Math 280B Homework Problems Winter 2019

Problems are from Resnick, S. A Probability Path, Birkhauser, or from the lecture notes. The problems from the lecture notes are restated here. In the lecture note problems listed in the assignments, you should look up the corresponding problem in the lecture notes where more context is given including extra standing assumptions for the problem. This context and standing assumptions are not always extracted out when I construct the homework sheets.

2.1 Homework 1. Due Monday, January 14, 2019
- Look at Lecture note Exercise. 3.5 13.3 – 13.4.
- Look at from Resnick § 5.10: 8, 18, 19, 22
- Hand in Lecture note Exercises: 3.1 3.2 3.3 3.6 3.7
- Hand in from Resnick § 5.10: 9

2.2 Homework 2. Due Wednesday, January 23, 2019
- Look at Lecture note Exercise 3.8 3.9 3.10 3.11 3.12 3.13 3.15 3.17
- Look at from Resnick § 4.6: 3, 5
- Hand in Lecture note Exercises: 3.10 3.11 3.12 3.13 3.14 3.16 3.18 3.21
- Hand in from Resnick § 4.6: 6, 19

2.3 Homework 3. Due Monday, January 28, 2019
You should work alone on this homework set! Please ask questions about these problems in class.
- Look at Lecture note Exercise: Read Proposition 12.30, 3.19*
- Look at from Resnick § 4.6: 28, 29.
- Hand in Lecture note Exercises: 3.38, 3.20

2.4 Homework 4. Due Monday, February 4, 2019
- Look at Lecture note Exercise: 3.27, 3.28, 3.29
- Look at from Resnick § 5.10: #39
- Hand in Lecture note Exercise 3.4 3.22 3.23 3.24 3.25 3.26

2.5 Homework 5. Due Monday, February 11, 2019
- Look at Lecture note Exercise: 3.30 3.31 3.32 3.33 3.34 3.35 3.36 3.37
- Hand in Lecture note Exercise 3.31 3.32 3.33 3.34 3.35 3.36 3.37

* Exercise 3.19 has been replaced by Exercise 3.38 in the problems to be handed in as Exercise 3.19 was already given last quarter. Sorry about that.
Lecture Note Problems

Exercise 3.1. Let \((\Omega, \mathcal{B}, P)\) be a probability space and \(X : \Omega \to \mathbb{R}\) is a random variable. Prove the following assertions.

1. If \(a < b\) and \(a, b \in \text{median (} X \text{) , then} \ [a, b] \subset \text{median (} X \text{) .}
2. If
\[
m_− := \inf \{m \in \mathbb{R} : P (X \leq m) \geq 1/2\} \quad \text{and} \quad m_+ := \sup \{m \in \mathbb{R} : P (X \geq m) \geq 1/2\},
\]
show \(-\infty < m_- \leq m_+ < \infty\) and \(\text{median (} X \text{) = [m_-, m_+] . [In particular median (} X \text{) \neq \emptyset.]}
3. If \(I = [a, b]\) is a bounded closed interval such that \(P (X \in I) \geq 1/2,\) then either \(\{a, b\} \cap \text{median (} X \text{) \neq \emptyset\) or \(\text{median (} X \text{) \subset (} a, b \text{).}

Exercise 3.2. Let \((\Omega, \mathcal{B}, P)\) be a probability space, \(X \in L^1 (P : \mathbb{R}) ,\) and \(m \in \text{median (} X \text{) . Show for any} 1 \leq p < \infty \text{ and} a \in \mathbb{R} \text{ that}
\[
|m - a|^p \leq 2E ||X - a||^p.
\]
[In particular, by taking \(a = EX\) and \(p = 2\) it follows that\footnote{One may remove the factor of 2 here by using the next problem along with Hölder’s or Jensen’s inequalities which has not yet been covered.}]
\[
|m - EX| \leq \sqrt{2 \text{Var (} X \text{)}}
\]
and if \(\text{Var (} X \text{)} \) is small the medians of \( X \) and mean, \(EX,\) must be close. ] Hint: you may find it useful to consider two case, \(m > a\) and \(m < a.\)

Exercise 3.3 (\(L^1\)-minimization). Let \((\Omega, \mathcal{B}, P)\) be a probability space, \(X \in L^1 (P : \mathbb{R}) ,\) and \(m \in \text{median (} X \text{) . Show
\[
E |X - a| \geq E |X - m| \quad \text{for all} \quad a \in \mathbb{R}.
\]
HINTS:
1. Let \(Y = X - m\) and \(\alpha = a - m,\) then \(0 \in \text{median (} Y \text{) = median (} X \text{) -} m\) and Eq. 3.2 is then equivalent to proving
\[
E |Y - \alpha| \geq E |Y| \quad \text{for all} \quad \alpha \in \mathbb{R}.
\]
2. A simple exercise shows for \(\alpha > 0\) that
\[
\begin{align*}
|Y - \alpha| - |Y| &= -\alpha1_{Y \geq \alpha} + (\alpha - 2Y)1_{0 < Y < \alpha} + \alpha1_{Y \leq 0} \\
&\geq -\alpha1_{Y > 0} + \alpha1_{Y \leq 0}.
\end{align*}
\]
3. Integrate Eq. 3.5 to find to prove Eq. 3.3 for \(\alpha > 0.\)
4. If \(\alpha < 0\) show that you may use item 3. by replacing \(Y\) by \(-Y.\)

Exercise 3.4 (A law of rare events). Let \(S_{n,p} \sim \text{Binomial (} n, p \text{) ,} k \in \mathbb{N},\)
\(p_n = \lambda_n / n\) where \(\lambda_n \to \lambda > 0\) as \(n \to \infty.\) Show that
\[
\lim_{n \to \infty} P (S_{n,p_n} = k) = \frac{\lambda^k}{k!} e^{-\lambda} = P (\text{Poisson (} \lambda \text{) =} k).
\]
[See Exercise 10.14 for the definitions of the distributions being used here.]

Interpretation. Given a large \(n \in \mathbb{N},\) \(k \in \mathbb{N}\) with \(k<<n,\) and \(p = O (1/n),\) then
\[
P (\text{Binomial (} n, p \text{) =} k) \cong P (\text{Poisson (} pn \text{) =} k) = \frac{(pn)^k}{k!} e^{-pn}.
\]
(We will come back to the Poisson distribution and the related Poisson process later on.)
Exercise 3.5. Let \( \{X_i\}_{i=1}^\infty \) and \( \{Y_i\}_{i=1}^\infty \) be two sequences of random variables such that \( \{X_i\}_{i=1}^\infty \overset{d}{=} \{Y_i\}_{i=1}^\infty \). Let \( \{S_n\}_{n=1}^\infty \) and \( \{T_n\}_{n=1}^\infty \) be defined by, \( S_n := X_1 + \cdots + X_n \) and \( T_n := Y_1 + \cdots + Y_n \). Prove the following assertions.

1. Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^k \) is a \( \mathcal{B}_{\mathbb{R}^n}/\mathcal{B}_{\mathbb{R}^k} \) measurable function, then \( f(X_1, \ldots, X_n) \overset{d}{=} f(Y_1, \ldots, Y_n) \).
2. Use your result in item 1. to show \( \{S_n\}_{n=1}^\infty \overset{d}{=} \{T_n\}_{n=1}^\infty \).

Hint: Apply item 1. with \( k = n \) after making a judicious choice for \( f : \mathbb{R}^n \to \mathbb{R}^n \).

Exercise 3.6 (Folland Problem 2.48 on p. 69.). Let \( X = Y = \mathbb{N}, \mathcal{B} = 2^\mathbb{N} \) and \( \mu = \nu = \sum_{n=1}^\infty \delta_n \) be counting measure on \((X, \mathcal{B})\) and \((Y, \mathcal{B})\). If \( f : X \times Y \to \mathbb{R} \) is defined by

\[
f(m, n) = 1_{m=n} - 1_{m=n+1}
\]

show \( \int_{\mathbb{N} \times \mathbb{N}} |f| d(\mu \otimes \nu) = \infty \) while \( \int_{\mathbb{N}} d\mu(m) \int_{\mathbb{N}} d\nu(n) f(m, n) \) and \( \int_{\mathbb{N}} d\nu(n) \int_{\mathbb{N}} d\mu(m) f(m, n) \) both exist but are unequal.

Exercise 3.7 (Folland Problem 2.56 on p. 77.). Let \( a \in (0, \infty), f \in L^1((0, a), dm) \), and

\[
g(x) = \int_x^a \frac{f(t)}{t} dt \text{ for } x \in (0, a).
\]

Show \( g \in L^1((0, a), dm) \) and

\[
\int_0^a g(x) dx = \int_0^a f(t) dt.
\]

Exercise 3.8 (Folland Problem 2.60 on p. 77.). If \( \Gamma(x) \) is the \( \Gamma \)-function as in Definition 10.45, show

\[
\Gamma(x) / \Gamma(x+y) = \int_0^1 (1-r)^{x-1} r^{y-1} \, dr \text{ for all } x, y > 0.
\]

Hint: write \( \Gamma(x) / \Gamma(x+y) \) as a double integral and then make a couple of change of variables.

Exercise 3.9. Continuing the setup in Theorem 14.13, show that

\[
\int_\Omega |f \circ T| |\det T'| dm < \infty
\]

and if \( f \in L^1(T(\Omega), dm^d) \), then Eq. (14.8) holds.

Exercise 3.10. Let \( P \) be the probability measure on \( \Omega := \mathbb{R}^d \) defined by

\[
dP(x) : = \left( \frac{1}{2\pi} \right)^{d/2} e^{-\frac{1}{2} x \cdot x} \, dx = \prod_{i=1}^d \left( \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} \right). \]

Show that \( N : \Omega \to \mathbb{R}^d \) defined by \( N(x) = x \) is Gaussian and satisfies Eq. (14.29) with \( Q = I \) and \( \mu = 0 \). Also show that

\[
\mu_i = \mathbb{E} N_i \text{ and } \delta_{ij} = \text{Cov} (N_i, N_j) \text{ for all } 1 \leq i, j \leq d. \quad (3.6)
\]

Hint: use Exercise 10.17 and (of course) Fubini’s theorem.

Exercise 3.11. Let \( A \) be any real \( m \times d \) matrix and \( \mu \in \mathbb{R}^m \) and set \( X := AN+b \) where \( \Omega = \mathbb{R}^d \), \( P \), and \( N \) are as in Exercise 3.10. Show that \( X \) is Gaussian by showing Eq. (14.29) holds with \( Q = AA^T \) (\( A^T \) is the transpose of the matrix \( A \)) and \( \mu = b \). Also show that

\[
\mu_i = \mathbb{E} X_i \text{ and } Q_{ij} = \text{Cov} (X_i, X_j) \text{ for all } 1 \leq i, j \leq m. \quad (3.7)
\]

Exercise 3.12. Suppose that \( Q \) is a positive definite (for simplicity) \( d \times d \) real matrix and \( \mu \in \mathbb{R}^m \) and let \( \Omega := \mathbb{R}^d \), \( P \), and \( N \) be as in Exercise 3.10. By Exercise 3.11 we know that \( X = Q^{1/2} N + \mu \) is a Gaussian random vector satisfying Eq. (14.29). Use the multi-dimensional change of variables formula to show

\[
\text{Law}_P(X) (dy) = \frac{1}{\sqrt{\det(2\pi Q)}} \exp \left( -\frac{1}{2} Q^{-1} (y-\mu) \cdot (y-\mu) \right) dy.
\]

Exercise 3.13 (Gaussian random vectors are “highly” integrable.). Suppose that \( X : \Omega \to \mathbb{R}^d \) is a Gaussian random vector, say \( X \overset{d}{=} N(Q, \mu) \). Let \( \|x\| := \sqrt{x \cdot x} \) and \( m := \max \{ Qx : \|x\| = 1 \} \) be the largest eigenvalue of \( Q \). Then \( \mathbb{E} e^{\varepsilon \|x\|^2} \leq 1 \) for every \( \varepsilon < \frac{1}{2m} \).

Exercise 3.14. Prove the assertion made in Remark 14.35. Hint: explicitly compute \( \mathbb{E} [e^{i(\lambda_1 X + \lambda_2 Y)}] \).

Exercise 3.15. Let \( X, Y \) be two random variables on \( (\Omega, \mathcal{B}, P) \).

1. Show that \( X \) and \( Y \) are independent iff \( \text{Cov} (f(X), g(Y)) = 0 \) (i.e. \( f(X) \) and \( g(Y) \) are uncorrelated) for bounded measurable functions, \( f, g : \mathbb{R} \to \mathbb{R} \).
2. If \( X, Y \in L^2(P) \) and \( X \) and \( Y \) are independent, then \( \text{Cov} (X, Y) = 0 \).
3. Show by example that if $X, Y \in L^2(P)$ and $\text{Cov}(X,Y) = 0$ does not necessarily imply that $X$ and $Y$ are independent. **Hint:** try taking $(X,Y) = (X,ZX)$ where $X$ and $Z$ are independent simple random variables such that $EZ = 0$ similar to Remark 14.35.

**Exercise 3.16 (A correlation inequality).** Suppose that $X$ is a random variable and $f, g : \mathbb{R} \to \mathbb{R}$ are two increasing functions such that both $f(X)$ and $g(X)$ are square integrable, i.e. $\mathbb{E}|f(X)|^2 + \mathbb{E}|g(X)|^2 < \infty$. Show $\text{Cov}(f(X), g(X)) \geq 0$. **Hint:** let $Y$ be another random variable which has the same law as $X$ and is independent of $X$. Then consider

$$\mathbb{E}[(f(Y) - f(X)) \cdot (g(Y) - g(X))].$$

**Exercise 3.17.** Suppose that $X \overset{d}{=} N(0,a^2)$ and $Y \overset{d}{=} N(0,b^2)$ and $X$ and $Y$ are independent. Show by direct computation using the formulas for the distributions of $X$ and $Y$ that $X + Y = N(0,a^2 + b^2)$.

**Exercise 3.18.** Show that the sum, $N_1 + N_2$, of two independent Poisson random variables, $N_1$ and $N_2$, with parameters $\lambda_1$ and $\lambda_2$ respectively is again a Poisson random variable with parameter $\lambda_1 + \lambda_2$. (You could use generating functions or do this by hand.) In short Poi $(\lambda_1) \overset{+}{=} \overset{d}{=} \overset{+}{=} \text{Poi}(\lambda_1 + \lambda_2).

**Exercise 3.19 (Gamma Distributions).** Let $X$ be a positive random variable. For $k, \theta > 0$, we say that $X \overset{d}{=} \text{Gamma}(k, \theta)$ if

$$(X,P) (dx) = f(x;k,\theta) \, dx \text{ for } x > 0,$$

where

$$f(x;k,\theta) := \frac{x^{k-1} \, e^{-x/\theta}}{\theta^k \Gamma(k)} \text{ for } x > 0, \text{ and } k,\theta > 0.$$ 

Find the moment generating function (see Definition 10.58), $M_X(t) = \mathbb{E}[e^{tX}]$ for $t < \theta^{-1}$. Differentiate your result in $t$ to show

$$\mathbb{E}[X^m] = k(k+1)\ldots(k+m-1)\theta^m \text{ for all } m \in \mathbb{N}_0.$$ 

In particular, $\mathbb{E}[X] = k\theta$ and $\text{Var}(X) = k\theta^2$. (Notice that when $k = 1$ and $\theta = \lambda^{-1}$, $X \overset{d}{=} \text{E}(\lambda)$.)

**Exercise 3.20.** Use the following outline to give a proof of Theorem 15.35.

1. First show that $x^n \leq 1 + x^4$ for all $x \geq 0$ and $1 \leq p \leq 4$. Use this to conclude;

$$\mathbb{E}|X_n|^p \leq 1 + \mathbb{E}|X_n|^4 < \infty \text{ for } 1 \leq p \leq 4.$$ 

Thus $\gamma := \mathbb{E}[|X_n - \mu|^4]$ and the standard deviation $(\sigma^2)$ of $X_n$ defined by,

$$\sigma^2 := \mathbb{E}[X_n^2] - \mu^2 = \mathbb{E}[(X_n - \mu)^2] < \infty,$$

are finite constants independent of $n$.

2. Show for all $n \in \mathbb{N}$ that

$$\mathbb{E}\left[\left(\frac{S_n - \mu}{n}\right)^4\right] = \frac{1}{n^4}(n\gamma + 3n(n-1)\sigma^4) = \frac{1}{n^2}\left[n^{-1}\gamma + 3 \left(1 - n^{-1}\right) \sigma^4\right].$$

(Thus $\frac{S_n}{n} \to \mu$ in $L^4(P)$.)

3. Use item 2. and Chebyshev’s inequality to show

$$P\left(\left|\frac{S_n - \mu}{\varepsilon}\right| > \varepsilon\right) \leq \frac{n^{-1}\gamma + 3 \left(1 - n^{-1}\right) \sigma^4}{\varepsilon^4 n^2}.$$ 

4. Use item 3. and the first Borel Cantelli Lemma 10.15 to conclude

$$\lim_{n \to \infty} \frac{S_n}{n} = \mu \text{ a.s.}$$

**Exercise 3.21.** Suppose that $(\Omega,B,P)$ is a probability space, $Y : \Omega \to \mathbb{R}$ is a random variable and $c \in \mathbb{R}$ is a constant. Then $Y = c \text{ a.s.}$ iff $Y \overset{d}{=} c$.

**Exercise 3.22.** Show $m_n(\Delta_n(T)) = T^n/n!$ where $m_n$ is Lebesgue measure on $B_{2^n}$. 

**Exercise 3.23.** If $n \in \mathbb{N}$ and $g : \Delta_n \to \mathbb{R}$ bounded (non-negative) measurable, then

$$\mathbb{E}[g(W_1,\ldots,W_n)] = \int_{\Delta_n} g(w_1,\ldots,w_n) \lambda^n e^{-\lambda w} dw_1 \ldots dw_n. \quad (3.8)$$

**Exercise 3.24.** Show $N_t \overset{d}{=} \text{Poi}(\lambda t)$ for all $t > 0$.

**Exercise 3.25.** Suppose that $X_1,\ldots,X_n$ are non-negative random variables such that $P(X_i = X_j) = 0$ for all $i \neq j$. Show;

4 The non-negativity of the $X_i$ are not really necessary here but this is all we need to consider.
3 Lecture Note Problems

1. If $f: \Delta_n \to \mathbb{R}$ is bounded (non-negative) measurable, then

$$E \left[ f \left( \tilde{X}_1, \ldots, \tilde{X}_n \right) \right] = \sum_{\sigma \in S_n} E \left[ f \left( X_{\sigma 1}, \ldots, X_{\sigma n} \right) : X_{\sigma 1} < X_{\sigma 2} < \cdots < X_{\sigma n} \right].$$

where $S_n$ is the permutation group on $\{1, 2, \ldots, n\}$.

2. If we further assume that $\{X_1, \ldots, X_n\}$ are i.i.d. random variables, then

$$E \left[ f \left( \tilde{X}_1, \ldots, \tilde{X}_n \right) \right] = n! \cdot E \left[ f \left( X_1, \ldots, X_n \right) : X_1 < X_2 < \cdots < X_n \right].$$

(3.9)

(It is not important that $f \left( \tilde{X}_1, \ldots, \tilde{X}_n \right)$ is not defined on the null set,

$\cup_{i \neq j} \{X_i = X_j\}.$

3. $f: \mathbb{R}_n^+ \to \mathbb{R}$ is a bounded (non-negative) measurable symmetric function

(i.e. $f(w_{\sigma 1}, \ldots, w_{\sigma n}) = f(w_1, \ldots, w_n)$ for all $\sigma \in S_n$ and $(w_1, \ldots, w_n) \in \mathbb{R}_n^+$) then

$$E \left[ f \left( \tilde{X}_1, \ldots, \tilde{X}_n \right) \right] = E \left[ f \left( X_1, \ldots, X_n \right) \right].$$

(3.10)

4. Suppose that $Y_1, \ldots, Y_n$ is another collection of non-negative random variables such that $P(Y_i = Y_j) = 0$ for all $i \neq j$ such that

$$E \left[ f \left( X_1, \ldots, X_n \right) \right] = E \left[ f \left( Y_1, \ldots, Y_n \right) \right]$$

for all bounded (non-negative) measurable symmetric functions from $\mathbb{R}_n^+ \to \mathbb{R}$. Show that $\left( \tilde{X}_1, \ldots, \tilde{X}_n \right) \overset{d}{=} \left( \tilde{Y}_1, \ldots, \tilde{Y}_n \right)$.

**Hint:** if $g: \Delta_n \to \mathbb{R}$ is a bounded measurable function, define $f: \mathbb{R}_n^+ \to \mathbb{R}$ by;

$$f(y_1, \ldots, y_n) = \sum_{\sigma \in S_n} 1_{y_{\sigma 1} < y_{\sigma 2} < \cdots < y_{\sigma n}} g(y_{\sigma 1}, y_{\sigma 2}, \ldots, y_{\sigma n})$$

and then show $f$ is symmetric.

**Exercise 3.26.** Let $t \in \mathbb{R}_+$ and $\{U_i\}_{i=1}^n$ be i.i.d. uniformly distributed random variables on $[0, t]$. Show that the order statistics, $\left( \tilde{U}_1, \ldots, \tilde{U}_n \right)$, of $(U_1, \ldots, U_n)$ has the same distribution as $(W_1, \ldots, W_n)$ given $N_i = n$. (Thus, given $N_i = n$, the collection of points, $\{W_1, \ldots, W_n\}$, has the same distribution as the collection of points, $\{U_1, \ldots, U_n\}$, in $[0, t]$.)

**Exercise 3.27 (A Generalized Poisson Process I).** Suppose that $(S, B_S, \mu)$ is a finite measure space with $\mu(S) < \infty$. Define $\Omega = \bigcup_{n=0}^\infty S^n$ where $S^0 = \{\ast\}$, were $\ast$ is some arbitrary point. Define $B_{\Omega}$ to be those sets, $B = \bigcup_{n=0}^\infty B_n$ where $B_n \in B_S$, $B_S := B_{S}^{\otimes n} \times \sigma$ – the product $\sigma$ – algebra on $S^n$. Now define a probability measure, $P$, on $(\Omega, B_{\Omega})$ by

$$P(B) := e^{-\mu(S)} \sum_{n=0}^\infty \frac{1}{n!} \mu^{\otimes n}(B_n)$$

where $\mu^{\otimes 0}(\{\ast\}) = 1$ by definition. (We denote $P$ schematically by $P := e^{-\mu(S)}e^{\mu(S)}$.) Finally for ever $\omega \in \Omega$, let $N_\omega$, be the point measure on $(S, B_S)$ defined by; $N_\omega = 0$ and

$$N_\omega = \sum_{i=1}^n \delta_{s_i} \text{ if } \omega = (s_1, \ldots, s_n) \in S^n \text{ for } n \geq 1.$$
Exercise 3.31. Let \((\Omega, \mathcal{B}, \mu)\) be a measure space, \(p \in [1, \infty]\), and suppose that \(0 \leq f \in L^1(\mu), 0 \leq f_n \in L^1(\mu)\) for all \(n\), \(f_n \overset{\text{L}}{\to} f\), and \(\int f_n d\mu \to \int f d\mu\). Then \(f_n \to f\) in \(L^1(\mu)\). [In particular, if \(f, f_n \in L^p(\mu)\) and \(f_n \to f\) in \(L^p(\mu)\), then \(\|f_n\|^p \to \|f\|^p\) in \(L^1(\mu)\).

Exercise 3.32. (In this exercise the reader should refer to Lemma 17.26 for context and the notation used here.) Let \((\Omega, \mathcal{B}, \mu)\) be a measure space and \(\mathcal{B}_0\) be a sub-\(\sigma\)-algebra of \(\mathcal{B}\). Further suppose that to every \(B \in \mathcal{B}\) there exists \(A \in \mathcal{B}_0\) such that \(\mu(B \Delta A) = 0\). Show for all \(1 \leq p < \infty\) that \(i(L^p(\Omega, \mathcal{B}_0, \mu)) = L^p(\Omega, \mathcal{B}, \mu)\), i.e. to each \(f \in L^p(\Omega, \mathcal{B}, \mu)\) there exists a \(g \in L^p(\Omega, \mathcal{B}_0, \mu)\) such that \(f = g\) a.e. Hints: 1. verify the last assertion for simple functions in \(L^p(\Omega, \mathcal{B}_0, \mu)\). 2. then make use of Theorem 9.41 and Exercise 9.6.

Exercise 3.33. Suppose that \(1 \leq p < \infty\), \((\Omega, \mathcal{B}, \mu)\) is a \(\sigma\)-finite measure space and \(\mathcal{B}_0\) is a sub-\(\sigma\)-algebra of \(\mathcal{B}\). Show that \(i(L^p(\Omega, \mathcal{B}_0, \mu)) = L^p(\Omega, \mathcal{B}, \mu)\) implies; to every \(B \in \mathcal{B}\) there exists \(A \in \mathcal{B}_0\) such that \(\mu(B \Delta A) = 0\).

Exercise 3.34. Suppose \(A\) is an index set, \(\{f_\alpha\}_{\alpha \in A}\) and \(\{g_\alpha\}_{\alpha \in A}\) are two collections of random variables and \(C \subseteq (0, \infty)\). If \(\{g_\alpha\}_{\alpha \in A}\) is uniformly integrable and \(|f_\alpha| \leq C|g_\alpha|\) for all \(\alpha \in A\), show \(\{f_\alpha\}_{\alpha \in A}\) is uniformly integrable as well. [An an example which occurs in the dominated convergence theorem is when \(g_\alpha = g \in L^1(\mu)\) for all \(\alpha \in A\).]

Exercise 3.35. Let \(\infty > a, b > 1\) with \(a^{-1} + b^{-1} = 1\). Give a calculus proof of the inequality
\[
st \leq \frac{s^a}{a} + \frac{t^b}{b}\]
for all \(s, t \geq 0\).

Hint: by taking \(s = xt^{b/a}\), show that it suffices to prove
\[
x \leq \frac{x^a}{a} + \frac{1}{b}\]
for all \(x \geq 0\).

and then maximize the function \(f(x) = x - x^a/a\) for \(x \in [0, \infty)\).

Exercise 3.36. Suppose that \((\Omega, \mathcal{B}, P)\) is a probability space and \(\{X_n\}_{n=1}^\infty\) is a sequence of uncorrelated (i.e. \(\text{Cov}(X_n, X_m) = 0\) if \(m \neq n\)) square integrable random variables such that \(\mu = \mathbb{E}X_n\) and \(\sigma^2 = \text{Var}(X_n)\) for all \(n\). Let \(S_n := X_1 + \cdots + X_n\). Show \(\|\frac{S_n}{\sqrt{n}} - \mu\|^2 = \frac{\sigma^2}{n} \to 0\) as \(n \to \infty\).

Exercise 3.37. Suppose that \(\{X_n\}_{n=1}^\infty\) are i.i.d. integrable random variables and \(S_n := X_1 + \cdots + X_n\) and \(\mu := \mathbb{E}X_n\). Show, \(\frac{S_n}{\sqrt{n}} \to \mu\) in \(L^1(P)\) as \(n \to \infty\). (Incidentally, this shows that \(\left\{\frac{S_n}{\sqrt{n}}\right\}_{n=1}^\infty\) is U.I. Hint: for \(M \in (0, \infty)\), let \(X^M_i := X_i \cdot 1_{|X_i| \leq M}\) and \(S^M_n := X^M_1 + \cdots + X^M_n\) and use Exercise 3.36 to see that \(\frac{S^M_n}{\sqrt{n}} \to \mu\) in \(L^1(P)\) as \(n \to \infty\).)

Exercise 3.38. Let \(\mu, \nu, \gamma\) be three probability measure on \((\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})\).

Show:
1. \(\mu * \nu = \nu * \mu\).
2. \(\mu * (\nu * \gamma) = (\mu * \nu) * \gamma\). (So it is now safe to write \(\mu * \nu * \gamma\) for either side of this equation.)
3. \((\mu * \delta_x) (A) = \mu (A - x)\) for all \(x \in \mathbb{R}^n\) where \(\delta_x (A) := 1_A (x)\) for all \(A \in \mathcal{B}_{\mathbb{R}^n}\) and in particular \(\mu * \delta_0 = \mu\).