Math 280A Homework Problems Fall 2018

Problems are from Resnick, S. A Probability Path, Birkhauser, or from the lecture notes. The problems from the lecture notes are restated here. In the lecture note problems listed in the assignments, you should look up the corresponding problem in the lecture notes where more context is given including extra standing assumptions for the problem. This context and standing assumptions are not always extracted out when I construct the homework sheets.

1.1 Homework 1. Due Friday, October 5, 2018

- Read over Lecture notes Chapter 1.
- Lecture note Exercises: 2.1, 2.2, and 2.3.

1.2 Homework 2. Due Friday, October 12, 2018

- Lecture note Exercises: 2.1, 2.2, 2.3, 2.4, 2.5, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12.
- Look at Resnick, p. 20-27: 9, 12, 17, 23.
- Hand in Resnick, p. 20-27: 5, 18, 40*.

*Notes on Resnick's #40: (i) $B((0,1])$ should be $B([0,1))$ in the statement of this problem, (ii) $k$ is an integer, (iii) $r \geq 2$.

1.3 Homework 3. Due Friday, October 19, 2018

- Look at Resnick, p. 20-27: 9, 12, 17, 23.
- Hand in Resnick, p. 20-27: 5, 18, 40*.

1.4 Homework 4. Due a, October 26, 2018

- Look at lecture note exercises: 2.21, 2.28.
- Hand in lecture note exercises: 2.22, 2.23, 2.24, 2.25, 2.26, 2.27.

1.5 Homework 5. Due Friday, November 2, 2018

- Hand in Resnick exercises: § 2.6, #7* and § 2.6, #13.

*Hint: For Resnick #7 you might label the coupons as $\{1, 2, \ldots, N\}$ and let $A_i$ be the event that the collector does not have the $i^{th}$ coupon after buying $n$ boxes of cereal.

1.6 Homework 6. Due Friday, November 9, 2018

- Hand in Lecture note exercises: 2.30, 2.31, 2.41.
- Hand in lecture note exercises: 2.32, 2.33, 2.34, 2.35, 2.36, 2.37, 2.38.

*Notes. * In #6, the random variable $X$ is understood to take values in the extended real numbers.

**In #18, I would write the left side in terms of an expectation.

1.7 Homework 7. Due Wednesday, November 21, 2018

- Hand in Lecture note exercises: 2.39, 2.40, 2.42, 2.43, 2.44, 2.45, 2.46.
- Hand in from Resnick, p. 155–166: 6b, 7, 8, 12, 17, 21.
- Hand in from Resnick, p. 155–166: 3, 6, 13, 16, 30, 37.
Lecture Note Problems

Exercise 2.1 (Prove the Fubini Proposition \[1.15\]). Suppose \( \{a_{k,n}\}_{n=1}^{\infty} \subset \mathbb{R} \) such that
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |a_{k,n}| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{k,n}| < \infty.
\]
Then
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{k,n} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{k,n}.
\]

Hint: Let \( a_{k,n}^+ := \max(0, a_{k,n}) \) and \( a_{k,n}^- := \max(-a_{k,n}, 0) \) and observe that: \( a_{k,n} = a_{k,n}^+ - a_{k,n}^- \) and \( |a_{k,n}^+| + |a_{k,n}^-| = |a_{k,n}| \). Now apply Tonelli’s theorem (Proposition \[1.11\]) with \( a_{k,n} \) replaced by \( a_{k,n}^+ \) and \( a_{k,n}^- \). You should be careful to verify that \( \{a_{k,n}\}_{n=1}^{\infty} \) is summable for each \( k \) and that \( \{S_k = \sum_{n=1}^{\infty} a_{k,n}\}_{k=1}^{\infty} \) is summable so that \( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{k,n} = \sum_{k=1}^{\infty} S_k \) exits, etc. etc.

Exercise 2.2 (Prove Fatou’s Lemma in Proposition \[1.17\]). Suppose that for each \( n \in \mathbb{N} \), \( \{h_n(i)\}_{i=1}^{\infty} \) is any sequence in \([0, \infty]\), then
\[
\sum_{i=1}^{\infty} \liminf_{n \to \infty} h_n(i) \leq \liminf_{n \to \infty} \sum_{i=1}^{\infty} h_n(i).
\]

Hint: apply the MCT by applying the monotone convergence theorem with \( f_n(i) := \inf_{m \geq n} h_m(i) \).

Exercise 2.3 (Prove DCT as in Proposition \[1.18\]). Suppose that for each \( n \in \mathbb{N} \), \( \{f_n(i)\}_{i=1}^{\infty} \subset \mathbb{R} \) is a sequence and \( \{g_n(i)\}_{i=1}^{\infty} \) is a sequence in \([0, \infty]\) such that:
1. \( \sum_{i=1}^{\infty} g_n(i) < \infty \) for all \( n \),
2. \( f(i) := \lim_{n \to \infty} f_n(i) \) and \( g(i) := \lim_{n \to \infty} g_n(i) \) exists for each \( i \),
3. \( |f_n(i)| \leq g_n(i) \) for all \( i \) and \( n \),
4. \( \lim_{n \to \infty} \sum_{i=1}^{\infty} g_n(i) = \sum_{i=1}^{\infty} g(i) < \infty \).

Then
\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} f_n(i) = \sum_{i=1}^{\infty} \lim_{n \to \infty} f_n(i) = \sum_{i=1}^{\infty} f(i).
\]

Hint: Apply Fatou’s lemma twice. Once with \( h_n(i) = g_n(i) + f_n(i) \) and once with \( h_n(i) = g_n(i) - f_n(i) \).

Exercise 2.4. Suppose that \( B \subset Y \), show that \( B \setminus (\cup_{i \in I} A_i) = \cap_{i \in I} (B \setminus A_i) \).

Exercise 2.5. Let \( \{B_i\}_{i \in I} \) be another collection of subsets of \( Y \). Show \( \cup_{i \in I} A_i \setminus \cap_{i \in I} B_i \subset \cup_{i \in I} (A_i \setminus B_i) \) and then use this inclusion twice to show \( \cup_{i \in I} A_i \setminus \cap_{i \in I} B_i \subset \cup_{i \in I} (A_i \setminus B_i) \).

Exercise 2.6 (Triangle inclusion for sets). If \( A, B, C \) are subsets of \( X \), show \( A \setminus (B \cup C) = (B \setminus A) \cap (C \setminus A) \) and use this identity twice to show
\[
A \cup (B \cup C) \subset [A \setminus B] \cup [B \setminus C] \cup [C \setminus A] \quad (2.1)
\]

Exercise 2.7. Find a function \( f : X = \{a, b, c\} \to Y = \{1, 2\} \) and subsets \( C \) and \( D \) of \( X \) such that
\[
f(C \cap D) \neq f(C) \cap f(D) \quad \text{and} \quad f(C^c) \neq \overline{f(C)}^c.
\]

Exercise 2.8. Suppose that \( E_i \subset 2^X \) for \( i = 1, 2 \). Show that \( \mathcal{A}(E_1) = \mathcal{A}(E_2) \) if \( E_1 \subset \mathcal{A}(E_2) \) and \( E_2 \subset \mathcal{A}(E_1) \). Similarly show, \( \sigma(E_1) = \sigma(E_2) \) iff \( E_1 \subset \sigma(E_2) \) and \( E_2 \subset \sigma(E_1) \). Give a simple example where \( \mathcal{A}(E_1) = \mathcal{A}(E_2) \) while \( E_1 \neq E_2 \).

Exercise 2.9. Verify the Borel \( \sigma \) – algebra, \( \mathcal{B}_R \), is generated by any of the following collection of sets:
1. \( E_1 := \{(a, \infty) : a \in \mathbb{R}\} \), 2. \( E_2 := \{(a, \infty) : a \in \mathbb{Q}\} \) or 3. \( E_3 := \{[a, \infty) : a \in \mathbb{Q}\} \).

Hint: make use of the ideas in Exercise 2.8.

Exercise 2.10 (Look at but do not hand in.). Let \( X \) be a set, \( I \) be an infinite index set, and \( \mathcal{E} = \{A_i\}_{i \in I} \) be a partition of \( X \). Prove the algebra, \( \mathcal{A}(\mathcal{E}) \), and that \( \sigma \) – algebra, \( \sigma(\mathcal{E}) \), generated by \( \mathcal{E} \) are given by
\[
\mathcal{A}(\mathcal{E}) = \{\cup_{i \in I} A_i : A \subset I \text{ with } \#(A) < \infty \text{ or } \#(A^c) < \infty\} \quad (2.2)
\]
and
\[
\sigma(\mathcal{E}) = \{\cup_{i \in I} A_i : A \subset I \text{ with } A \text{ countable or } A^c \text{ countable}\} \quad (2.3)
\]
respectively. Here we are using the convention that \( \cup_{i \in A} A_i := \emptyset \) when \( A = \emptyset \). In particular if \( I \) is countable, then
\[
\sigma(\mathcal{E}) = \{\cup_{i \in I} A_i : A \subset I\}.
\]
Exercise 2.11. Let $\tau$ be a topology on a set $X$ and $A = A(\tau)$ be the algebra generated by $\tau$. Show $A$ is the collection of subsets of $X$ which may be written as finite union of sets of the form $F \cap V$ where $F$ is closed and $V$ is open.

Exercise 2.12. Let $A \subset 2^X$ and $B \subset 2^Y$ be elementary class. Show the collection

$$S := A \times B := \{ A \times B : A \in A \text{ and } B \in B \}$$

is also an elementary class.

Exercise 2.13 (Look at but do not hand in.) Show;

1. $A_n$ is an algebra for each $n \in \mathbb{N}$,
2. $A_n \subset A_{n+1}$ for all $n$, and
3. $A \subset 2^\Omega$ is an algebra of subsets of $\Omega$. (In fact, you might show that $A = \bigcup_{n=1}^\infty A_n$ is an algebra whenever $\{A_n\}_{n=1}^\infty$ is an increasing sequence of algebras.)

Exercise 2.14 (Consistency Conditions). [Look at only, do not hand in.] If $p_n$ is defined as above, show:

1. $\sum_{s \in S} p_1 (s) = 1$ and
2. for all $n \in \mathbb{N}$ and $(s_1, \ldots, s_n) \in S^n$,

$$p_n (s_1, \ldots, s_n) = \sum_{s \in S} p_{n+1} (s_1, \ldots, s_n, s).$$

These conditions are basically equivalent to the statements that $Q_1 (S) = 1$ and $Q_{n+1} (B \times S) = Q_n (B)$ for all $n \in \mathbb{N}$ and $B \subset S^n$.

Exercise 2.15 (Converse to 2.14). Suppose for each $n \in \mathbb{N}$ we are given functions, $p_n : S^n \rightarrow [0, 1]$ such that the consistency conditions in Exercise 2.14 hold. Then there exists a unique finitely additive probability measure, $P$ on $A$ such that Eq. (5.6) holds for all $n \in \mathbb{N}$ and $(s_1, \ldots, s_n) \in S^n$ and such that $P|_{A_n}$ is a $\sigma$-additive measure on $A_n$ for all $n \in \mathbb{N}$.

Exercise 2.16 ($A$ – measurable simple functions). As in Example 4.19 let $A \subset 2^\Omega$ be a finite algebra and $\{B_1, \ldots, B_k\}$ be the partition of $\Omega$ associated to $A$. Show that a function, $f : \Omega \rightarrow \mathbb{C}$, is an $A$ – simple function iff $f$ is constant on $B_i$ for each $i$. Thus any $A$ – simple function is of the form,

$$f = \sum_{i=1}^k \alpha_i 1_{B_i},$$

for some $\alpha_i \in \mathbb{C}$.

Exercise 2.17. Let $P$ is a finitely additive probability measure on an algebra $A \subset 2^\Omega$ and for $A,B \in A$ let $\rho (A, B) := P (A \Delta B)$ where $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Show:

1. $\rho (A, B) = E |1_A - 1_B|$ and then use this (or not) to show
2. $\rho (A, C) \leq \rho (A, B) + \rho (B, C)$ for all $A, B, C \in A$.

Remark: it is now easy to see that $\rho : A \times A \rightarrow [0, 1]$ satisfies the axioms of a metric except for the condition that $\rho (A, B) = 0$ does not imply that $A = B$ but only that $A = B$ modulo a set of probability zero.

Exercise 2.18. For $1 \leq k \leq n$, show:

1. (as functions on $\Omega$) that

$$\binom{N}{k} = \sum_{A \subset \{1,2,\ldots,n\} : |A| = k} 1_{\cap_{i \in A} A_i},$$

where by definition

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!} & \text{if } k > m \\ 1 & \text{if } k = 0 \end{cases}$$

2. Conclude from Eq. (2.5) that for all $z \in \mathbb{C}$,

$$(1 + z)^N = 1 + \sum_{k=1}^n z^k 1_{\cap_{i \in [1,k]} A_i}$$

provided $(1 + z)^0 = 1$ even when $z = -1$.

3. Conclude from Eq. (2.5) that $S_k = \mathbb{E} \binom{N}{k}$.

Exercise 2.19. Taking expectations of Eq. (2.7) implies,

$$\mathbb{E} \left[ (1 + z)^N \right] = 1 + \sum_{k=1}^n S_k z^k.$$  

Show that setting $z = -1$ in Eq. (2.8) gives another proof of the inclusion exclusion formula. Hint: use the definition of the expectation to write out $\mathbb{E} \left[ (1 + z)^N \right]$ explicitly.

Exercise 2.20. Let $1 \leq m \leq n$. In this problem you are asked to compute the probability that there are exactly $m$ – coincidences. Namely you should show,
Exercise 2.22 (Simple conditional expectation). Let \( X \in \mathbb{S}(\mathcal{A}) \) and, for simplicity, assume all functions are real valued. Prove the following assertions:

1. (Orthogonal Projection Property 1.) If \( Z \in \mathbb{S}(\mathcal{A}) \), then
   \[
   \mathbb{E}[XZ] = \mathbb{E}[X]\mathbb{E}[Z] = \mathbb{E}[\mathcal{E}_A X \cdot Z] \tag{2.10}
   \]
   and
   \[
   (\mathcal{E}_A Z)(\omega) = \begin{cases} 
   Z(\omega) & \text{if } P(A_\omega) > 0 \\
   0 & \text{if } P(A_\omega) = 0
   \end{cases}. \tag{2.11}
   \]
   [Applying Eq. (2.11) with \( Z = \mathcal{E}_A X \) then shows \( \mathcal{E}_A [\mathcal{E}_A X] = \mathcal{E}_A X \) for all \( X \in \mathbb{S}(\mathcal{B}) \).]

2. (Orthogonal Projection Property 2.) If \( Y \in \mathbb{S}(\mathcal{A}) \) satisfies \( \mathbb{E}[XZ] = \mathbb{E}[YZ] \) for all \( Z \in \mathbb{S}(\mathcal{A}) \), then \( Y(\omega) = \bar{X}(\omega) \) whenever \( P(A_\omega) > 0 \). In particular, \( P(X \neq \bar{X}) = 0 \). \textbf{Hint}: use item 1. to compute \( \mathbb{E}[|X - \bar{X}|^2] \).

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Hint: differentiate Eq. (2.25) \( m \) times with respect to \( z \) and then evaluate the result at \( z = -1 \). In order to do this you will find it useful to derive formulas for:

\[
\frac{d^m}{dz^m}|_{z=-1}(1 + z)^n \quad \text{and} \quad \frac{d^m}{dz^m}|_{z=-1}z^k.
\]

Exercise 2.21 (Do not hand in). Prove the following statements.

1. For all \( f \in \mathbb{S}(\mathcal{A}) \),
   \[
   |\mathbb{E}_\mu f| \leq \mu(\Omega) \|f\|_u. \tag{2.9}
   \]

2. If \( f \in \mathbb{S} \) and \( f_n \in \mathbb{S} := \mathbb{S}(\mathcal{A}) \) such that \( \lim_{n \to \infty} \|f - f_n\|_u = 0 \), show \( \lim_{n \to \infty} \mathbb{E}_\mu f_n \) exists. Also show that defining \( \mathbb{E}_\mu f := \lim_{n \to \infty} \mathbb{E}_\mu f_n \) is well defined, i.e. you must show that \( \lim_{n \to \infty} \mathbb{E}_\mu f_n = \lim_{n \to \infty} \mathbb{E}_\mu g_n \) if \( g_n \in \mathbb{S} \) such that \( \lim_{n \to \infty} \|f - g_n\|_u = 0 \).

3. Show \( \mathbb{E}_\mu : \mathbb{S} \to \mathbb{C} \) is still linear and still satisfies Eq. (2.9).

4. Show \( |f| \in \mathbb{S} \) if \( f \in \mathbb{S} \) and that Eq. (2.25) is still valid, i.e. \( |\mathbb{E}_\mu f| \leq \mathbb{E}_\mu |f| \) for all \( f \in \mathbb{S} \).

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Exercise 2.23. Suppose that \( S, T, U \) are sets and \( X : \Omega \to S, Y : \Omega \to T \), and \( Z : \Omega \to U \) are \( \mathcal{B} \)-simple functions such that \( Z \) is independent of \( (X, Y) : \Omega \to S \times T \). Show that \( Z \) is independent of \( X \). [In words, if knowing values of both \((X, Y)\) does not change the likelihood that \( Z = u \) then knowing the value of just \( X \) does not change the likelihood that \( Z = u \) either.]

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3. (Best Approximation Property.) For any \( Y \in \mathbb{S}(\mathcal{A}) \),
   \[
   \mathbb{E}[|X - \bar{X}|^2] \leq \mathbb{E}[|X - Y|^2] \tag{2.12}
   \]
   with equality iff \( \bar{X} = Y \) almost surely (a.s. for short), where \( \bar{X} = Y \) a.s. iff \( P(X \neq Y) = 0 \). In words, \( \bar{X} = \mathcal{E}_A X \) is the best (“\( L^2 \)”) approximation to \( X \) by an \( \mathcal{A} \)-measurable random variable.

4. (Contraction Property.) \( \mathbb{E}[|X|] \leq \mathbb{E}[|X|] \). (It is typically \textbf{not} true that \( |\bar{X}\omega| \leq |X\omega| \) for all \( \omega \).)

5. (Pull Out Property.) If \( Z \in \mathbb{S}(\mathcal{A}) \), then
   \[
   \mathcal{E}_A [ZX] = \mathcal{E}_A X.
   \]

\textbf{Remark 2.1.} The cleanest way to see that \( \mathcal{E}_A \) in Eq. (2.50) is an orthogonal projection is to let
   \[
   (X, Y) := \mathbb{E}[XY] \quad \text{for all } X, Y \in \mathbb{S}(\mathcal{B})
   \]
   and observe that \((\cdot, \cdot)\) satisfies the axioms of inner product except for possibly the axiom that \( (X, X) = 0 \) implies \( X = 0 \). What is true is that if \( (X, X) = 0 \), then \( X = 0 \) a.s., i.e. \( P(X \neq 0) = 0 \). To avoid technicalities associate with these “null” sets, let us suppose that \( P(B_i) > 0 \) for each \( i \). In this case \[ \left\{ \frac{1_B}{\sqrt{P(B)}} \right\}_i \]
   is an orthonormal basis for the subspace \( \mathbb{S}(\mathcal{A}) \subset \mathbb{S}(\mathcal{B}) \). Therefore orthogonal projection from \( \mathbb{S}(\mathcal{B}) \) onto \( \mathbb{S}(\mathcal{A}) \) is given by
   \[
   X \to \sum_i \left( X \cdot \frac{1_B}{\sqrt{P(B)}} \right) \frac{1_B}{\sqrt{P(B)}} = \sum_i \frac{\mathbb{E}[X1_B]}{P(B)} 1_B,
   \]
   which is precisely the formula for \( \mathcal{E}_A X \).

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Exercise 2.24. Suppose that \( X_i : \Omega \to S_i \) is an \( \mathcal{B} \)-simple function for each \( 1 \leq i \leq n \). Show \( \{X_i\}_{i=1}^n \) are independent iff
   \[
   P(\cap_{i=1}^n \{X_i = s_i\}) = \prod_{i=1}^n P(X_i = s_i) \tag{2.13}
   \]
   for all \((s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n \).


Exercise 2.25. Suppose that $X_i : \Omega \to S_i$ is an $B$—simple function for each $1 \leq i \leq n$. Show the following are equivalent:

1. $\{X_i\}_{i=1}^{n}$ are independent.
2. For all choices of functions, $f_i : S_i \to \mathbb{R}$ with $1 \leq i \leq n$,
   \[
   \mathbb{E} \left[ \prod_{i=1}^{n} f_i (X_i) \right] = \prod_{i=1}^{n} \mathbb{E} [ f_i (X_i) ] .
   \]
3. For all choices of subsets $A_i \subset S_i$ with $1 \leq i \leq n$,
   \[
   P ( \cap_{i=1}^{n} \{ X_i \in A_i \} ) = \prod_{i=1}^{n} P ( \{ X_i \in A_i \} ) .
   \]

Exercise 2.26. Suppose now that $S$ is a finite set, $\Omega = S^n$, $B = 2^\Omega$, and $X_i : \Omega \to S$ is projection onto the $i^{th}$ factor of $\Omega$, i.e. $X_i (\omega) = \omega_i \in S$ for all $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$.

Let $P$ be a probability measure on $(\Omega, B)$. Show $\{X_i\}_{i=1}^{n}$ are $P$—independent iff $P$ there are functions, $q_i : S \to [0,1]$ (for $1 \leq i \leq n$) such that $\sum_{s \in S} q_i (s) = 1$ and

\[
P ( \{ s \} ) = \prod_{i=1}^{n} q_i (s_i) \quad \text{for all } s \in S .
\]

Exercise 2.27 (A Weak Law of Large Numbers). Suppose that $A \subset \mathbb{R}$ is a finite set, $n \in \mathbb{N}$, $\Omega = A^n$, $p (\omega) = \prod_{i=1}^{n} q (\omega_i)$ where $q : A \to [0,1]$ such that $\sum_{\lambda \in A} q (\lambda) = 1$, and let $P : 2^\Omega \to [0,1]$ be the probability measure defined as in Eq. (2.15) with $S$ replaced by $A$. Further let $X_i (\omega) = \omega_i$ for $i = 1, 2, \ldots, n$, $\xi := EX_i$, $\sigma^2 := \mathbb{E} (X_i - \xi)^2$, and

\[
S_n = \frac{1}{n} (X_1 + \cdots + X_n) .
\]

1. Show, $\xi = \sum_{\lambda \in A} \lambda \cdot q (\lambda)$ and
   \[
   \sigma^2 = \sum_{\lambda \in A} (\lambda - \xi)^2 q (\lambda) = \sum_{\lambda \in A} \lambda^2 q (\lambda) - \xi^2 .
   \]
2. Show, $\mathbb{E} S_n = \xi$.
3. Let $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Show
   \[
   \mathbb{E} [(X_i - \xi) (X_j - \xi)] = \delta_{ij} \sigma^2 .
   \]
4. Using $S_n - \xi$ may be expressed as, $\frac{1}{n} \sum_{i=1}^{n} (X_i - \xi)$, show
   \[
   \mathbb{E} (S_n - \xi)^2 = \frac{1}{n} \sigma^2 .
   \]

5. Conclude using Eq. (2.18) and Chebyshev’s Inequality (Remark 5.29) that
   \[
   P (|S_n - \xi| \geq \varepsilon) \leq \frac{\sigma^2}{n \varepsilon^2} .
   \]

So for large $n$, $S_n$ is concentrated near $\xi = EX_i$ with probability approaching 1 for $n$ large. This is a version of the weak law of large numbers.

Exercise 2.28 (Bernoulli Random Variables). Let $A = \{0,1\}$, $X : A \to \mathbb{R}$ be defined by $X (0) = 0$ and $X (1) = 1$, $x \in [0,1]$, and define $Q = x \delta_1 + (1-x) \delta_0$, i.e. $Q (\{0\}) = 1-x$ and $Q (\{1\}) = x$. Verify,

\[
\xi (x) := E_Q X = x \quad \text{and} \quad \sigma^2 (x) := E_Q (X - x)^2 = (1-x) x \leq 1/4 .
\]

Exercise 2.29. Suppose $A \subset 2^\Omega$ is an algebra and $\mu$ and $\nu$ are two finite $\sigma$-additive measures on $B = \sigma (A)$ such that $\mu = \nu$ on $A$. Show $\mu = \nu$ on $B$.

Exercise 2.30. Let $\mu$, $\mu$, $\Lambda$, and $B := B (\mu)$ be as in Theorem 6.20. Further suppose that $B_0 \subset 2^\Omega$ is a $\sigma$-algebra such that $A \subset B_0 \subset B$ and $\nu : B_0 \to [0, \mu (\Omega)]$ is a $\sigma$-additive measure on $B_0$ such that $\nu = \mu$ on $A$. Show that $\nu = \mu$ on $B_0$ as well.

Exercise 2.31. Suppose $S$ is a finite set, $\Omega = S^N$, and $B := \sigma (A)$ as above. Show every sequence of probability measures $\{P_n\}_{n=1}^{\infty}$ possesses a subsequence $\{P_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k \to \infty} P_{n_k} (A) = P (A)$ for all $A \in A$ where $P$ is a probability measure on $(\Omega, B)$. Hints: 1) note that $A$ is a countable collection of subsets of $\Omega$, and 2) use Cantor's diagonalization argument.

Exercise 2.32. If $(X, \rho)$ is a metric space and $\mu$ is a finite measure on $(X, B_X)$, then for all $A \in B_X$ and $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $\mu (V \setminus F) = \mu (F \triangle V) < \varepsilon$. Here are some suggestions.

1. Let $B_0$ denote those $\Lambda \subset X$ such that for all $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $\delta_\mu (F, V) = \mu (V \setminus F) < \varepsilon$.
2. Show $B_0$ contains all closed (or open if you like) sets using Corollary 6.47.
3. Show $B_0$ is a $\sigma$-algebra. [You may find Proposition 6.50 to be helpful in this step.]
4. Explain why this proves the result.
**Exercise 2.33.** Let \((X, \rho)\) be a metric space and \(\mu\) be a measure on \((X, B_X)\). If there exists open sets, \(\{V_n\}_{n=1}^\infty\), of \(X\) such that \(V_n \uparrow X\) and \(\mu(V_n) < \infty\) for all \(n\), then for all \(A \in B_X\) and \(\varepsilon > 0\) there exists a closed set \(F\) and open set \(V\) such that \(F \subset A \subset V\) and \(d_\mu(F, V) = \mu(V \setminus F) < \varepsilon\). **Hints:**

1. Show it suffices to prove; for all \(\varepsilon > 0\) and \(A \in B_X\), there exists an open set \(V \subset X\) such that \(A \subset V\) and \(\mu(V \setminus A) < \varepsilon\).
2. Now you must verify the assertion above holds. For this, you may find it useful to apply Exercise 2.32 to the measures, \(\mu_n : B_X \to [0, \mu(V_n)]\), defined by \(\mu_n(A) := \mu(A \cap V_n)\) for all \(A \in B_X\). The \(\varepsilon > 0\) in Exercise 2.32 should be replaced by judiciously chosen small quantities \(\varepsilon_n > 0\) depending on \(n\), for example \(\varepsilon_n = \varepsilon 2^{-n}\) will work.

**Exercise 2.34.** If \(f : X \to Y\) is a function and \(\mathcal{F} \subset 2^Y\) and \(\mathcal{B} \subset 2^X\) are \(\sigma\)–algebras (algebras), then \(f^* \mathcal{F}\) and \(f_* \mathcal{B}\) are \(\sigma\)–algebras (algebras).

**Exercise 2.35.** Given \(x \in \mathbb{R}\setminus \{0\}\) let

\[
x + B := \{x + y : y \in B\}
\]

and \(x \cdot B := \{xy : y \in B\}\).

Show \(x + B\) and \(x \cdot B\) are in \(\mathcal{B}_\mathbb{R}\) for all \(B \in \mathcal{B}_\mathbb{R}\). **Hint:** take \(\mathcal{E} = \{(a, b) : -\infty < a < b < \infty\}\) and apply Lemma 9.3 with \(f(y) = y - x\) and \(f(y) = y/x\) respectively.

**Exercise 2.36 (Look at but do not hand in).** Prove Corollary 9.12, i.e. suppose that \((X, \mathcal{M})\) is a measurable space. Then the following conditions on a function \(f : X \to \mathbb{R}\) are equivalent:

1. \(f\) is \((\mathcal{M}, \mathcal{B}_\mathbb{R})\) – measurable,
2. \(f^{-1}((a, \infty)) \in \mathcal{M}\) for all \(a \in \mathbb{R}\),
3. \(f^{-1}((a, \infty)) \in \mathcal{M}\) for all \(a \in \mathbb{Q}\),
4. \(f^{-1}((-\infty, a]) \in \mathcal{M}\) for all \(a \in \mathbb{R}\).

**Hint:** See Exercise 2.49

**Exercise 2.37.** Show that every monotone function \(f : \mathbb{R} \to \mathbb{R}\) is \((\mathcal{B}_\mathbb{R}, \mathcal{B}_\mathbb{R})\) – measurable.

**Exercise 2.38.** Let \((X, \mathcal{M})\) be a measure space and \(f_n : X \to \mathbb{R}\) be a sequence of measurable functions on \(X\). Show that \(\{x : \lim_{n \to \infty} f_n(x) \text{ exists in } \mathbb{R}\} \in \mathcal{M}\). Similarly show the same holds if \(\mathbb{R}\) is replaced by \(\mathbb{C}\).

**Exercise 2.39.** Let \(X\) be a set, \(I\) an index set, and suppose there is a collection of measurable spaces \(\{(Y_\alpha, \mathcal{F}_\alpha) : \alpha \in I\}\) and functions \(f_\alpha : X \to Y_\alpha\) for all \(\alpha \in I\). If \(\mathcal{E}_\alpha \subset \mathcal{F}_\alpha\) for each \(\alpha \in I\) such that \(\mathcal{F}_\alpha := \sigma(\mathcal{E}_\alpha)\), show

\[
\sigma(f_\alpha : \alpha \in I) = \sigma(\cup_{\alpha \in I} f_\alpha^* \mathcal{F}_\alpha) = \sigma(\cup_{\alpha \in I} f_\alpha^* \mathcal{E}_\alpha).
\]

**Exercise 2.40.** Let \(X\) be a set, \(I\) be an index set, and suppose there is a collection of measurable spaces \(\{(Y_\alpha, \mathcal{F}_\alpha) : \alpha \in I\}\) and functions \(f_\alpha : X \to Y_\alpha\) for all \(\alpha \in I\). If \(Z\) is another set and \(G : Z \to X\) is a function then \(G^* \sigma(f_\alpha : \alpha \in I) = \sigma(f_\alpha \circ G : \alpha \in I)\).

**Exercise 2.41.** If \(\mathcal{M}\) is the \(\sigma\)–algebra generated by \(\mathcal{E} \subset 2^X\), then \(\mathcal{M}\) is the union of the \(\sigma\)–algebras generated by countable subsets \(\mathcal{F} \subset \mathcal{E}\).

**Exercise 2.42.** Suppose that \((Y_1, \mathcal{F}_1)\) and \((Y_2, \mathcal{F}_2)\) are measurable spaces and \(\mathcal{E}_i\) is a subset of \(\mathcal{F}_i\) such that \(Y_1 \in \mathcal{E}_1\) and \(\mathcal{F}_1 = \sigma(\mathcal{E}_1)\) for \(i = 1, 2\). Show \(\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma(\mathcal{E})\) where \(\mathcal{E} := \{A_1 \times A_2 : A_i \in \mathcal{E}_i\} \subset 2^{Y_1 \times Y_2}\).

**Exercise 2.43.** Suppose that \((Y_1, \mathcal{F}_1)\) and \((Y_2, \mathcal{F}_2)\) are measurable spaces and \(\emptyset \neq B_i \subset Y_i\) for \(i = 1, 2\). Show

\[
[\mathcal{F}_1 \otimes \mathcal{F}_2]_{B_1 \times B_2} = [\mathcal{F}_1]_{B_1} \otimes [\mathcal{F}_2]_{B_2}\.
\]

**Exercise 2.44 (Abstract Change of Variables Formula).** Let \((X, \mathcal{M}, \mu)\) be a measure space, \((Y, \mathcal{F})\) be a measurable space and \(f : X \to Y\) be a measurable map. Recall that \(\nu = f_* \mu : \mathcal{F} \to [0, \infty]\) defined by \(\nu(A) := \mu(f^{-1}(A))\) for all \(A \in \mathcal{F}\) is a measure on \(\mathcal{F}\).

1. Show

\[
\int_Y g d\nu = \int_X (g \circ f) d\mu
\]

for all measurable functions \(g : Y \to [0, \infty]\). **Hint:** see the hint from Exercise 10.6

2. Show a measurable function \(g : Y \to \mathbb{C}\) is in \(L^1(\nu)\) iff \(g \circ f \in L^1(\mu)\) and that Eq. (2.21) holds for all \(g \in L^1(\nu)\).

**Exercise 2.45 (A Weak Law of Large Numbers).** Assume \(\{X_n\}_{n=1}^\infty\) is a sequence if uncorrelated square integrable random variables which are identically distributed, i.e. \(X_n \overset{d}{=} X_m\) for all \(m, n \in \mathbb{N}\). Let \(S_n := \sum_{k=1}^n X_k, \mu := E X_k\) and \(\sigma^2 := \text{Var}(X_k)\) (these are independent of \(k\)). Show;
\[ \mathbb{E} \left[ \frac{S_n}{n} \right] = \mu, \]
\[ \mathbb{E} \left( \frac{S_n}{n} - \mu \right)^2 = \text{Var} \left( \frac{S_n}{n} \right) = \frac{\sigma^2}{n}, \quad \text{and} \]
\[ P \left( \left| \frac{S_n}{n} - \mu \right| > \varepsilon \right) \leq \frac{\sigma^2}{n\varepsilon^2} \]

for all \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). (Compare this with Exercise 2.27.)

**Exercise 2.46.** For any function \( f \in L^1(m) \), show \( x \in \mathbb{R} \rightarrow \int_{[-\infty,x]} f(t) \, dm(t) \) is continuous in \( x \). Also find a finite measure, \( \mu \), on \( \mathcal{B} \mathbb{R} \) such that \( x \rightarrow \int_{[-\infty,x]} f(t) \, d\mu(t) \) is not continuous.
Resnick Problems