Math 280A Homework Problems Fall 2018

Problems are from Resnick, S. A Probability Path, Birkhauser, or from the lecture notes. The problems from the lecture notes are restated here.

1.1 Homework 1. Due Friday, October 5, 2018

- Read over Lecture notes Chapter 1
- Lecture note Exercises: 2.1, 2.2, and 2.3

1.2 Homework 2. Due Friday, October 12, 2018

- Lecture note Exercises: 2.4, 2.5, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12
- Look at Resnick, p. 20-27: 9, 12, 17, 23
- Hand in Resnick, p. 20-27: 5, 18, 40*

*Notes on Resnick’s #40: (i) $B((0,1])$ should be $B([0,1))$ in the statement of this problem, (ii) $k$ is an integer, (iii) $r \geq 2$.

1.3 Homework 3. Due Friday, October 18, 2013

- Look at Resnick, p. 20-27: and 19, 27, 30, 36
- Look at Lecture note Exercises: 2.13, 2.14
- Hand in Lecture note Exercises: 2.15, 2.16, 2.17, 2.18, 2.19, 2.20, 2.21
Exercise 2.1 (Prove the Fubini Proposition [1.15]). Suppose \( \{a_{kn}\}_{k,n=1}^{\infty} \subset \mathbb{R} \) such that
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |a_{kn}| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{kn}| < \infty.
\]
Then
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{kn}.
\]

**Hint:** Let \( a_{kn}^+ := \max(a_{kn}, 0) \) and \( a_{kn}^- := \max(-a_{kn}, 0) \) and observe that: \( a_{kn} = a_{kn}^+ - a_{kn}^- \) and \( |a_{kn}^+| + |a_{kn}^-| = |a_{kn}| \). Now apply Tonelli’s theorem (Proposition 1.11) with \( a_{kn} \) replaced by \( a_{kn}^+ \) and \( a_{kn}^- \). You should be careful to verify that \( \{a_{kn}\}_{n=1}^{\infty} \) is summable for each \( k \) and that \( \{S_k = \sum_{n=1}^{\infty} a_{kn}\}_{k=1}^{\infty} \) is summable so that \( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{kn} = \sum_{k=1}^{\infty} S_k \) exits, etc. etc.

Exercise 2.2 (Prove Fatou’s Lemma in Proposition [1.17]). Suppose that for each \( n \in \mathbb{N} \), \( \{h_n(i)\}_{i=1}^{\infty} \) is any sequence in \([0, \infty] \), then
\[
\lim_{n \to \infty} \inf_{i=1}^{\infty} h_n(i) \leq \lim_{n \to \infty} \inf_{i=1}^{\infty} h_n(i).
\]

**Hint:** apply the MCT by applying the monotone convergence theorem with \( f_n(i) := \inf_{m \geq n} h_m(i) \).

Exercise 2.3 (Prove DCT as in Proposition [1.18]). Suppose that for each \( n \in \mathbb{N} \), \( \{f_n(i)\}_{i=1}^{\infty} \subset \mathbb{R} \) is a sequence and \( \{g_n(i)\}_{i=1}^{\infty} \) is a sequence in \([0, \infty] \) such that:
1. \( \sum_{i=1}^{\infty} g_n(i) < \infty \) for all \( n \),
2. \( f(i) = \lim_{n \to \infty} f_n(i) \) and \( g(i) = \lim_{n \to \infty} g_n(i) \) exists for each \( i \),
3. \( |f_n(i)| \leq g_n(i) \) for all \( i \) and \( n \),
4. \( \lim_{n \to \infty} \sum_{i=1}^{\infty} g_n(i) = \sum_{i=1}^{\infty} g(i) < \infty \).

Then
\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} f_n(i) = \sum_{i=1}^{\infty} f_n(i) = \sum_{i=1}^{\infty} f(i).
\]

**Hint:** Apply Fatou’s lemma twice. Once with \( h_n(i) = g_n(i) + f_n(i) \) and once with \( h_n(i) = g_n(i) - f_n(i) \).

Exercise 2.4. Suppose that \( B \subset Y \), show that \( B \setminus \bigcup_{i \in I} A_i = \cap_{i \in I} (B \setminus A_i) \).

Exercise 2.5. Let \( \{B_i\}_{i \in I} \) be another collection of subsets of \( Y \). Show \( \bigcup_{i \in I} A_i \setminus \bigcup_{i \in I} B_i \subset \bigcup_{i \in I} (A_i \setminus B_i) \) and then use this inclusion twice to show \( \bigcup_{i \in I} A_i \cap \bigcup_{i \in I} B_i \subset \bigcup_{i \in I} (A_i \cap B_i) \).

Exercise 2.6 (Triangle inclusion for sets). If \( A, B, C \) are subsets of \( X \), show \( A \cup \bigcap \{B \setminus C\} \subset \bigcup \{A \setminus C\} \) and use this identity twice to show
\[
A \cup C \subset [A \cup B] \cup [B \cup C].
\]

Exercise 2.7. Find a function \( f : X = \{a,b,c\} \to Y = \{1,2\} \) and subsets \( C \) and \( D \) of \( X \) such that
\[
f(C \cap D) \neq f(C) \cap f(D) \quad \text{and} \quad f(C^c) \neq [f(C)]^c.
\]

Exercise 2.8. Suppose that \( \mathcal{E}_i \subset 2^X \) for \( i = 1, 2 \). Show that \( \mathcal{A}(\mathcal{E}_1) = \mathcal{A}(\mathcal{E}_2) \iff \mathcal{E}_1 \subset \mathcal{A}(\mathcal{E}_2) \) and \( \mathcal{E}_2 \subset \mathcal{A}(\mathcal{E}_1) \). Similarly show, \( \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) \) iff \( \mathcal{E}_1 \subset \sigma(\mathcal{E}_2) \) and \( \mathcal{E}_2 \subset \sigma(\mathcal{E}_1) \). Give a simple example where \( \mathcal{A}(\mathcal{E}_1) = \mathcal{A}(\mathcal{E}_2) \) while \( \mathcal{E}_1 \neq \mathcal{E}_2 \).

Exercise 2.9. Verify the Borel \( \sigma \) – algebra, \( \mathcal{B}_1 \), is generated by any of the following collection of sets:
1. \( \mathcal{E}_1 := \{(a, \infty) : a \in \mathbb{R} \} \), 2. \( \mathcal{E}_2 := \{(a, \infty) : a \in \mathbb{Q} \} \) or 3. \( \mathcal{E}_3 := \{(a, \infty) : a \in \mathbb{Q} \} \).

**Hint:** make use of the ideas in Exercise 2.8

Exercise 2.10 (Look at but do not hand in.). Let \( X \) be a set, \( I \) be an infinite index set, and \( \mathcal{E} = \{A_i\}_{i \in I} \) be a partition of \( X \). Prove the algebra, \( \mathcal{A}(\mathcal{E}) \) , and that \( \sigma \) – algebra, \( \sigma(\mathcal{E}) \), generated by \( \mathcal{E} \) are given by
\[
\mathcal{A}(\mathcal{E}) = \{\bigcup_{A_i \in I} A_i : A \subset I \text{ with } \#(A) < \infty \text{ or } \#(A^c) < \infty\}
\]
and
\[
\sigma(\mathcal{E}) = \{\bigcup_{A_i \in I} A_i : A \subset I \text{ with } A \text{ countable or } A^c \text{ countable}\}
\]
respectively. Here we are using the convention that \( \bigcup_{A_i \in I} A_i := \emptyset \) when \( A = \emptyset \). In particular if \( I \) is countable, then
\[
\sigma(\mathcal{E}) = \{\bigcup_{A_i \in I} A_i : A \subset I \}.
\]
Exercise 2.11. Let \( \tau \) be a topology on a set \( X \) and \( \mathcal{A} = \mathcal{A}(\tau) \) be the algebra generated by \( \tau \). Show \( \mathcal{A} \) is the collection of subsets of \( X \) which may be written as finite union of sets of the form \( F \cap V \) where \( F \) is closed and \( V \) is open.

Exercise 2.12. Let \( \mathcal{A} \subset 2^X \) and \( \mathcal{B} \subset 2^Y \) be elementary class. Show the collection
\[
S := \mathcal{A} \times \mathcal{B} := \{ A \times B : A \in \mathcal{A} \text{ and } B \in \mathcal{B} \}
\]
is also an elementary class.

Exercise 2.13 (Look at but do not hand in.). Show:
1. \( \mathcal{A}_n \) is an algebra for each \( n \in \mathbb{N} \),
2. \( \mathcal{A}_n \subset \mathcal{A}_{n+1} \) for all \( n \), and
3. \( \mathcal{A} \subset 2^\Omega \) is an algebra of subsets of \( \Omega \). (In fact, you might show that \( \mathcal{A} = \bigcup_{n=1}^\infty \mathcal{A}_n \) is an algebra whenever \( \{ \mathcal{A}_n \}_{n=1}^\infty \) is an increasing sequence of algebras.)

Exercise 2.14 (Consistency Conditions). [Look at only, do not hand in.] If \( p_n \) is defined as above, show:
1. \( \sum_{s \in S} p_1(s) = 1 \) and
2. for all \( n \in \mathbb{N} \) and \( (s_1, \ldots, s_n) \in S^n \),
\[
p_n(s_1, \ldots, s_n) = \sum_{s \in S} p_{n+1}(s_1, \ldots, s_n, s).
\]

These conditions are basically equivalent to the statements that \( Q_1(S) = 1 \) and \( Q_{n+1}(B \times S) = Q_n(B) \) for all \( n \in \mathbb{N} \) and \( B \subset S^n \).

Exercise 2.15 (Converse to 2.14). Suppose for each \( n \in \mathbb{N} \) we are given functions, \( p_n : S^n \to [0,1] \) such that the consistency conditions in Exercise 2.14 hold. Then there exists a unique finitely additive probability measure, \( P \) on \( \mathcal{A} \) such that Eq. \((5.6)\) holds for all \( n \in \mathbb{N} \) and \( (s_1, \ldots, s_n) \in S^n \) and such that \( P|_{\mathcal{A}_n} \) is a \( \sigma \)-additive measure on \( \mathcal{A}_n \) for all \( n \in \mathbb{N} \).

Exercise 2.16 (\( \mathcal{A} \) – measurable simple functions). As in Example 4.19 let \( \mathcal{A} \subset 2^\Omega \) be a finite algebra and \( \{ B_1, \ldots, B_k \} \) be the partition of \( \Omega \) associated to \( \mathcal{A} \). Show that a function, \( f : \Omega \to \mathbb{C} \), is an \( \mathcal{A} \) – simple function iff \( f \) is constant on \( B_i \) for each \( i \). Thus any \( \mathcal{A} \) – simple function is of the form,
\[
f = \sum_{i=1}^k \alpha_i 1_{B_i}, \tag{2.4}
\]
for some \( \alpha_i \in \mathbb{C} \).

Exercise 2.17. Let \( P \) is a finitely additive probability measure on an algebra \( \mathcal{A} \subset 2^\Omega \) and for \( A, B \in \mathcal{A} \) let \( \rho(A, B) := P(A \triangle B) \) where \( A \triangle B = (A \setminus B) \cup (B \setminus A) \). Show:
1. \( \rho(A, B) = \mathbb{E}|1_A - 1_B| \) and then use this (or not) to show
2. \( \rho(A, C) \leq \rho(A, B) + \rho(B, C) \) for all \( A, B, C \in \mathcal{A} \).

Remark: it is now easy to see that \( \rho : \mathcal{A} \times \mathcal{A} \to [0,1] \) satisfies the axioms of a metric except for the condition that \( \rho(A, B) = 0 \) does not imply that \( A = B \) but only that \( A = B \) modulo a set of probability zero.

Exercise 2.18. For \( 1 \leq k \leq n \), show:
1. (as functions on \( \Omega \)) that
\[
\binom{N}{k} = \sum_{A \subset \{1,2,\ldots,n\} \atop |A| = k} 1_{\cap_{i \in A} A_i}, \tag{2.5}
\]
where by definition
\[
\binom{m}{k} = \begin{cases} 
0 & \text{if } k > m \\
\frac{m}{k!(m-k)!} & \text{if } 1 \leq k \leq m \\
1 & \text{if } k = 0
\end{cases}
\]
2. Conclude from Eq. \((2.5)\) that for all \( z \in \mathbb{C} \),
\[
(1 + z)^N = 1 + \sum_{k=1}^n z^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} 1_{A_{i_1} \cap \ldots \cap A_{i_k}} \tag{2.7}
\]
provided \( (1 + z)^0 = 1 \) even when \( z = -1 \).
3. Conclude from Eq. \((2.5)\) that \( S_k = \mathbb{E} \rho(A_k)^N \).

Exercise 2.19. Taking expectations of Eq. \((2.7)\) implies,
\[
\mathbb{E} [(1 + z)^N] = 1 + \sum_{k=1}^n S_k z^k. \tag{2.8}
\]
Show that setting \( z = -1 \) in Eq. \((2.8)\) gives another proof of the inclusion exclusion formula. \textbf{Hint:} use the definition of the expectation to write out \( \mathbb{E} [(1 + z)^N] \) explicitly.

Exercise 2.20. Let \( 1 \leq m \leq n \). In this problem you are asked to compute the probability that there are exactly \( m \) – coincidences. Namely you should show,
\[ P(N = m) = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} S_k \]
\[ = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} \sum_{1 \leq i_1 < \cdots < i_k \leq n} P(A_{i_1} \cap \cdots \cap A_{i_k}) \]

**Hint:** Differentiate Eq. (2.8) \( m \) times with respect to \( z \) and then evaluate the result at \( z = -1 \). In order to do this you will find it useful to derive formulas for:
\[ \frac{d^m}{dz^m}|_{z=-1} (1 + z)^n \quad \text{and} \quad \frac{d^m}{dz^m}|_{z=-1} z^k. \]

**Exercise 2.21 (Simple conditional expectation).** Let \( X \in \mathbb{S}(\mathcal{B}) \) and, for simplicity, assume all functions are real valued. Prove the following assertions:

1. **(Orthogonal Projection Property 1.)** If \( Z \in \mathbb{S}(\mathcal{A}) \), then
   \[ \mathbb{E}[XZ] = \mathbb{E}[X \cdot Z] = \mathbb{E}[\mathbb{E}_AX \cdot Z] \]  
   \[ (\mathbb{E}_AX)(\omega) = \begin{cases} Z(\omega) & \text{if } P(A_\omega) > 0 \\ 0 & \text{if } P(A_\omega) = 0. \end{cases} \]
   In particular, \( \mathbb{E}_A[\mathbb{E}_AZ] = \mathbb{E}_A Z. \)
   This basically says that \( \mathbb{E}_A \) is orthogonal projection from \( \mathbb{S}(\mathcal{B}) \) onto \( \mathbb{S}(\mathcal{A}) \) relative to the inner product
   \[ (f, g) = \mathbb{E}[fg] \quad \text{for all } f, g \in \mathbb{S}(\mathcal{B}). \]

2. **(Orthogonal Projection Property 2.)** If \( Y \in \mathbb{S}(\mathcal{A}) \) satisfies, \( \mathbb{E}[XZ] = \mathbb{E}[YZ] \) for all \( Z \in \mathbb{S}(\mathcal{A}) \), then \( Y(\omega) = \bar{X}(\omega) \) whenever \( P(A_\omega) > 0 \). In particular, \( P(Y \neq \bar{X}) = 0. \)** Hint:** use item 1. to compute \( \mathbb{E}[(\bar{X} - Y)^2] \).

3. **(Best Approximation Property.)** For any \( Y \in \mathbb{S}(\mathcal{A}) \),
   \[ \mathbb{E}[(X - \bar{X})^2] \leq \mathbb{E}[(X - Y)^2] \]
   with equality iff \( \bar{X} = Y \) almost surely (a.s. for short), where \( \bar{X} = Y \) a.s. iff \( P(\bar{X} \neq Y) = 0 \). In words, \( \bar{X} = \mathbb{E}_AX \) is the best ("L2") approximation to \( X \) by an \( \mathcal{A} \) - measurable random variable.

4. **(Contraction Property.)** \( \mathbb{E}|X| \leq \mathbb{E}|X| \). (It is typically **not** true that \( |X(\omega)| \leq |X(\omega)| \) for all \( \omega \).)

5. **(Pull Out Property.)** If \( Z \in \mathbb{S}(\mathcal{A}) \), then \( \mathbb{E}_A[ZX] = Z\mathbb{E}_A X. \)
Resnick Problems