Math 280C homeworks: Spring 2019

1.1 Homework 1. Due Wednesday, April 10, 2019
- Look at Lecture note Exercise 2.2, 2.3 (done in class), 2.4, 2.5
- Hand in Lecture note Exercise 2.1, 2.6, 2.7, 2.8, 2.9, 2.10

1.2 Homework 2. Due Wednesday, April 17, 2019
- Look at Lecture note Exercise 2.12, 2.14
- Look at Resnick Chapter 10: #14
- Hand in Lecture note Exercise 2.11, 2.13, 2.15, 2.16

1.3 Homework 3. Due Wednesday, April 24, 2019
- Look at Lecture note Exercise 2.19, 2.27, 2.25, 2.26
- Hand in Lecture note Exercise 2.20, 2.21, 2.22, 2.23, 2.24
- Hand in Resnick Chapter 10: 10.17 and 10.19

*For Resnick 10.19, please define $X_{n+1}/X_n = Z_{n+1}$ where

$$Z_{n+1} = \begin{cases} 
X_{n+1}/X_n & X_n \neq 0 \\
1 & X_n = 0 = X_{n+1} \\
\infty \cdot X_{n+1} & X_n = 0 \text{ and } X_{n+1} \neq 0.
\end{cases}$$

1.4 Homework 4. Due Wednesday, May 1, 2019
- Look at Lecture note Exercise 2.17, 2.18
- Hand in Lecture note Exercise 2.27, 2.28, 2.29, 2.30, 2.31, 2.32

1.5 Homework 5. Due Wednesday, May 8, 2019
- Look at Lecture note Exercise 2.39
- Hand in Lecture note Exercise 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.40

1.6 Homework 6. Due Wednesday, May 15, 2019
- Look at Lecture note Exercise 2.43, 2.52
- Hand in Resnick Chapter 9: #5, #6, #9 a-e., #11 (Exercise 2.40 may be useful here.)
- Hand in Lecture note Exercise 2.41, 2.42, 2.44, 2.45, 2.46, 2.47
Lecture Note Problems

Exercise 2.1 (Jump - Hold Description I). Let $S$ be a countable set \( \{ \Omega, B, \{ B_n \}_{n=0}^\infty, P, \{ Y_n \}_{n=0}^\infty \} \) be a Markov chain with transition kernel, \( \{ q (x, y) \}_{x, y \in S} \) and let \( \nu (x) := P (Y_0 = x) \) for all \( x \in S \). For simplicity let us assume there are no absorbing states \(^1\) (i.e. \( q (x, x) < 1 \) for all \( x \in S \)) and then define,

\[
\tilde{q} (x, y) := \begin{cases} 
q (x, y) & \text{if } x \neq y \\
0 & \text{if } x = y.
\end{cases}
\]

Let \( j_k \) denote the time of the \( k \)-th jump of the chain \( \{ Y_n \}_{n=0}^\infty \) so that

\[
\begin{align*}
\begin{cases}
\begin{aligned}
j_1 & := \inf \{ n > 0 : Y_n \neq Y_0 \} \\
j_{k+1} & := \inf \{ n > j_k : Y_n \neq Y_j \}
\end{aligned}
\end{cases}
\end{align*}
\]

with the convention that \( j_0 = 0 \). Further let \( \sigma_k := j_k - j_{k-1} \) denote the time spent between the \((k - 1)\)-th and \( k \)-th jump of the chain \( \{ Y_n \}_{n=0}^\infty \). Show;

1. For \( \{ x_k \}_{k=0}^n \subset S \) with \( x_k \neq x_{k-1} \) for \( k = 1, \ldots, n \) and \( m_1, \ldots, m_k \in \mathbb{N} \), show

\[
P (\bigcap_{k=0}^n \{ Y_{j_k} = x_k \} ) \cap \bigcap_{k=1}^n \{ \sigma_k = m_k \} \\
= \nu (x_0) \prod_{k=1}^n q (x_{k-1}, x_k) \cdot \tilde{q} (x_{k-1}, x_k).
\]

(2.1)

2. Summing the previous formula on \( m_1, \ldots, m_k \in \mathbb{N} \), conclude

\[
P (\bigcap_{k=0}^n \{ Y_{j_k} = x_k \} ) = \nu (x_0) \cdot \prod_{k=1}^n \tilde{q} (x_{k-1}, x_k),
\]

i.e. this shows \( \{ Y_{j_k} \}_{k=0}^\infty \) is a Markov chain with transition kernel, \( \tilde{q} \).

3. Conclude, relative to the conditional probability measure, \( P (\cdot | \bigcap_{k=0}^n \{ Y_{j_k} = x_k \} ) ) \), that \( \{ \sigma_k \}_{k=1}^n \) are independent geometric

\[
\sigma_k \overset{\text{d}}{=} \text{Geo} (1 - q (x_{k-1}, x_k - 1)) \text{ for } 1 \leq k \leq n,
\]

see Exercises [10.14] and [2.2]

\(^1\) A state \( x \) is absorbing if \( q (x, x) = 1 \) since in this case there is no chance for the chain to leave \( x \) once it hits \( x \).

Exercise 2.2. Let \( \sigma \) be a geometric random variable with parameter \( p \in (0, 1] \), i.e. \( P (\sigma = n) = (1 - p)^{n-1} p \) for all \( n \in \mathbb{N} \). Show, for all \( n \in \mathbb{N} \) that

\[
P (\sigma > n) = (1 - p)^n \text{ for all } n \in \mathbb{N}
\]

and then use this to conclude that

\[
P (\sigma > m + n | \sigma > n) = P (\sigma > m) \text{ for all } m, n \in \mathbb{N}.
\]

[This shows that the geometric distributions are the discrete analogue of the exponential distributions.]

Exercise 2.3. Suppose that \( S = \{ 1, 2, \ldots, n \} \) and \( A \) is a matrix such that \( A_{ij} \geq 0 \) for \( i \neq j \) and \( \sum_j A_{ij} = 0 \) for all \( i \). Show

\[
Q_t = e^{tA} := \sum_{n=0}^\infty \frac{t^n}{n!} A^n
\]

is a time homogeneous Markov kernel.

Hints: 1. To show \( Q_t (i, j) \geq 0 \) for all \( t \geq 0 \) and \( i, j \in S \), write \( Q_t = e^{-t\lambda} e^{t(M + A)} \) where \( \lambda > 0 \) is chosen so that \( \lambda I + A \) has only non-negative entries. 2. To show \( \sum_{i \in S} Q_t (i, j) = 1 \), compute \( \frac{d}{dt} Q_t (1) \).

Exercise 2.4. Let \( \{ T_k \}_{k=1}^\infty \) be i.i.d. exponential random variables with intensity \( \lambda \) and \( \{ \sigma_k \}_{k=1}^n \) be independent geometric random variables with \( \sigma_k = \text{Geo} (b_k) \) for some \( b_k \in (0, 1] \). Further assume that \( \{ \sigma_k \}_{k=1}^n \cup \{ T_k \}_{k=1}^\infty \) are independent. We also let

\[
W_0 = 0, \quad W_n := T_1 + \ldots + T_n,
\]

\[
j_0 = 0, \quad j_k := \sigma_1 + \ldots + \sigma_k,
\]

\[
S_\ell := W_{j_\ell} - W_{j_{\ell-1}} \text{ for } 1 \leq \ell \leq n.
\]

Show \( \{ S_\ell \}_{\ell=1}^n \) are independent exponential random variables with \( S_\ell \overset{\text{d}}{=} \exp (b_\ell \lambda) \) for all \( 1 \leq \ell \leq n \).

Exercise 2.5. Keeping the notation of Example [22.52] and [22.53]. Use Corollary [22.57] to show again that \( P_x (T_B < \infty) = (q/p) \) for all \( x > 0 \) and \( E_x T_0 = x/(q - p) \) for \( x < 0 \). You should do so without making use of the extraneous hitting times, \( T_n \) for \( n \neq 0 \).
Exercise 2.6. Let \( x \in X \). Show:

1. for all \( n \in \mathbb{N}_0 \),
   \[
   P_x (\tau_x > n + 1) = \sum_{y \neq x} p (x, y) P_y (T_x > n). \tag{2.3}
   \]

2. Use Eq. \( \text{(2.5)} \) to conclude that if \( P_y (T_x = \infty) = 0 \) for all \( y \neq x \) then
   \( P_x (\tau_x = \infty) = 0 \), i.e. \( \{X_n\} \) will return to \( x \) when started at \( x \).

3. Sum Eq. \( \text{(2.5)} \) on \( n \in \mathbb{N}_0 \) to show
   \[
   E_x [\tau_x] = P_x (\tau_x > 0) + \sum_{y \neq x} p (x, y) E_y [T_x]. \tag{2.4}
   \]

4. Now suppose that \( S \) is a finite set and \( P_y (T_x = \infty) < 1 \) for all \( y \neq x \), i.e. there is a positive chance of hitting \( x \) from any \( y \neq x \) in \( S \). Explain how Eq. \( \text{(2.6)} \) combined with Lemma \( \text{22.42} \) (or see Corollary \( \text{22.59} \)) shows that
   \( E_x [\tau_x] < \infty \).

Exercise 2.7 (2nd order recurrence relations). Let \( a, b, c \) be real numbers with \( a \neq b \neq c \), \( \alpha, \beta \in \mathbb{Z} \cup \{\pm \infty\} \) with \( \alpha < \beta \), and suppose \( \{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\} \) solves the second order homogeneous recurrence relation:
   \[
   au (x + 1) + bu (x) + cu (x - 1) = 0 \tag{2.5}
   \]
   for \( \alpha < x < \beta \). Show:

1. for any \( \lambda \in \mathbb{C} \),
   \[
   a \lambda^{x+1} + b \lambda^x + c \lambda^{x-1} = \lambda^x \tag{2.6}
   \]
   where \( p (\lambda) = a \lambda^2 + b \lambda + c \) is the characteristic polynomial associated to Eq. \( \text{(2.7)} \).

Let \( \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) be the roots of \( p (\lambda) \) and suppose for the moment
   \[
   b^2 - 4ac \neq 0. \tag{2.7}
   \]
   From Eq. \( \text{(2.7)} \) it follows that for any choice of \( A_{\pm} \in \mathbb{R} \), the function,
   \[
   w (x) := A_+ \lambda_+^x + A_- \lambda_-^x,
   \]
   solves Eq. \( \text{(2.7)} \) for all \( x \in \mathbb{Z} \).

2. Show there is a unique choice of constants, \( A_{\pm} \in \mathbb{R} \), such that the function \( u (x) \) is given by
   \[
   u (x) := A_+ \lambda_+^x + A_- \lambda_-^x \quad \text{for all} \quad \alpha \leq x \leq \beta. \tag{2.8}
   \]

3. Now suppose that \( b^2 = 4ac \) and \( A_0 := -b / (2a) \) is the double root of \( p (\lambda) \).

   Show for any choice of \( A_0 \) and \( A_1 \) in \( \mathbb{R} \) that
   \[
   w (x) := (A_0 + A_1 x) \lambda_0^x,
   \]
   solves Eq. \( \text{(2.7)} \) for all \( x \in \mathbb{Z} \). Hint: Differentiate Eq. \( \text{(2.8)} \) with respect to \( \lambda \) and then set \( \lambda = \lambda_0 \).

4. Again show that any function \( u \) solving Eq. \( \text{(2.7)} \) is of the form \( u (x) = (A_0 + A_1 x) \lambda_0^x \) for \( \alpha \leq x \leq \beta \) for some unique choice of constants \( A_0, A_1 \in \mathbb{R} \).

Exercise 2.8. Let \( w_x := P_x \left( X_{T_{a,b}} = b \right) := P \left( X_{T_{a,b}} = b | X_0 = x \right) \).

1. Use first step analysis to show for \( a < x < b \) that
   \[
   w_x = \frac{1}{2} (w_{x+1} + w_{x-1}) \tag{2.7}
   \]
   provided we define \( w_a = 0 \) and \( w_b = 1 \).

2. Use the results of Exercise 2.7 to show
   \[
   P_x \left( X_{T_{a,b}} = b \right) = w_x = \frac{1}{b-a} (x-a). \tag{2.8}
   \]

3. Let
   \[
   T_b := \begin{cases} \min \{ n : X_n = b \} & \text{if } \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}
   \]
   be the first time \( \{X_n\} \) hits \( b \). Explain why \( \{X_{T_{a,b}} = b\} \subset \{T_b < \infty\} \) and use this along with Eq. \( \text{(2.10)} \) to conclude that \( P_x \left( T_b < \infty \right) = 1 \) for all \( x < b \). (By symmetry this result holds true for all \( x \in \mathbb{Z} \).)

Exercise 2.9. The goal of this exercise is to give a second proof of the fact that \( P_x \left( T_b < \infty \right) = 1 \). Here is the outline:

1. Let \( w_x := P_x \left( T_b < \infty \right) \). Again use first step analysis to show that \( w_x \) satisfies Eq. \( \text{(2.9)} \) for all \( x \) with \( w_0 = 1 \).

2. Use Exercise 2.7 to show that there is a constant, \( c \), such that
   \[
   w_x = c (x-b) + 1 \quad \text{for all } x \in \mathbb{Z}. \tag{2.9}
   \]

3. Explain why \( c \) must be zero to again show that \( P_x \left( T_b < \infty \right) = 1 \) for all \( x \in \mathbb{Z} \).

Exercise 2.10. Let \( T = T_{a,b} \) and \( u_x := E_x T := E \left[ T | X_0 = x \right] \).

1. Use first step analysis to show for \( a < x < b \) that
   \[
   u_x = \frac{1}{2} (u_{x+1} + u_{x-1}) + 1 \tag{2.9}
   \]
   with the convention that \( u_a = 0 = u_b \).

\footnote{The fact that \( P_x \left( T_b < \infty \right) = 1 \) is also follows from Example \( \text{15.82} \) above.}
2. Show that
\[ u_x = A_0 + A_1 x - x^2 \] (2.10)
solves Eq. (2.11) for any choice of constants $A_0$ and $A_1$.

3. Choose $A_0$ and $A_1$ so that $u_x$ satisfies the boundary conditions, $u_a = 0 = u_b$.
Use this to conclude that
\[ \mathbb{E}_x T_{a,b} = -ab + (b + a) x - x^2 = -a (b - x) + bx - x^2. \] (2.11)

**Exercise 2.11.** For $\theta \in \mathbb{R}$ let
\[ f_\theta (n, x) := \bar{Q}^{-n} e^{\theta x} = (pe^\theta + qe^{-\theta})^{-n} e^{\theta x} \]
so that $Q f_\theta (n + 1, \cdot) = f_\theta (n, \cdot)$ for all $\theta \in \mathbb{R}$. Compute;

1. $f_\theta^{(k)} (n, x) := \left( \frac{d}{dx} \right)^k f_\theta (n, x)$ for $k = 1, 2$.
2. Use your results to show,
\[ M_n^{(1)} := S_n - n (p - q) \]
and
\[ M_n^{(2)} := (S_n - n (p - q))^2 - 4npq \]
are martingales.

(If you are ambitious you might also find $M_n^{(3)}$.)

**Exercise 2.12 (Very similar to above example?).** Suppose $\{M_n\}_{n=0}^\infty$ is a square integrable martingale. Show;

1. $\mathbb{E} \left[ M_{n+1}^2 - M_n^2 | B_n \right] = \mathbb{E} \left[ (M_{n+1} - M_n)^2 | B_n \right]$. Conclude from this that the Doob decomposition of $M_n^2$ is of the form,
\[ M_n^2 = N_n + A_n \]
where
\[ A_n := \sum_{1 \leq k \leq n} \mathbb{E} \left[ (M_k - M_{k-1})^2 | B_{k-1} \right]. \]
2. If we further assume that $M_k - M_{k-1}$ is independent of $B_{k-1}$ for all $k = 1, 2, \ldots$, explain why,
\[ A_n = \sum_{1 \leq k \leq n} \mathbb{E} (M_k - M_{k-1})^2. \]

**Exercise 2.13 (Martingale problem I).** Suppose that $\{X_n\}_{n=0}^\infty$ is an $(S, S)$-valued adapted process on some filtered probability space $(\Omega, \mathcal{B}, \{B_n\}_{n=0}^\infty, P)$ and $Q$ is a probability kernel on $S$. To each $f : S \to \mathbb{R}$ which is bounded and measurable, let
\[ M_n^f := f (X_n) - \sum_{k<n} (Qf (X_k) - f (X_k)) = f (X_n) - \sum_{k<n} ((Q - I) f) (X_k). \]
Show;

1. If $\{X_n\}_{n=0}^\infty$ is a time homogeneous Markov chain with transition kernel, $Q$, then $\{M_n^f\}_{n=0}^\infty$ is a martingale for each $f \in S_n$.
2. Conversely if $\{M_n^f\}_{n=0}^\infty$ is a martingale for each $f \in S_n$, then $\{X_n\}_{n=0}^\infty$ is a time homogeneous Markov chain with transition kernel, $Q$.

**Exercise 2.14.** Suppose $\tau$ is a stopping time, $(S, S)$ is a measurable space, and $Z : \Omega \to S$ is a function. Show that $Z$ is $\mathcal{B}_\tau / S$ measurable iff $Z|_{\{\tau = n\}}$ is $(\mathcal{B}_n)_{\{\tau = n\}} / S$ measurable for all $n \in \mathbb{N}_0$.

**Exercise 2.15.** Suppose $\sigma$ and $\tau$ are two stopping times. Show;

1. $\{\sigma < \tau\}, \{\sigma = \tau\}$, and $\{\sigma \leq \tau\}$ are all in $\mathcal{B}_\sigma \cap \mathcal{B}_\tau$.
2. $\mathcal{B}_{\sigma \wedge \tau} = \mathcal{B}_\sigma \cap \mathcal{B}_\tau$.
3. $\mathcal{B}_{\sigma \lor \tau} = \mathcal{B}_\sigma \lor \mathcal{B}_\tau := \sigma (\mathcal{B}_\sigma \cup \mathcal{B}_\tau)$ and
4. $\mathcal{B}_0 = \mathcal{B}_{\sigma \wedge \tau}$ on $C$ where $C$ is any one of the following three sets; $\{\sigma \leq \tau\}$, $\{\sigma < \tau\}$, or $\{\sigma = \tau\}$.

*As an example, since
\[ \{\sigma \leq \tau\} \cap \{\sigma \wedge \tau = n\} = \{\sigma \leq \tau\} \cap \{\sigma = n\} = \{n \leq \tau\} \cap \{\sigma = n\} \in \mathcal{B}_n \]
for all $n \in \mathbb{N}_0$, it follows that

**Exercise 2.16.** Show, by example, that it is not necessarily true that
\[ \mathbb{E}_{\mathcal{G}_1} \mathbb{E}_{\mathcal{G}_2} = \mathbb{E}_{\mathcal{G}_1 \wedge \mathcal{G}_2} \]
for arbitrary $\mathcal{G}_1$ and $\mathcal{G}_2$—sub-sigma algebras of $\mathcal{B}$.

**Hint:** it suffices to take $(\Omega, B, P)$ with $\Omega = \{1, 2, 3\}$, $B = 2^\Omega$, and $P (\{j\}) = \frac{1}{3}$ for $j = 1, 2, 3$.

**Exercise 2.17 (Rademacher’s theorem).** Let $\Omega := (0, 1)$, $B := B_{[0,1]}$, $P = m$ be Lebesgue measure, and $f \in L^1 (P)$. To each partition $\Pi := \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\}$ of $(0, 1)$ we let $B_{\Pi} := \sigma (J_i := (x_{i-1}, x_i] : 1 \leq i \leq n)$.\footnote{In fact, you will likely show in your proof that every set in $B_{\sigma} \lor B_\tau$ may be written as a disjoint union of a set from $B_{\sigma}$ with a set from $B_\tau$.}
1. Show \( \mathbb{E} [ f | B_H ] (x) = \sum_{i=1}^{n} \frac{1}{x_i - x_{i-1}} \left[ f \left( x_{i-1} \right) - f \left( x_i \right) \right] \) for a.e. \( x \in \Omega \).

2. For \( f \in C ([0, 1], \mathbb{R}) \), let

\[
I_n (x) := \sum_{i=1}^{n} \frac{\Delta_i f}{\Delta_i} \mathbf{1}_{x_i - x_{i-1}} (x)
\]  

(2.12)

where \( \Delta_i f := f (x_i) - f (x_{i-1}) \) and \( \Delta_i := x_i - x_{i-1} \). Show if \( \Pi' \) is another partition of \( \Omega \) which refines \( \Pi \), i.e., \( \Pi \subset \Pi' \), then

\[
f = \mathbb{E} [ f | B_H ] \text{ a.s.}
\]

3. Show for any \( a, b \in \Pi \) with \( a < b \) that

\[
\frac{f (b) - f (a)}{b - a} = \frac{1}{b - a} \int_{a}^{b} I_n (x) \, dx.
\]  

(2.13)

**Hint:** consider the partition \( \Pi_0 := \{ 0 < a < b < 1 \} \).

Now let \( B_n := B_{\Pi_n} \) and where \( \Pi_n := \left\{ \frac{k}{2^n} \right\}_{k=0}^{2^n} \) an observe your have now shown \( g_n := I_n \) is a martingale.

4. Let us now further suppose that \( |f (y) - f (x)| \leq K |y - x| \) for all \( x, y \in [0, 1] \), i.e. \( f \) is Lipschitz. From Eq. (2.10) it follows that \( |g_n| := |I_n| \leq K \) so that \( \{ g_n \}_{n=1}^{\infty} \) is a bounded martingale. Use this along with Eq. (2.12) and Theorem 23.70 to conclude there exists \( g \in L^\infty (\mathbb{P}) \) such that

\[
f (b) - f (a) = \int_{a}^{b} g (x) \, dx \text{ for all } 0 \leq a < b \leq 1.
\]

[You may be interested to know that under these hypothesis, \( f' (x) \) exists a.e. and \( g (x) = f' (x) \) a.e.. Thus this a version of the fundamental theorem of calculus.]

**Exercise 2.18.** Suppose that \( \{ M_n \}_{n=1}^{\infty} \) is a decreasing sequence of closed subspaces of a Hilbert space, \( H \). Let \( M_H := \cap_{n=1}^{\infty} M_n \). Show \( \lim_{n \to \infty} P_{M_n} x = P_{M_H} x \) for all \( x \in H \). [**Hint:** you might make use of Exercise 18.5.]

**Exercise 2.19.** Let \( (M_n)_{n=0}^{\infty} \) be a martingale with \( M_0 = 0 \) and \( E[M_n^2] < \infty \) for all \( n \). Show that for all \( \lambda > 0 \),

\[
P \left( \max_{1 \leq m \leq n} M_m \geq \lambda \right) \leq \frac{E[M_n^2]}{E[M_n^2] + \lambda^2}.
\]

**Hints:** First show that for any \( c > 0 \) that \( \{ X_n := (M_n + c)^2 \}_{n=0}^{\infty} \) is a submartingale and then observe,

\[
\left\{ \max_{1 \leq m \leq n} M_m \geq \lambda \right\} \subset \left\{ \max_{1 \leq m \leq n} X_n \geq (\lambda + c)^2 \right\}.
\]

Now use Doob\'s Maximal inequality (Proposition 23.46) to estimate the probability of the last set and then choose \( c \) so as to optimize the resulting estimate you get for \( P \left( \max_{1 \leq m \leq n} M_m \geq \lambda \right) \). (Notice that this result applies to \(-M_n\) as well so it also holds that;

\[
P \left( \min_{1 \leq m \leq n} M_m \geq -\lambda \right) \leq \frac{E[M_n^2]}{E[M_n^2] + \lambda^2} \text{ for all } \lambda > 0.
\]

**Exercise 2.20.** Let \( (Z_n)_{n=1}^{\infty} \) be independent random variables, \( S_0 = 0 \) and \( S_n := Z_1 + \cdots + Z_n \), and \( f_n (\lambda) := \mathbb{E} [ e^{i \lambda Z_n} ] \). Suppose \( \mathbb{E} e^{i \lambda S_n} = \prod_{n=1}^{N} f_n (\lambda) \) converges to a continuous function, \( F (\lambda) \), as \( N \to \infty \). Show for each \( \lambda \in \mathbb{R} \) that

\[
P \left( \lim_{n \to \infty} e^{i \lambda S_n} \text{ exists} \right) = 1.
\]  

(2.14)

**Hints:**

1. Show it is enough to find an \( \varepsilon > 0 \) such that Eq. (2.18) holds for \( |\lambda| \leq \varepsilon \).

2. Choose \( \varepsilon > 0 \) such that \( |F (\lambda) - 1| < 1/2 \) for \( |\lambda| \leq \varepsilon \). For \( |\lambda| \leq \varepsilon \), show \( M_n (\lambda) := e^{i \lambda S_n} / \sqrt{n} \) is a bounded complex martingale relative to the filtration, \( \mathcal{B}_n = \sigma (Z_1, \ldots, Z_n) \).

**Exercise 2.21.** For \( a < 0 < b \) with \( a, b \in \mathbb{Z} \), let \( \tau = \sigma_a \wedge \sigma_b \). Explain why \( \tau \) is regular for \( S \). Use this to show \( P (\tau = \infty) = 0 \). [**Hint:** make use of Remark 23.76 and the fact that \( |S_n - S_{n-1}| = |Z_n| = 1 \) for all \( n \).

**Exercise 2.22.** In this exercise, you are asked to use the central limit Theorem 15.50 to prove again that \( P (\tau = \infty) = 0 \), Exercise 2.21. [**HINTS:** Use the central limit theorem to show

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f (x) e^{-x^2/2} \, dx \geq f (0) P (\tau = \infty)
\]  

(2.15)

for all \( f \in C^3 (\mathbb{R} \to [0, \infty)) \) with \( M := \sup_{x \in \mathbb{R}} |f^{(3)} (x)| < \infty \). Use this inequality to conclude that \( P (\tau = \infty) = 0 \). [**Hint:** consider \( \mathbb{E} \left[ f \left( \frac{S_n}{\sqrt{n}} \right) \right] \).

**Exercise 2.23.** Show

\[
P (\sigma_b < \sigma_a) = \frac{|a|}{b + |a|}
\]  

(2.16)

Please use the obvious generalization of a martingale for complex valued processes.

It will be useful to observe that the real and imaginary parts of a complex martingales are real martingales.
and use this to conclude \( P(\sigma_b < \infty) = 1 \), i.e. every \( b \in \mathbb{N} \) is almost surely visited by \( S_n \). (This last result also follows by the Hewitt-Savage Zero-One Law, see Example 15.82 where it is shown \( b \) is visited infinitely often.)

**Hint:** Using properties of martingales and Exercise 2.21 compute \( \lim_{n \to \infty} E[S_n \wedge \sigma_b] \) in two different ways.

**Exercise 2.24.** Let \( \tau := \sigma_a \wedge \sigma_b \). In this problem you are asked to show \( E[\tau] = |a|b \) with the aid of the following outline.

1. Use Exercise 2.12 above to conclude \( N_n := S_n^2 - n \) is a martingale.
2. Now show
   \[
   0 = EN_0 = EN_{\tau \wedge n} = ES_n^2 \wedge n - E[\tau \wedge n].
   \]
3. Now use DCT and MCT along with Exercise 2.23 to compute the limit as \( n \to \infty \) in Eq. (2.22) to find
   \[
   E[\sigma_a \wedge \sigma_b] = E[\tau] = b|a|.
   \]
4. By considering the limit, \( a \to -\infty \) in Eq. (2.23), show \( E[\sigma_b] = \infty \).

**Exercise 2.25.** Verify,
\[
M_n := S_n - n(p - q)
\]
and
\[
N_n := N_n^2 - \sigma^2n
\]
are martingales, where \( \sigma^2 = 1 - (p - q)^2 \). (This should be simple; see either Exercise 2.12 or Exercise 2.11.)

**Exercise 2.26.** Using Exercise 2.25 show
\[
E(\sigma_a \wedge \sigma_b) = \left( b \left[ 1 - \left( \frac{q}{p} \right)^a \right] + a \left[ \frac{q}{p} \right]^{b-1} \right) (p-q)^{-1}.
\]
By considering the limit of this equation as \( a \to -\infty \), show
\[
E[\sigma_b] = \frac{b}{p-q}
\]
and by considering the limit as \( b \to \infty \), show \( E[\sigma_a] = \infty \).

**Exercise 2.27.** Let \( S_n \) be the total assets of an insurance company in year \( n \in \mathbb{N} \). Assume \( S_0 > 0 \) is a constant and that for all \( n \geq 1 \) that \( S_n = S_{n-1} + \xi_n \), where \( \xi_n = c - Z_n \) and \( \{Z_n\}_{n=1}^{\infty} \) are i.i.d. random variables having the normal distribution with mean \( \mu < c \) and variance \( \sigma^2 \). (The number \( c \) is to be interpreted as the yearly premium.) Let \( R = \{S_n \leq 0 \text{ for some } n\} \) be the event that the company eventually becomes bankrupt, i.e. is Ruined. Show

\[
P(\text{Ruin}) = P(R) \leq e^{-2(c-\mu)S_0/\sigma^2}.
\]

**Outline:**

1. Show that \( \lambda = -2(c-\mu)/\sigma^2 < 0 \) satisfies, \( E[e^{\lambda \xi_n}] = 1 \).
2. With this \( \lambda \) show
   \[
   Y_n := \exp(\lambda S_n) = e^{\lambda S_0} \prod_{j=1}^{n} e^{\lambda \xi_j}
   \]
   is a non-negative \( \mathcal{F}_n = \sigma(Z_1, \ldots, Z_n) \) martingale.
3. Use a martingale convergence theorem to argue that \( \lim_{n \to \infty} Y_n = Y_\infty \) exists a.s. and then use Fatou’s lemma to show \( EY_\tau \leq e^{\lambda S_0} \), where
   \[
   \tau = \inf\{n : S_n \leq 0\}
   \]
is the time of the company’s ruin.
4. Finally conclude that
\[
P(R) \leq E[Y_\tau : \tau < \infty] \leq EY_\tau \leq e^{\lambda S_0} = e^{-2(c-\mu)S_0/\sigma^2}.
\]

**Exercise 2.28.** Suppose that \( Z \) is exponentially integrable and \( \psi(\theta) := \ln M(\theta) = \ln E[e^{\theta Z}] \). Show
\[
\psi'(\theta) = E_{\theta}Z \text{ and } \psi''(\theta) = \text{Var}_{\theta}(Z).
\]
[Use Proposition 10.59 in order to give a short solution to this problem.]

**Exercise 2.29.** Let \( Z \overset{d}{=} N(0, \sigma^2) \) and \( t > 0 \). By Lemma 10.47 we know that
\[
P(Z \geq t) = P(\sigma N \geq t) = P(N \geq t/\sigma) \leq ce^{-\frac{t^2}{2\sigma^2}}
\]
where \( c = 1/2 \). The goal of this exercise is to use Proposition 25.4 to prove this same bound above but with \( c = 1 \). In more detail show:

1. Recall that Gaussian integration formulas implies,
   \[
   M(\theta) = E_{\theta}e^{\theta Z} = e^{\frac{1}{2} \theta^2 \sigma^2}
   \]
   and so \( \psi(\theta) = \frac{1}{2} \theta^2 \sigma^2 \).
2. Show
   \[
   \theta \to \theta t - \psi(\theta) = \theta t - \frac{1}{2} \theta^2 \sigma^2
   \]
is maximized at \( \theta = t/\sigma^2 \) and that
   \[
   \psi^*(t) = \sup_{\theta \in \mathbb{R}} \left( \theta t - \frac{1}{2} \theta^2 \sigma^2 \right) = \frac{t^2}{2\sigma^2}.
   \]
This assertion along with Proposition 25.4 verifies the tail bound in Eq. (2.26) with \( c = 1 \).
3. Show Law $P_{\theta_i}(Z) \triangleq N(\sigma^2, \theta \sigma^2)$ – a normal random variable with variance $\sigma^2$ and mean $\theta \sigma^2$. Hence when $\theta = \theta_i = t/\sigma^2$,

$$\text{Law} P_{\theta_i}(Z) \triangleq N(\sigma^2, t). \quad (2.22)$$

4. Conclude that

$$\frac{1}{2} = P_{\theta_i}(Z \geq t) = \frac{1}{M(\theta_i)} E \left[ 1_{Z \geq \theta} e^{\theta Z} \right]$$

and explain (using Eq. (25.3)) that this inequality then implies Eq. (2.26) with $c = 1/2$.

**Exercise 2.30.** Suppose $-\infty < a < b < \infty$ and $Z$ is a random variable such that $a \leq Z \leq b$. Let $\mu = E Z$ and $\psi(\theta) = \ln E[e^{\theta Z}]$.

1. Use Taylor’s theorem along with Exercise 2.28 to show for any $\theta \in \mathbb{R}$, there exists $\theta^*$ between 0 and $\theta$ such that

$$\psi(\theta) = \theta \mu + \frac{1}{2} \theta^2 \text{Var}_{\theta^*}(Z).$$

2. Use item 1. to show

$$\psi(\theta) \leq \theta \mu + \frac{(b - a)^2}{8} \theta^2$$

by showing $\text{Var}_{\theta^*}(Z) \leq (b - a)^2/4$. **Hint:** this variance inequality holds no matter the distribution of $Z$ as long as $a \leq Z \leq b$ a.s.

3. Use items 1. and 2. to prove **Hoeffding’s inequality**, i.e.

$$E[\exp(\theta(Z - \mu))] \leq \exp \left( \frac{(b - a)^2 \theta^2}{8} \right) \quad \forall \theta \in \mathbb{R}. \quad (2.23)$$

4. Then use this and Lemma 25.7 to prove the Chernoff type bound,

$$P(Z - \mu \geq t) \leq e^{-\frac{t^2}{2(a - b)^2}} \quad \forall \ t > 0$$

5. Show, by applying the previous inequality with $Z$ replaced by $-Z$, that

$$P(Z - \mu \leq -t) \leq e^{-\frac{t^2}{2(b - a)^2}} \quad \forall \ t > 0$$

By adding the two previous bounds it follows that

$$P(|Z - \mu| \geq t) \leq 2e^{-\frac{t^2}{2(a - b)^2}} \quad \forall \ t > 0.$$

**Exercise 2.31.** Suppose that $-\infty < a_j < b_j < \infty$ and $\{Z_j\}_{j=1}^n$ are independent random variables with $a_j \leq Z_j \leq b_j$ for $1 \leq j \leq n$. If $S = \sum_{j=1}^n Z_j$, $\mu = \mathbb{E} S$, and $v = \sum_{j=1}^n (b_j - a_j)^2$, show

$$E[\exp(\theta(S - \mu))] \leq e^{\frac{\theta^2 v}{2}} \quad (2.24)$$

and

$$P(S - \mu \geq t) \leq e^{-\frac{t^2}{8v}} \quad \forall \ t > 0$$

Use this result to conclude, if $L = \max_j (b_j - a_j)$, then $v \leq n L^2$ and

$$P\left(\frac{S_n - \mu}{n} \geq t\right) \leq e^{-\frac{t^2}{2nL^2}} \quad \forall \ t > 0.$$

**Exercise 2.32.** **Prove Theorem 26.1**

**Exercise 2.33 (Resnik 7.1).** Does $\sum_n 1/n$ converge? Does $\sum_n (-1)^n/n$ converge? Let $\{X_n\}$ be iid with $P[X_n = \pm 1] = 1/2$. Does $\sum_n X_n/n$ converge? [See Example 26.42 below for a more thorough investigation of this sort.]

**Exercise 2.34 (Two Series Theorem – Resnik 7.15).** Prove that the three series theorem reduces to a two series theorem when the random variables are positive. That is, if $X_n \geq 0$ are independent, then $\sum_n X_n < \infty$ a.s. iff for any $c > 0$ we have

$$\sum_n P(X_n > c) < \infty \quad (2.25)$$

and

$$\sum_n E[X_n 1_{X_n \leq c}] < \infty, \quad (2.26)$$

that is it is unnecessary to verify the convergence of the second series in Theorem 26.43 involving the variances.

**Exercise 2.35.** Let $P$ denote the set of probability measures on $(\Omega, \mathcal{B})$. Show $d_{TV}$ is a complete metric on $P$.

**Exercise 2.36.** Suppose that $\mu, \nu$, and $\gamma$ are probability measures on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Show $d_{TV}(\mu * \nu, \mu * \gamma) \leq d_{TV}(\nu, \gamma)$. Use this fact along with Exercise 2.35 to show,

$$d_{TV}(\mu_1 * \mu_2 * \cdots * \mu_n, \nu_1 * \nu_2 * \cdots * \nu_n) \leq \sum_{i=1}^n d_{TV}(\mu_i, \nu_i)$$

for all choices probability measures, $\mu_i$ and $\nu_i$ on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. 
Exercise 2.37. Suppose that $\Omega$ is a (at most) countable set, $B := 2^\Omega$, and \{ $\mu_n$ \}_{n=0}^\infty$ are probability measures on $(\Omega, B)$. Show

$$d_{TV} (\mu_n, \mu_0) = \frac{1}{2} \sum_{\omega \in \Omega} |\mu_n (\{ \omega \}) - \mu_0 (\{ \omega \})|$$

and $\lim_{n \to \infty} d_{TV} (\mu_n, \mu_0) = 0$ iff $\lim_{n \to \infty} \mu_n (\{ \omega \}) = \mu_0 (\{ \omega \})$ for all $\omega \in \Omega$.

Exercise 2.38. Let $\mu_p (\{ 1 \}) = p$ and $\mu_p (\{ 0 \}) = 1 - p$ and $\nu_\lambda (\{ 1 \}) := e^{-\lambda \frac{\nu}{n!}}$ for all $n \in \mathbb{N}_0$.

1. Find $d_{TV} (\mu_p, \mu_q)$ for all $0 \leq p, q \leq 1$.
2. Show $d_{TV} (\mu_p, \nu_p) = p (1 - e^{-p})$ for all $0 \leq p \leq 1$. From this estimate and the estimate,

$$1 - e^{-p} = \int_0^p e^{-x} dx \leq \int_0^p 1 dx = p,$$

(2.27)

it follows that $d_{TV} (\mu_p, \nu_p) \leq p^2$ for all $0 \leq p \leq 1$.

3. Show

$$d_{TV} (\nu_\lambda, \nu_\gamma) \leq |\lambda - \gamma|$$

for all $\lambda, \gamma \in \mathbb{R}_+$. (2.28)

Hints: (Andy Parrish’s method – a former 280 student.)

a) Observe that for any $n \in \mathbb{N}$ we have $\nu_\lambda$ and $\nu_\gamma$ are equal to the $n$ – fold convolutions of $\nu_\lambda/n$ and $\nu_\gamma/n$ and use this to conclude

$$d_{TV} (\nu_\lambda, \nu_\gamma) \leq nd_{TV} (\nu_\lambda/n, \nu_\gamma/n).$$

(2.29)

b) Using item 2. of this exercise, show

$$|d_{TV} (\nu_\lambda/n, \nu_\gamma/n) - d_{TV} (\mu_\lambda/n, \mu_\gamma/n)| \leq C n^{-2}.$$

c) Finally make use of your results in item 1. part b. in order to let $n \to \infty$ in Eq. (2.35).

Exercise 2.39. Let $(S_1, \rho_1)$ and $(S_2, \rho_2)$ be separable metric spaces and $B_{S_1}$ and $B_{S_2}$ be the Borel $\sigma$ – algebras on $S_1$ and $S_2$ respectively. Prove the analogue of Lemma 9.29 namely show $B_{S_1 \times S_2} = B_{S_1} \otimes B_{S_2}$. Hint: you may find Exercise 9.10 helpful.

Exercise 2.40. Let $(S_1, \rho_1)$ and $(S_2, \rho_2)$ be separable metric spaces and $B_{S_1}$ and $B_{S_2}$ be the Borel $\sigma$ – algebras on $S_1$ and $S_2$ respectively. Further suppose that $\{ \mu_n \} \cup \{ \mu \} \subset \mathcal{P} (S_1)$ and $\{ \nu_n \} \cup \{ \nu \} \subset \mathcal{P} (S_2)$. Show; if $\mu_n \Rightarrow \mu$ and $\nu_n \Rightarrow \nu$, then $\mu_n \otimes \nu_n \Rightarrow \mu \otimes \nu$. Hint: You may find it useful to use Skorohod’s Theorem 28.8.

Exercise 2.41. To each finite and compactly supported measure, $\nu$, on $(\mathbb{R}, \mathcal{B})$ show there exists a sequence $\{ \nu_n \}_{n=1}^\infty$ of finitely supported finite measures on $(\mathbb{R}, \mathcal{B}_k)$ such that $\nu_n \Rightarrow \nu$. Here we say $\nu$ is compactly supported if there exists $M < \infty$ such that $\nu (\{ x : |x| \geq M \}) = 0$ and we say $\nu$ is finitely supported if there exists a finite subset, $A \subset \mathbb{R}$ such that $\nu (\mathbb{R} \setminus A) = 0$.

Exercise 2.42. Use Theorem 28.20 to give a proof of half of Theorem 28.16 when $S = \mathbb{R}^d$ and $A \subset \mathcal{P} (S)$, i.e; show; if $A$ is weakly sequentially compact then $A$ is tight. Hint: start by showing that if $A$ were not tight, then there would exist an $\varepsilon > 0$ and $\mu_n \in A$ so that $\mu_n (C_n) < 1 - \varepsilon$ for all $n \in \mathbb{N}$.

Exercise 2.43. Let $(S, \rho)$ be a separable metric space, $S_0 \subset S$ be a countable dense set, and $\{ x_n \}_{n=1}^\infty \cup \{ x \} \subset S$. Show $\lim_{n \to \infty} \rho (x_n, x) = 0$ iff $\lim_{n \to \infty} \rho (x, y) = \rho (x, y)$ for all $y \in S_0$.

Exercise 2.44 (Continuous Mapping Theorem II). Let $(S_1, \rho_1)$ and $(S_2, \rho_2)$ be separable metric spaces and $B_{S_1}$ and $B_{S_2}$ be the Borel $\sigma$ – algebras on $S_1$ and $S_2$ respectively. Let further suppose that $\{ \mu_n \} \cup \{ \mu \}$ are probability measures on $(S_1, B_{S_1})$ such that $\mu_n \Rightarrow \mu$. If $f : S_1 \to S_2$ is a Borel measurable function such that $\mu (D (f)) = 0$ (see Notation 28.22), then $f_n \mu_n \Rightarrow f \mu$ where $f_n \mu := f \circ f^{-1}$.

Exercise 2.45. Let $\{ X_n \}_{n=1}^\infty$ be an i.i.d. sequence of random variables with zero mean and $\text{Var} (X_n) = 1$ and $E [X_n^3] < \infty$ (so that Corollary 15.43 applies). For $t \geq 0$, let $W_n (t) := \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor}$ where $\lfloor nt \rfloor$ is the nearest integer to $nt$ less than or equal to $nt$ and $S_n := \sum_{k \leq m} X_k$ where $S_0 = 0$ by definition. Show that $W_n \overset{d}{\to} B$ where $\{ B (t) : t \geq 0 \}$ is a Brownian motion as defined in Definition 22.26. You might use the following outline.

1. For any $0 \leq s < t < \infty$, explain why $W_n (t) - W_n (s) \Rightarrow N (0, (t - s) \cdot)$.

(You may find Slutzky’s Theorem 28.25 useful here.)

2. Given $A := \{ 0 = t_0 < t_1 < t_2 < \cdots < t_K \} \subset \mathbb{R}_+$ argue that $\{ W_n (t_i) - W_n (t_{i-1}) \}_{i=1}^K$ are independent and then show

$$\{ W_n (t_i) - W_n (t_{i-1}) \}_{i=1}^K \Rightarrow \{ B (t_i) - B (t_{i-1}) \}_{i=1}^K \text{ as } n \to \infty.$$  

3. Now show that $\{ W_n (t_i) \}_{i=1}^K \Rightarrow \{ B (t_i) \}_{i=1}^K \text{ as } n \to \infty$. Hint; use Exercise 2.44.

Exercise 2.46 (Lemma 29.17 generalization). Suppose now $X : (\Omega, \mathcal{B}, \mathbb{P}) \to \mathbb{R}^d$ is a random vector and $f_X (\lambda) := E [e^{i \lambda \cdot X}]$ is its characteristic function. Show for $a > 0$,
\[
P(|X|_\infty \geq a) \leq 2 \left( \frac{a}{4} \right)^d \int_{[-2/a,2/a]^d} (1 - f_X(\lambda)) d\lambda
\]
\[
= 2 \left( \frac{a}{4} \right)^d \int_{[-2/a,2/a]^d} (1 - \text{Re} f_X(\lambda)) d\lambda
\]
(2.30)

where \(|X|_\infty = \max_i |X_i|\) and \(d\lambda = d\lambda_1, \ldots, d\lambda_d\).

**Exercise 2.47.** For \(x, \lambda \in \mathbb{R}\), let (also see Eq. (2.41))
\[
\varphi(\lambda, x) := \begin{cases} 
\frac{e^{i\lambda x} - 1 - i\lambda x}{x^2} & \text{if } x \neq 0 \\
-\frac{1}{2} \lambda^2 & \text{if } x = 0.
\end{cases}
\]
(2.31)

Let \(\{x_k\}_{k=1}^n \subset \mathbb{R} \setminus \{0\}\), \(\{Z_k\}_{k=1}^n \cup \{N\}\) be independent random variables with \(N \overset{d}{=} N(0, 1)\) and \(Z_k\) being Poisson random variables with mean \(a_k > 0\), i.e.
\[P(Z_k = n) = e^{-a_k} \frac{a_k^n}{n!}\] for \(n = 0, 1, 2, \ldots\). With \(Y := \sum_{k=1}^n x_k (Z_k - a_k) + \alpha N\), show
\[
f_Y(\lambda) := \mathbb{E}[e^{i\lambda Y}] = \exp \left( \int_{\mathbb{R}} \varphi(\lambda, x) d\nu(x) \right)
\]
where \(\nu\) is the discrete measure on \((\mathbb{R}, \mathcal{B}_{\mathbb{R}})\) given by
\[
\nu = \alpha^2 \delta_0 + \sum_{k=1}^n a_k x_k^2 \delta_{x_k}.
\]
(2.32)

**Remark:** It is easy to see that \(\varphi(\lambda, 0) = \lim_{x \to 0} \varphi(\lambda, x)\). In fact by Taylor's theorem with integral remainder we have
\[
\varphi(\lambda, x) = -\lambda^2 \int_0^1 e^{it\lambda x} (1 - t) dt.
\]
(2.33)

From this formula it is clear that \(\varphi\) is a smooth function of \((\lambda, x)\).
Solutions for selected problems from Resnick