in Lecture note Exercise 3.31, 3.32, 3.34, 3.38, 3.39

2.6 Homework 6. Due Wednesday, February 20, 2019

- Look at Lecture note Exercise: 3.43, 3.44
- Hand in Lecture note Exercise 3.36, 3.40, 3.42, 3.45, 3.46

2.7 Homework 7. Due Monday, February 25, 2019

- Look at Lecture note Exercise: 3.48, 3.49, 3.51
- Hand in Lecture note Exercise 3.35, 3.41, 3.47, 3.50, 3.52

2.8 Homework 8. Due Monday, March 4, 2019

- Look at Lecture note Exercise
- Look at from Resnick §6.7: 7 (Hint: Observe that $X_n \overset{d}{=} \sigma_n N(0, 1)$.
- Hand in Lecture note Exercise 3.53, 3.54, 3.55, 3.56, 3.57, 3.58, 3.59
Lecture Note Problems

Exercise 3.1. Let \((\Omega, \mathcal{B}, P)\) be a probability space and \(X : \Omega \to \mathbb{R}\) is a random variable. Prove the following assertions.

1. If \(a < b\) and \(a, b \in \text{median} (X)\), then \([a, b] \subset \text{median} (X)\).
2. If
   \[
   m_+ := \inf \{m \in \mathbb{R} : P (X \leq m) \geq 1/2\} \quad \text{and} \quad m_- := \sup \{m \in \mathbb{R} : P (X \geq m) \geq 1/2\},
   \]
   show \(-\infty < m_- \leq m_+ < \infty\) and \(\text{median} (X) = [m_-, m_+]\). [In particular \(\text{median} (X) \neq \emptyset\).]
3. If \(I = [a, b]\) is a bounded closed interval such that \(P (X \in I) \geq 1/2\), then either \([a, b] \cap \text{median} (X) \neq \emptyset\) or \(\text{median} (X) \subset (a, b)\).

Exercise 3.2. Let \((\Omega, \mathcal{B}, P)\) be a probability space, \(X \in L^1 (P : \mathbb{R})\), and \(m \in \text{median} (X)\). Show for any \(1 \leq p < \infty\) and \(a \in \mathbb{R}\) that
   \[
   |m - a|^p \leq 2E [|X - a|^p]. \tag{3.1}
   \]
   [In particular, by taking \(a = EX\) and \(p = 2\) it follows that\(^1\)]
   \[
   |m - EX| \leq \sqrt{2 \text{Var} (X)}
   \]
   and if \(\text{Var} (X)\) is small the medians of \(X\) and mean, \(EX\), must be close. \] Hint: you may find it useful to consider two case, \(m > a\) and \(m < a\).

Exercise 3.3 (\(L^1\)-minimization). Let \((\Omega, \mathcal{B}, P)\) be a probability space, \(X \in L^1 (P : \mathbb{R})\), and \(m \in \text{median} (X)\). Show
   \[
   E |X - a| \geq E |X - m| \quad \text{for all } a \in \mathbb{R}. \tag{3.2}
   \]

Hints:

1. Let \(Y = X - m\) and \(a = a - m\), then \(0 \in \text{median} (Y) = \text{median} (X) - m\) and Eq. (3.2) is then equivalent to proving
   \[
   E |Y - \alpha| \geq E |Y| \quad \text{for all } \alpha \in \mathbb{R}. \tag{3.3}
   \]
   \(\alpha\) One may remove the factor of 2 here by using the next problem along with Hölder’s or Jensen’s inequalities which has not yet been covered.

2. A simple exercise shows for \(\alpha > 0\) that
   \[
   |Y - \alpha| - |Y| = -\alpha Y_{\geq \alpha} + (\alpha - 2Y) 1_{0 < Y < \alpha} + \alpha Y_{\leq \alpha} \tag{3.4}
   \]
   \[
   \geq -\alpha Y_{> 0} + \alpha Y_{\leq 0}. \tag{3.5}
   \]

3. Integrate Eq. (3.5) to find to prove Eq. (3.3) for \(\alpha > 0\).
4. If \(\alpha < 0\) show that you may use item 3. by replacing \(Y\) by \(-Y\).

Exercise 3.4 (A law of rare events). Let \(S_{n,p} \sim \text{Binomial}(n, p)\), \(k \in \mathbb{N}\), \(p_n = \lambda_n / n\) where \(\lambda_n \to \lambda > 0\) as \(n \to \infty\). Show that
   \[
   \lim_{n \to \infty} P (S_{n,p} = k) = \frac{\lambda^k}{k!} e^{-\lambda} = P (\text{Poisson} (\lambda) = k). \tag{3.6}
   \]
[See Exercise 10.14 for the definitions of the distributions being used here.]

Interpretation. Given a large \(n \in \mathbb{N}\), \(k \in \mathbb{N}\) with \(k << n\), and \(p = O (1/n)\), then
   \[
   P (\text{Binomial} (n, p) = k) \approx P (\text{Poisson} (pn) = k) = \frac{(pn)^k}{k!} e^{-pn}.
   \]
(We will come back to the Poisson distribution and the related Poisson process later on.)
Exercises 3.5. Let \( \{X_i\}_{i=1}^{\infty} \) and \( \{Y_i\}_{i=1}^{\infty} \) be two sequences of random variables such that \( \{X_i\}_{i=1}^{\infty} \overset{d}{=} \{Y_i\}_{i=1}^{\infty} \). Let \( \{S_n\}_{n=1}^{\infty} \) and \( \{T_n\}_{n=1}^{\infty} \) be defined by, \( S_n := X_1 + \cdots + X_n \) and \( T_n := Y_1 + \cdots + Y_n \). Prove the following assertions.

1. Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^k \) is a \( \mathcal{B}_{\mathbb{R}^n}/\mathcal{B}_{\mathbb{R}^k} \) measurable function, then
   \( f(X_1, \ldots, X_n) \overset{d}{=} f(Y_1, \ldots, Y_n) \).

2. Use your result in item 1. to show \( \{S_n\}_{n=1}^{\infty} \overset{d}{=} \{T_n\}_{n=1}^{\infty} \).

   Hint: Apply item 1. with \( k = n \) after making a judicious choice for \( f : \mathbb{R}^n \to \mathbb{R}^n \).

Exercises 3.6 (Folland Problem 2.48 on p. 69.). Let \( X = Y = \mathbb{N} \), \( \mathcal{B} = 2^\mathbb{N} \) and \( \mu = \nu = \sum_{n=1}^{\infty} \delta_n \) be counting measure on \( (X, \mathcal{B}) \) and \( (Y, \mathcal{B}) \). If \( f : X \times Y \to \mathbb{R} \) is defined by

\[
f(m, n) = \sum_{n=1}^{m-1} - \sum_{m=n+1}^{\infty}
\]

show \( \int_{\mathbb{N} \times \mathbb{N}} |f| \, d(\mu \otimes \nu) = \infty \) while \( \int_{\mathbb{N}} \sum_{n=1}^{\infty} d\mu(n) \int_{\mathbb{N}} f(m, n) \, d\nu(n) \) and \( \int_{\mathbb{N}} \sum_{n=1}^{\infty} d\nu(n) \int_{\mathbb{N}} f(m, n) \, d\mu(n) \) both exist but are unequal.

Exercises 3.7 (Folland Problem 2.56 on p. 77.). Let \( a \in (0, \infty) \), \( f \in L^1((0, a), dm) \), and

\[
g(x) = \int_x^a \frac{f(t)}{t} \, dt \text{ for } x \in (0, a).
\]

Show \( g \in L^1((0, a), dm) \) and

\[
\int_0^a g(x) \, dx = \int_0^a f(t) \, dt.
\]

Exercises 3.8 (Folland Problem 2.60 on p. 77.). If \( \Gamma(\cdot) \) is the \( \Gamma \)-function as in Definition 10.45 show

\[
\Gamma(x) \Gamma(y) / \Gamma(x+y) = \int_0^1 (1-r)^{x-1} r^{y-1} \, dr \text{ for all } x, y > 0.
\]

Hint: write \( \Gamma(x) \Gamma(y) / \Gamma(x+y) \) as a double integral and then make a couple of change of variables.

Exercises 3.9. Continuing the setup in Theorem 14.13 show that \( f \in L^1(T(\Omega), m^d) \) iff

\[
\int_{\Omega} |f \circ T| \cdot \det T' \, dm < \infty
\]

and if \( f \in L^1(T(\Omega), m^d) \), then Eq. 14.8 holds.

Exercises 3.10. Let \( P \) be the probability measure on \( \Omega := \mathbb{R}^d \) defined by

\[
dP(x) := \left( \frac{1}{2\pi} \right)^{d/2} e^{-\frac{1}{2} x \cdot x} \, dx = \prod_{i=1}^d \left( \frac{1}{\sqrt{2\pi}} \right)^{1/2} e^{-x_i^2/2} \, dx_i.
\]

Show that \( N : \Omega \to \mathbb{R}^d \) defined by \( N(x) = x \) is Gaussian and satisfies Eq. 14.29 with \( Q = I \) and \( \mu = 0 \). Also show

\[
\mu_i = \mathbb{E}N_i \text{ and } \delta_{ij} = \text{Cov}(N_i, N_j) \text{ for all } 1 \leq i, j \leq d.
\]

Hint: use Exercise 10.17 and (of course) Fubini’s theorem.

Exercises 3.11. Let \( A \) be any real \( m \times d \) matrix and \( \mu \in \mathbb{R}^m \) and let \( \Omega := \mathbb{R}^d \), \( P \), and \( N \) are as in Exercise 3.10. Show that \( X \) is Gaussian by showing Eq. 14.29 holds with \( Q = AA^\top \) (\( A^\top \) is the transpose of the matrix \( A \)) and \( \mu = b \). Also show that

\[
\mu_i = \mathbb{E}X_i \text{ and } Q_{ij} = \text{Cov}(X_i, X_j) \text{ for all } 1 \leq i, j \leq m.
\]

Exercises 3.12. Suppose that \( Q \) is a positive definite (for simplicity) \( d \times d \) real matrix and \( \mu \in \mathbb{R}^d \) and let \( \Omega = \mathbb{R}^d \), \( P \), and \( N \) be as in Exercise 3.10. By Exercise 3.11 we know that \( X = Q^{1/2}N + \mu \) is a Gaussian random vector satisfying Eq. 14.29. Use the multi-dimensional change of variables formula to show

\[
\text{Law}_P(X)(dy) = \frac{1}{\sqrt{\det(2\pi Q)}} \exp \left( -\frac{1}{2} Q^{-1}(y - \mu) \cdot (y - \mu) \right) dy.
\]

Exercises 3.13 (Gaussian random vectors are “highly” integrable.). Suppose that \( X : \Omega \to \mathbb{R}^d \) is a Gaussian random vector, say \( X \overset{d}{=} N(\mu, Q) \). Let \( \|x\| := \sqrt{x \cdot x} \) and \( m := \max \{|Qx \cdot x : \|x\| = 1\} \) be the largest eigenvalue \( \epsilon^2 \) of \( Q \).

Then \( \mathbb{E}[e^{\epsilon\|X\|^2}] < \infty \) for every \( \epsilon < \frac{1}{2m} \).

Exercises 3.14. Prove the assertion made in Remark 14.35 Hint: explicitly compute \( \mathbb{E}[e^{(\lambda_1 X + \lambda_2 Y)^2}] \).

Exercises 3.15. Let \( X, Y \) be two random variables on \( (\Omega, \mathcal{B}, P) \).

1. Show that \( X \) and \( Y \) are independent iff \( \text{Cov}(f(X), g(Y)) = 0 \) (i.e. \( f(X) \) and \( g(Y) \) are uncorrelated) for bounded measurable functions, \( f, g : \mathbb{R} \to \mathbb{R} \).

2. If \( X, Y \in L^2(P) \) and \( X \) and \( Y \) are independent, then \( \text{Cov}(X, Y) = 0 \).

\( ^2 \) For those who know about operator norms observe that \( m = \|Q\| \) in this case.
3. Show by example that if $X, Y \in L^2(P)$ and $\text{Cov}(X,Y) = 0$ does not necessarily imply that $X$ and $Y$ are independent. **Hint:** try taking $(X,Y) = (X,ZX)$ where $X$ and $Z$ are independent simple random variables such that $EZ = 0$ similar to Remark [14.35]

**Exercise 3.16 (A correlation inequality).** Suppose that $X$ is a random variable and $f, g : \mathbb{R} \to \mathbb{R}$ are two increasing functions such that both $f(X)$ and $g(X)$ are square integrable, i.e. $\mathbb{E}[|f(X)|^2 + |g(X)|^2] < \infty$. Show $\text{Cov}(f(X), g(X)) \geq 0$. **Hint:** let $Y$ be another random variable which has the same law as $X$ and is independent of $X$. Then consider

$$
\mathbb{E}[(f(Y) - f(X)) \cdot (g(Y) - g(X))].
$$

**Exercise 3.17.** Suppose that $X \overset{d}{=} N(0, a^2)$ and $Y \overset{d}{=} N(0, b^2)$ and $X$ and $Y$ are independent. Show by direct computation using the formulas for the distributions of $X$ and $Y$ that $X + Y = N(0, a^2 + b^2)$.

**Exercise 3.18.** Show that the sum, $N_1 + N_2$, of two independent Poisson random variables, $N_1$ and $N_2$, with parameters $\lambda_1$ and $\lambda_2$ respectively is again a Poisson random variable with parameter $\lambda_1 + \lambda_2$. (You could use generating functions or do this by hand.) In short $\text{Poi}(\lambda_1) + \text{Poi}(\lambda_2) \overset{d}{=} \text{Poi}(\lambda_1 + \lambda_2)$.

**Exercise 3.19 (Gamma Distributions).** Let $X$ be a positive random variable. For $k, \theta > 0$, we say that $X \overset{d}{=} \text{Gamma}(k, \theta)$ if

$$(X, P)(dx) = f(x; k, \theta) \, dx \text{ for } x > 0,$$

where

$$f(x; k, \theta) := x^{k-1} e^{-x/\theta} / \theta^k \Gamma(k) \text{ for } x > 0, \text{ and } k, \theta > 0.$$  

Find the moment generating function (see Definition [10.58]), $M_X(t) = \mathbb{E}[e^{tX}]$ for $t < \theta^{-1}$. Differentiate your result in $t$ to show

$$\mathbb{E}[X^m] = k(k+1) \ldots (k+m-1) \theta^m \text{ for all } m \in \mathbb{N}_0.$$ 

In particular, $\mathbb{E}[X] = k\theta$ and $\text{Var}(X) = k\theta^2$. (Notice that when $k = 1$ and $\theta = \lambda^{-1}$, $X \overset{d}{=} \mathcal{E}(\lambda)$.)

**Exercise 3.20.** Use the following outline to give a proof of Theorem [16.35].

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3 We now no longer assume $k$ is an integer.

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1. First show that $x^n \leq 1 + x^4$ for all $x \geq 0$ and $1 \leq p \leq 4$. Use this to conclude;

$$
\mathbb{E}[|X_n|^p] \leq 1 + \mathbb{E}[|X_n|^4] < \infty \text{ for } 1 \leq p \leq 4.
$$

Thus $\gamma := \mathbb{E}[(X_n - \mu)^4]$ and the standard deviation ($\sigma^2$) of $X_n$ defined by,

$$
\sigma^2 := \mathbb{E}[X_n^2] - \mu^2 = \mathbb{E}[(X_n - \mu)^2] < \infty
$$

are finite constants independent of $n$.

2. Show for all $n \in \mathbb{N}$ that

$$
\mathbb{E}\left[\left(\frac{S_n}{n} - \mu\right)^4\right] = \frac{1}{n^4} \left(n\gamma + 3n(n-1)\sigma^4\right)
$$

$$
= \frac{1}{n^2} \left[n^{-1}\gamma + 3\left(1 - n^{-1}\right)\sigma^4\right].
$$

(Thus $S_n \overset{d}{\to} \mu$ in $L^4(P)$.)

3. Use item 2. and Chebyshev’s inequality to show

$$
P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \leq \frac{n^{-1}\gamma + 3\left(1 - n^{-1}\right)\sigma^4}{\varepsilon^4 n^2}.
$$

4. Use item 3. and the first Borel Cantelli Lemma [10.15] to conclude $\lim_{n \to \infty} \frac{S_n}{n} = \mu$ a.s.

**Exercise 3.21.** Suppose that $(\Omega, \mathcal{B}, P)$ is a probability space, $Y : \Omega \to \mathbb{R}$ is a random variable and $c \in \mathbb{R}$ is a constant. Then $Y = c$ a.s. iff $Y \overset{d}{=} c$.

**Exercise 3.22.** Show $m_n(\Delta_n(T)) = T^n/n!$ where $m_n$ is Lebesgue measure on $\mathcal{B}_{\mathbb{R}^n}$.

**Exercise 3.23.** If $n \in \mathbb{N}$ and $g : \Delta_n \to \mathbb{R}$ bounded (non-negative) measurable, then

$$
\mathbb{E}[g(W_1, \ldots, W_n)] = \int_{\Delta_n} g(w_1, w_2, \ldots, w_n) \lambda^n e^{-\lambda w_1} dw_1 \ldots dw_n. \tag{3.8}
$$

**Exercise 3.24.** Show $N_t \overset{d}{=} \text{Poi}(\lambda t)$ for all $t > 0$.

**Exercise 3.25.** Suppose that $X_1, \ldots, X_n$ are non-negative random variables such that $P(X_i = X_j) = 0$ for all $i \neq j$. Show;

---

4 The non-negativity of the $X_i$ are not really necessary here but this is all we need to consider.
1. If \( f : \Delta_n \rightarrow \mathbb{R} \) is bounded (non-negative) measurable, then

\[
E \left[ f \left( \hat{X}_1, \ldots, \hat{X}_n \right) \right] = \sum_{\sigma \in S_n} E \left[ f \left( X_{\sigma 1}, \ldots, X_{\sigma n} \right) : X_{\sigma 1} < X_{\sigma 2} < \cdots < X_{\sigma n} \right],
\]

where \( S_n \) is the permutation group on \( \{1, 2, \ldots, n\} \).

2. If we further assume that \( X_1, \ldots, X_n \) are i.i.d. random variables, then

\[
E \left[ f \left( \hat{X}_1, \ldots, \hat{X}_n \right) \right] = n! \cdot E \left[ f \left( X_1, \ldots, X_n \right) : X_1 < X_2 < \cdots < X_n \right].
\]

(3.9)

(It is not important that \( f \left( \hat{X}_1, \ldots, \hat{X}_n \right) \) is not defined on the null set, \( \cup_{i \neq j} \{ X_i = X_j \} \).

3. If \( f : \mathbb{R}^n_+ \rightarrow \mathbb{R} \) is a bounded (non-negative) measurable symmetric function (i.e. \( f(w_{1}, \ldots, w_{n}) = f(w_1, \ldots, w_n) \) for all \( \sigma \in S_n \) and \( (w_1, \ldots, w_n) \in \mathbb{R}^n_+ \) then

\[
E \left[ f \left( \hat{X}_1, \ldots, \hat{X}_n \right) \right] = E \left[ f \left( X_1, \ldots, X_n \right) \right].
\]

(3.10)

So for \( A \in \mathcal{B}_S \), we have \( N_s(A) = 0 \) and \( N_s(A) = \sum_{i=1}^{n} 1_A (s_i) \). Show:

1. For each \( A \in \mathcal{B}_S \), \( \omega \rightarrow N_s(A) \) is a Poisson random variable with intensity \( \mu (A) \), i.e. \( N(A) = \text{Poi} (\mu (A)) \).

2. If \( \{ A_k \}_{k=1}^{m} \subset \mathcal{B}_S \) are disjoint sets, the \( \{ \omega \rightarrow N_s(A_k) \}_{k=1}^{m} \) are independent random variables.

**Exercise 3.28 (A Generalized Poisson Process II).** Let \( (S, \mathcal{B}_S, \mu) \) be as in Exercise 3.27 \( \{ Y_i \}_{i=1}^{\infty} \) be i.i.d. \( S \)-valued Random variables with Law \( P_{\mu} (Y_i) = \mu (\cdot) / \mu (S) \) and \( \nu \) be a Poisson \( (\mu (S)) \) - random variable which is independent of \( \{ Y_i \} \). Show \( N := \sum_{i=1}^{\nu} \delta_{Y_i} \) is a Poisson process on \( (S, \mathcal{B}_S) \) with intensity measure, \( \mu \). Hints:

1. Assume that \( \{ A_k \}_{k=1}^{m} \subset \mathcal{B}_S \) is a measurable partition of \( S \) and show \( \{ N(A_k) \}_{k=1}^{m} \) are i.i.d. with \( N(A_k) = \text{Poi} (\mu (A_k)) \) for each \( k \).

2. Model your proof of item 1. on either of the proofs of Theorem 16.12.

**Exercise 3.29 (A Generalized Poisson Process III).** Suppose now that \( (S, \mathcal{B}_S, \mu) \) is a \( \sigma \) - finite measure space and \( S = \sum_{l=1}^{\infty} S_l \) is a partition of \( S \) such that \( 0 < \mu (S_l) < \infty \) for all \( l \). For each \( l \in \mathbb{N} \), using either of the construction above we may construct a Poisson point process, \( N_l \) on \( (S, \mathcal{B}_S) \) with intensity measure, \( \mu_l \) where \( \mu_l \) where \( \mu_l (A) := \mu (A \cap S_l) \) for all \( A \in \mathcal{B}_S \). We do this in such a way that \( \{ N_l \}_{l=1}^{\infty} \) are all independent. Show that \( N := \sum_{i=1}^{\infty} N_l \) is a Poisson point process on \( (S, \mathcal{B}_S) \) with intensity measure, \( \mu \). To be more precise observe that \( N \) is a random measure on \( (S, \mathcal{B}_S) \) which satisfies (as you should show):

1. For each \( A \in \mathcal{B}_S \) with \( \mu (A) < \infty \), show \( N(A) \overset{d}{=} \text{Poi} (\mu (A)) \). Also show \( N(A) = \infty \) a.s. if \( \mu (A) = \infty \).

2. If \( \{ A_k \}_{k=1}^{\infty} \subset \mathcal{B}_S \) are disjoint sets with \( \mu (A_k) < \infty \), show \( \{ N(A_k) \}_{k=1}^{m} \) are independent random variables.

**Exercise 3.30 (Fatou’s Lemma).** Let \( (\Omega, \mathcal{B}, \mu) \) be a measure space. If \( f_n \geq 0 \) and \( f_n \overset{\mu}{ightarrow} f \), then \( \int_\Omega f d\mu \leq \liminf_{n \rightarrow \infty} \int_\Omega f_n d\mu \).
Exercise 3.31. Let $(\Omega, B, \mu)$ be a measure space, $p \in [1, \infty)$, and suppose that $0 \leq f \in L^1(\mu), 0 \leq f_n \in L^1(\mu)$ for all $n$, $f_n \rightarrow f$, and $\int f_n d\mu \rightarrow \int f d\mu$. Then $f_n \rightarrow f$ in $L^1(\mu)$. [In particular, if $f, f_n \in L^P(\mu)$ and $f_n \rightarrow f$ in $L^P(\mu)$, then $|f_n|^p \rightarrow |f|^p$ in $L^1(\mu)$.]  

Exercise 3.32. (In this exercise the reader should refer to Lemma 17.29 for context and the notation used here.) Let $(\Omega, B, \mu)$ be a measure space and $B_0$ be a sub $-$ $\sigma$ - algebra of $B$. Further suppose that to every $B \in B$ there exists $A \in B_0$ such that $\mu(B \Delta A) = 0$. Show for all $1 \leq p < \infty$ that $i(L^p(\Omega, B_0, \mu)) = L^p(\Omega, B, \mu)$, i.e. to each $f \in L^p(\Omega, B, \mu)$ there exists a $g \in L^p(\Omega, B_0, \mu)$ such that $f = g$ a.e. Hints: 1. verify the last assertion for simple functions in $L^p(\Omega, B_0, \mu)$. 2. then make use of Theorem 9.41 and Exercise 9.6.  

Exercise 3.33. Suppose that $1 \leq p < \infty, (\Omega, B, \mu)$ is a $\sigma$ - finite measure space and $B_0$ is a sub $-$ $\sigma$ - algebra of $B$. Show that $i(L^p(\Omega, B_0, \mu)) = L^p(\Omega, B, \mu)$ implies; to every $B \in B$ there exists $A \in B_0$ such that $\mu(B \Delta A) = 0$.  

Exercise 3.34. Suppose $A$ is an index set, $(f_\alpha)_{\alpha \in A}$ and $(g_\alpha)_{\alpha \in A}$ are two collections of random variables and $C \in (0, \infty)$. If $(g_\alpha)_{\alpha \in A}$ is uniformly integrable and $|f_\alpha| \leq C|g_\alpha |$ for all $\alpha \in A$, show $(f_\alpha)_{\alpha \in A}$ is uniformly integrable as well. [An an example which occurs in the dominated convergence theorem is when $g_\alpha = g \in L^1(\mu)$ for all $\alpha \in A$.]  

Exercise 3.35. Prove that a subset $A \subset L^1(\mu)$ is uniformly integrable iff $A \subset L^1(\mu)$ is bounded and is GUAC. Hint: modify the proof of Proposition 17.53.  

Exercise 3.36 (Problem 5 on p. 196 of Resnick generalized). Suppose that $(X_n)_{n=1}^\infty$ is a sequence of integrable and random variables which are identically distributed. Show $(\frac{S_n}{n})_{n=1}^\infty$ is uniformly integrable where, as usual, $S_n := X_1 + \cdots + X_n$ for all $n \in \mathbb{N}$.  

Suggestions:  
1. First show $(X_n)_{n=1}^\infty$ are U.I. using the direct definition and the identically distributed assumption.  
2. Use the results of item 1. along with Proposition 17.53 to show $(\frac{S_n}{n})_{n=1}^\infty$ is uniformly integrable. [Via this method you will actually show; if $(X_n)_{n=1}^\infty$ is U.I. then so is $(\frac{S_n}{n})_{n=1}^\infty$.]  

Exercise 3.37. Let $\infty > a, b > 1$ with $a^{-1} + b^{-1} = 1$. Give a calculus proof of the inequality  
$$st \leq \frac{s^a}{a} + \frac{t^b}{b} \text{ for all } s, t \geq 0.$$  

Hint: by taking $s = x^b/a$, show that it suffices to prove $x \leq \frac{x^a}{a} + \frac{1}{b}$ for all $x \geq 0$.  

and then maximize the function $f(x) = x - x^a/a$ for $x \in [0, \infty).$  

Exercise 3.38. Suppose that $(\Omega, B, P)$ is a probability space and $(X_n)_{n=1}^\infty$ is a sequence of uncorrelated (i.e. Cov $(X_n, X_m) = 0$ if $m \neq n$) square integrable random variables such that $\mu = E X_n$ and $\sigma^2 = Var(X_n)$ for all $n$. Let $S_n := X_1 + \cdots + X_n$. Show $\frac{S_n}{n} \rightarrow \mu$ in $L^1(\mu)$ as $n \rightarrow \infty$. (Incidentally, this shows that $(\frac{S_n}{n})_{n=1}^\infty$ is U.I. Hint: for $M \in (0, \infty)$, let $X_i^M := X_i \cdot \mathbb{1}_{|X_i| \leq M}$ and $S_n^M := X_1^M + \cdots + X_n^M$ and use Exercise 3.38 to see that $\frac{S_n^M}{n} \rightarrow E X_1^M$ in $L^2(\mu) \subset L^1(\mu)$ for all $M$.  

Using this to show $\lim_{n \rightarrow \infty} \left\| \frac{S_n}{n} - E X_1 \right\|_1 = 0$ by getting good control on $\left\| \frac{S_n^M}{n} - \frac{S_n^M}{M} \right\|_1$ and $E X_n - E X_1^M$.  

Exercise 3.39. Suppose that $(X_n)_{n=1}^\infty$ are i.i.d. integrable random variables and $S_n := X_1 + \cdots + X_n$ and $\mu := E X_n$. Show, $\frac{S_n}{n} \rightarrow \mu$ in $L^1(\mu)$ as $n \rightarrow \infty$. (Incidentally, this shows that $(\frac{S_n}{n})_{n=1}^\infty$ is U.I. Hint: for $M \in (0, \infty)$, let $X_i^M := X_i \cdot \mathbb{1}_{|X_i| \leq M}$ and $S_n^M := X_1^M + \cdots + X_n^M$ and use Exercise 3.38 to see that $\frac{S_n^M}{n} \rightarrow E X_1^M$ in $L^2(\mu) \subset L^1(\mu)$ for all $M$.  

Using this to show $\lim_{n \rightarrow \infty} \left\| \frac{S_n}{n} - E X_1 \right\|_1 = 0$ by getting good control on $\left\| \frac{S_n^M}{n} - \frac{S_n^M}{M} \right\|_1$ and $E X_n - E X_1^M$.  

Exercise 3.40. Suppose $1 \leq p < \infty, (X_n)_{n=1}^\infty$ are i.i.d. random variables such that $E |X_n|^p < \infty, S_n := X_1 + \cdots + X_n$ and $\mu := E X_n$. Show, $\frac{S_n}{n} \rightarrow \mu$ in $L^p(\mu)$ as $n \rightarrow \infty$. Hint: explain why $\frac{1}{n} \sum_{i=1}^{n} |X_i|^p \rightarrow E |X_1|^p$ in $L^1(\mu)$ and then use this show this $\left\{ \frac{S_n}{n} \right\}_{n=1}^\infty$ is U.I. - this is not meant to be hard!  

Exercise 3.41. Suppose $M$ is a subset of $H$, then $M^{\perp} := \text{span} M$ where (as usual), span $M$ denotes all finite linear combinations of elements from $M$.  

Exercise 3.42. Show that a linear operator, $A : X \rightarrow Y$, is a bounded iff it is continuous.  

Exercise 3.43. Let $H, K, M$ be Hilbert spaces, $A, B \in L(H, K), C \in L(K, M)$ and $\lambda \in \mathbb{C}$. Show $(A + \lambda B)^* = A^* + \lambda B^*$ and $(CA)^* = A^* C^*$ in $L(M, H)$.  

Exercise 3.44. Let $H = \mathbb{C}^n$ and $K = \mathbb{C}^m$ equipped with the usual inner products, i.e. $\langle |z|^2 \rangle_{H} = z \cdot \bar{w}$ for $z, w \in H$. Let $A$ be an $m \times n$ matrix thought of as a linear operator from $H$ to $K$. Show the matrix associated to $A^* : K \rightarrow H$ is the conjugate transpose of $A$.  

Exercise 3.45. Suppose that $(M_n)_{n=1}^\infty$ is an increasing sequence of closed subspaces of a Hilbert space, $H$. Let $M$ be the closure of $M_0 := \cap_{n=1}^\infty M_n$. Show $\lim_{n \rightarrow \infty} P_{M_n} x = P_M x$ for all $x \in H$. Hint: first prove this for $x \in M_0$ and then for $x \in M$. Also consider the case where $x \in M^\perp$.  

3 Lecture Note Problems 9
Exercise 3.46 (A “Martingale” Convergence Theorem). Suppose that \( \{M_n\}_{n=1}^{\infty} \) is an increasing sequence of closed subspaces of a Hilbert space, \( H \), \( P_n := P_{M_n} \), and \( \{x_n\}_{n=1}^{\infty} \) is a sequence of elements from \( H \) such that \( x_n = P_n x \) for all \( n \in \mathbb{N} \). Show:

1. \( P_n x_n = x_n \) for all \( 1 \leq m \leq n < \infty \),
2. \( (x_n - x_m) \perp M_m \) for all \( n \geq m \),
3. \( \|x_n\| \) is increasing as \( n \) increases,
4. if \( \sup_n \|x_n\| = \lim_n \|x_n\| < \infty \), then \( x := \lim_n x_n \) exists in \( M \) and that \( x_n = P_n x \) for all \( n \in \mathbb{N} \). (Hint: show \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence.)

Exercise 3.47. Suppose \( f \in L^1(\Omega, \mathcal{B}, P) \) and \( f > 0 \) a.s. Show \( \mathbb{E}[f|\mathcal{G}] > 0 \) a.s. (i.e. show \( g > 0 \) a.s. for any version, \( g \), of \( \mathbb{E}[f|\mathcal{G}] \). Use this result to conclude if \( f \in (a, b) \) a.s. for some \( a, b \) such that \( -\infty < a < b < \infty \), then \( \mathbb{E}[f|\mathcal{G}] \in (a, b) \) a.s. More precisely you are to show that any version, \( g \), of \( \mathbb{E}[f|\mathcal{G}] \) satisfies, \( g \in (a, b) \) a.s.

Exercise 3.48 (Exercise 5.13 revisited.). Suppose \( (\Omega, \mathcal{B}, P) \) is a probability space and \( \mathcal{P} := \{A_i\}_{i=1}^{\infty} \subset \mathcal{B} \) is a partition of \( \Omega \). (Recall this means \( \Omega = \bigcup_{i=1}^{\infty} A_i \).) Let \( \mathcal{G} \) be the \( \sigma \)-algebra generated by \( \mathcal{P} \). Show:

1. \( B \in \mathcal{G} \) if \( B = \bigcup_{i \in A} A_i \) for some \( A \subset \mathbb{N} \).
2. \( g : \Omega \to \mathbb{R} \) is \( \mathcal{G} \)-measurable iff \( g = \sum_{i=1}^{\infty} \lambda_i 1_{A_i} \) for some \( \lambda_i \in \mathbb{R} \).
3. For \( f \in L^1(\Omega, \mathcal{B}, P) \), let \( \mathbb{E}[f|A_i] := \mathbb{E}[1_{A_i} f]/P(A_i) \) if \( P(A_i) \neq 0 \) and \( \mathbb{E}[f|A_i] = 0 \) otherwise. Show
   \[
   \mathbb{E}_G f = \sum_{i=1}^{\infty} \mathbb{E}[f|A_i] 1_{A_i} \quad \text{a.s.} \tag{3.11}
   \]

Exercise 3.49. Let \( (\Omega, \mathcal{B}, P) \) be a probability space, \( (X, \mathcal{M}) \) and \( (Y, \mathcal{N}) \) be measurable spaces, \( X : \Omega \to X \) and \( Y : \Omega \to Y \) are measurable functions. Let \( (\mathcal{M}, \mathcal{N}, (X, \mathcal{M}) \) be measurable spaces, \( (\Omega, \mathcal{F}, P) \) a probability space, and \( X : \Omega \to X \) and \( Y : \Omega \to Y \) be measurable functions. Further assume that \( \mathcal{G} \subset \mathcal{F} \) is a \( \sigma \)-algebra such that \( X \) is \( \mathcal{G}/\mathcal{M} \)-measurable and \( Y \) is independent of \( \mathcal{G} \). Then for any bounded \( (\mathcal{M} \otimes \mathcal{N}, \mathcal{B}_R) \)–measurable function \( f : X \times Y \to \mathbb{R} \) we have
   \[
   \mathbb{E}[f(X,Y)|\mathcal{G}] = h_f(X) = \mathbb{E}[f(x,Y)]|_{x=X} \text{ a.s.} \tag{3.12}
   \]
   where if \( \mu := \text{Law}_P(Y) \),
   \[
   h_f(x) := \mathbb{E}[f(x,Y)] = \int_Y f(x,y) d\mu(y) \ . \tag{3.13}
   \]
   [This exercise is essentially a special case of Exercise 19.6 below.]

Exercise 3.50. Suppose now that \((X,Y,Z)^{tr}\) is a mean zero Gaussian random vector with \( X \in \mathbb{R}^k \), \( Y \in \mathbb{R}^l \), and \( Z \in \mathbb{R}^m \). Show for all \( y \in \mathbb{R}^l \) and \( z \in \mathbb{R}^m \) that
   \[
   \mathbb{E}\left[\exp(i(y \cdot Y + z \cdot Z) | X\right] \tag{3.14}
   = \exp\left( -\text{Cov}(y \cdot W_1, z \cdot W_2) \right) \mathbb{E}[\exp(iy \cdot Y) | X] \cdot \mathbb{E}[\exp(iz \cdot Z) | X].
   \]

In performing these computations please use the following definitions,
   \[
   C := C_X := \mathbb{E}[X^{tr} X], \tag{3.15}
   A := \mathbb{E}\left[\begin{bmatrix} Y \\ Z \end{bmatrix}^{tr} \begin{bmatrix} Y \\ Z \end{bmatrix} C^{-1} \right] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{3.16}
   \]
   and
   \[
   W := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} Y \\ Z \end{bmatrix} - AX = \begin{bmatrix} Y - A_1 X \\ Z - A_2 X \end{bmatrix}. \tag{3.17}
   \]

Exercise 3.51. Suppose \( (\Omega, \mathcal{B}, P) \) and \((\Omega', \mathcal{B}', P')\) are two probability spaces, \( (X, \mathcal{M}) \) and \((Y, \mathcal{N})\) are measurable spaces, \( X : \Omega \to X, X' : \Omega' \to X, Y : \Omega \to Y, \) and \( Y' : \Omega \to Y \) are measurable functions such that \( P \circ (X,Y)^{-1} = P' \circ (X',Y')^{-1} \). Further set \( f : (X,Y) \to \mathbb{R} \) is a bounded measurable function and \( \hat{f} : (X,Y) \to \mathbb{R} \) is a measurable function such that \( f(X) = \mathbb{E}[f(X,Y)|X] \) \( P \)–a.s. then
   \[
   \mathbb{E}[f(X',Y')|X] = \hat{f}(X') \ \text{\( P' \)–a.s.}
   \]

Exercise 3.52. Let \( \{X_n\}_{n=1}^{\infty} \) be i.i.d. integrable random variables such that \( \mathbb{E}X_n = 0 \). Further set \( S_0 = 0 \) and for \( n \in \mathbb{N} \) let \( S_n := X_1 + \cdots + X_n \) and \( S_{n-1} := \sigma(S_n, S_{n+1}, S_{n+2}, \ldots) \). Show

\[
\mathbb{E}[X_1|\mathcal{B}_{-n}] = \mathbb{E}[X_1|S_{n+1}, S_{n+2}, \ldots] = S_n \quad \text{a.s.}
\]

[This problem will be used in Example 23.82 to give a proof of the strong law of large numbers.]

Hint: Use Exercise 3.51 to show
   \[
   \mathbb{E}[X_j|\mathcal{B}_{-n}] = \mathbb{E}[X_1|\mathcal{B}_{-n}] \text{ a.s. for all } j \leq n. \tag{3.18}
   \]

Exercise 3.53 (Martingale Convergence Theorem for \( p = 1 \) and \( 2.\)). Let \( (\Omega, \mathcal{B}, P) \) be a probability space and \( \mathcal{B}_n \) be an increasing sequence of sub-\( \sigma \)-algebras of \( \mathcal{B} \). Show;

1. The closure, \( M_n \) of \( \bigcup_{n=1}^{\infty} L^2(\Omega, \mathcal{B}_n, P) \) is \( L^2(\Omega, \mathcal{B}_\infty, P) \) where \( \mathcal{B}_\infty = \bigvee_{n=1}^{\infty} \mathcal{B}_n := \sigma(\bigcup_{n=1}^{\infty} \mathcal{B}_n) \). Hint: make use of Theorem 17.32.
2. For every $X \in L^2(\Omega, \mathcal{B}, P)$, $X_n := \mathbb{E}[X|\mathcal{B}_n] \to \mathbb{E}[X|\mathcal{B}_\infty]$ in $L^2(P)$. **Hint:** see Exercise 3.45.

3. For every $X \in L^1(\Omega, \mathcal{B}, P)$, $X_n := \mathbb{E}[X|\mathcal{B}_n] \to \mathbb{E}[X|\mathcal{B}_\infty]$ in $L^1(P)$. **Hint:** make use of item 2, by a truncation argument using the contractive properties of conditional expectations.

(Eventually we will show that $X_n = \mathbb{E}[X|\mathcal{B}_n] \to \mathbb{E}[X|\mathcal{B}_\infty]$ a.s. as well.)

**Exercise 3.54 (Martingale Convergence Theorem for general $p$).** Let $1 \leq p < \infty$, $(\Omega, \mathcal{B}, P)$ be a probability space, and $\{\mathcal{B}_n\}_{n=1}^\infty$ be an increasing sequence of sub-$\sigma$-algebras of $\mathcal{B}$. Show for all $X \in L^p(\Omega, \mathcal{B}, P)$, $X_n := \mathbb{E}[X|\mathcal{B}_n] \to \mathbb{E}[X|\mathcal{B}_\infty]$ in $L^p(P)$. **(Hint:** show that $\{\mathbb{E}[X|\mathcal{B}_n]\}_{n=1}^\infty$ is uniformly integrable and $\mathbb{E}[X|\mathcal{B}_n] \to \mathbb{E}[X|\mathcal{B}_\infty]$ with the aid of item 3. of Exercise 3.53.)

**Exercise 3.55 (Uniform Integrability).** Suppose that $(\Omega, \mathcal{B}, P)$ is a probability space and $A \subset L^1(P)$ is a uniformly integrable collection of random variables and let $\mathcal{G} \subset \mathcal{B}$ denote a generic sub-$\sigma$-algebra of $\mathcal{B}$. Show that

$$\hat{\mathcal{A}} = \{\mathbb{E}_\mathcal{G}X = \mathbb{E}[X|\mathcal{G}] : X \in A \text{ and } \mathcal{G} \subset \mathcal{B}\}$$

is again uniformly integrable. **(Hint:** you may find Exercise 3.35 useful here.)

**Exercise 3.56.** Suppose $M_i \subset (B_i)_b$ for $i = 1$ and $i = 3$ are multiplicative systems such that $B_i = \sigma(M_i)$. Show $B_1 \perp \perp B_3$ iff

$$\mathbb{E}(f \cdot g|B_2) = \mathbb{E}(f|B_2) \cdot \mathbb{E}(g|B_2) \text{ a.s. } \forall f \in M_1 \text{ and } g \in M_3. \quad (3.19)$$

**Hint:** Do this by two applications of the functional form of the multiplicative systems theorem, see Theorems 12.24 and 12.29 of Chapter 12. For the first application, fix an $f \in M_1$ and let

$$H := \{g \in (B_3)_b : \mathbb{E}(f \cdot g|B_2) = \mathbb{E}(f|B_2) \cdot \mathbb{E}(g|B_2) \text{ a.s.}\}.$$ 

(See the proof of Theorem 22.4 if you get stuck.)

**Exercise 3.57.** Suppose now that $(X, Y, Z)^T$ is a mean zero Gaussian random vector with $X \in \mathbb{R}^k$, $Y \in \mathbb{R}^l$, and $Z \in \mathbb{R}^m$ as in Exercise 3.50. Show $Y \perp \perp Z$ (see Definition 19.41) iff

$$\mathbb{E}[YZ^T] = \mathbb{E}[YX^T] C^{-1} \mathbb{E}[XZ^T],$$

where

$$C = C_X := \mathbb{E}[XX^T].$$

In solving this problem, please continue to use the notation setup in Exercise 3.50.

**Exercise 3.58.** Construct an example of a martingale, $\{M_n\}_{n=0}^\infty$ such that $\mathbb{E}|M_n| \to \infty$ as $n \to \infty$. [In particular, $\{M_n\}_{n=1}^\infty$ will be a martingale which is not of the form $M_n = \mathbb{E}_{B_n}X$ for some $X \in L^1(P)$.] **(Hint:** try taking $M_n = \sum_{k=0}^n Z_k$ for a judicious choice of $\{Z_k\}_{k=0}^\infty$ which you should take to be independent, mean zero, and having $\mathbb{E}|Z_n|$ growing rather rapidly.)

**Exercise 3.59.** Show that $M_n := 2^n 1_{[0,2^{-n}]}$ for $n \in \mathbb{N}$ as defined in Example 23.7 is a martingale.

**Exercise 3.60.** Let $\mu, \nu$, and $\gamma$ be three probability measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Show:

1. $\mu \ast \nu = \nu \ast \mu$.
2. $\mu \ast (\nu \ast \gamma) = (\mu \ast \nu) \ast \gamma$. (So it is now safe to write $\mu \ast \nu \ast \gamma$ for either side of this equation.)
3. $(\mu \ast \delta_x) (A) = \mu (A - x)$ for all $x \in \mathbb{R}^n$ where $\delta_x (A) := 1_A (x)$ for all $A \in \mathcal{B}_{\mathbb{R}^n}$ and in particular $\mu \ast \delta_0 = \mu$. 

Resnick Problems