Math 280C homeworks: Spring 2019

1.1 Homework 1. Due Wednesday, April 10, 2019

- **Look at** Lecture note Exercise 2.2, 2.3 (done in class), 2.4, 2.5
- **Hand in** Lecture note Exercise 2.1, 2.6, 2.7, 2.8, 2.9, 2.10
Lecture Note Problems

Exercise 2.1 (Jump - Hold Description I). Let $S$ be a countable set $(Ω, B, \{B_n\}_{n=0}^{∞}, P, \{Y_n\}_{n=0}^{∞})$ be a Markov chain with transition kernel, $(q(x, y))_{x,y \in S}$ and let $ν(x) := P(Y_0 = x)$ for all $x \in S$. For simplicity let us assume there are no absorbing states$^1$ (i.e. $q(x, x) < 1$ for all $x \in S$) and then define,

$$\tilde{q}(x, y) := \begin{cases} \frac{q(x, y)}{1-q(x, x)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Let $j_k$ denote the time of the $k^{th}$ jump of the chain $\{Y_n\}_{n=0}^{∞}$ so that

$$j_1 := \inf \{n > 0 : Y_n \neq Y_0\} \quad \text{and} \quad j_{k+1} := \inf \{n > j_k : Y_n \neq Y_{j_k}\}$$

with the convention that $j_0 = 0$. Further let $σ_k := j_k - j_{k-1}$ denote the time spent between the $(k - 1)^{st}$ and $k^{th}$ jump of the chain $\{Y_n\}_{n=0}^{∞}$. Show:

1. For $\{x_k\}_{k=0}^{n} \subset S$ with $x_k \neq x_{k-1}$ for $k = 1, \ldots, n$ and $m_1, \ldots, m_k \in \mathbb{N}$, show

$$P(\bigcap_{k=0}^{n} \{Y_{j_k} = x_k\}) \bigcap \bigcap_{k=1}^{n} \{σ_k = m_k\}) = ν(x_0) \prod_{k=1}^{n} q(x_{k-1}, x_k)^{m_k-1} (1 - q(x_{k-1}, x_k)) \cdot \tilde{q}(x_{k-1}, x_k).$$

(2.1)

2. Summing the previous formula on $m_1, \ldots, m_k \in \mathbb{N}$, conclude

$$P(\bigcap_{k=0}^{n} \{Y_{j_k} = x_k\}) = ν(x_0) \cdot \prod_{k=1}^{n} \tilde{q}(x_{k-1}, x_k),$$

i.e. this shows $\{Y_{j_k}\}_{k=0}^{∞}$ is a Markov chain with transition kernel, $\tilde{q}$.

3. Conclude, relative to the conditional probability measure, $P(\cdot | \bigcap_{k=0}^{n} \{Y_{j_k} = x_k\})$, that $\{σ_k\}_{k=1}^{n}$ are independent geometric

$$σ_k \overset{d}{=} \text{Geo}(1 - q(x_{k-1}, x_{k-1})) \text{ for } 1 \leq k \leq n,$$

see Exercises $10.14$ and $22.2$.

Exercise 2.2. Let $σ$ be a geometric random variable with parameter $p \in (0, 1]$, i.e. $P(σ = n) = (1 - p)^{n-1}p$ for all $n \in \mathbb{N}$. Show, for all $n \in \mathbb{N}$ that

$$P(σ > n) = (1 - p)^n \text{ for all } n \in \mathbb{N}$$

and then use this to conclude that

$$P(σ > m + n | σ > n) = P(σ > m) \text{ } \forall \text{ } m, n \in \mathbb{N}.$$
Exercise 2.6. Let \( x \in X \). Show:

1. for all \( n \in \mathbb{N}_0 \),

\[
P_x (\tau_x > n + 1) \leq \sum_{y \neq x} p(x, y) P_y (T_x > n).
\]

(2.3)

2. Use Eq. (2.5) to conclude that if \( P_y (T_x = \infty) = 0 \) for all \( y \neq x \) then \( P_x (\tau_x = \infty) = 0 \), i.e. \( \{X_n\} \) will return to \( x \) when started at \( x \).

3. Sum Eq. (2.5) on \( n \in \mathbb{N}_0 \) to show

\[
E_x [\tau_x] \leq P_x (\tau_x > 0) + \sum_{y \neq x} p(x, y) E_y [T_x].
\]

(2.4)

4. Now suppose that \( S \) is a finite set and \( P_y (T_x = \infty) < 1 \) for all \( y \neq x \), i.e. there is a positive chance of hitting \( x \) from any \( y \neq x \) in \( S \). Explain how Eq. (2.6) combined with Corollary 22.57 shows that \( E_x [\tau_x] < \infty \).

Exercise 2.7 (2nd order recurrence relations). Let \( a, b, c \) be real numbers with \( a \neq 0 \neq c \), \( \alpha, \beta \in \mathbb{Z} \cup \{\pm \infty\} \) with \( \alpha < \beta \), and suppose \( \{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\} \) solves the second order homogeneous recurrence relation:

\[
u(x + 1) + bu(x) + cu(x - 1) = 0
\]

(2.5)

for \( \alpha < x < \beta \). Show:

1. for any \( \lambda \in \mathbb{C} \),

\[
a \lambda^{x+1} + b \lambda^x + c \lambda^{x-1} = \lambda^{x-1} p(\lambda)
\]

(2.6)

where \( p(\lambda) = a \lambda^2 + b \lambda + c \) is the characteristic polynomial associated to Eq. (2.7).

Let \( \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) be the roots of \( p(\lambda) \) and suppose for the moment that \( b^2 - 4ac \neq 0 \). From Eq. (2.7) it follows that for any choice of \( A_{\pm} \in \mathbb{R} \), the function,

\[
w(x) := A_+ \lambda_+^x + A_- \lambda_-^x,
\]

solves Eq. (2.7) for all \( x \in \mathbb{Z} \).

2. Show there is a unique choice of constants, \( A_{\pm} \in \mathbb{R} \), such that the function \( u(x) \) is given by

\[
u(x) := A_+ \lambda_+^x + A_- \lambda_-^x \quad \text{for all } \alpha \leq x \leq \beta.
\]

3. Now suppose that \( b^2 = 4ac \) and \( \lambda_0 := -b / (2a) \) is the double root of \( p(\lambda) \).

Show for any choice of \( A_0 \) and \( A_1 \) in \( \mathbb{R} \) that

\[
w(x) := (A_0 + A_1 x) \lambda_0^x
\]

solves Eq. (2.7) for all \( x \in \mathbb{Z} \). Hint: Differentiate Eq. (2.8) with respect to \( \lambda \) and then set \( \lambda = \lambda_0 \).

4. Again show that any function \( u \) solving Eq. (2.7) is of the form \( u(x) = (A_0 + A_1 x) \lambda_0^x \) for \( \alpha \leq x \leq \beta \) for some unique choice of constants \( A_0, A_1 \in \mathbb{R} \).

Exercise 2.8. Let \( w_x := P_x (X_{T_{a,b}} = b) := P (X_{T_{a,b}} = b | X_0 = x) \).

1. Use first step analysis to show for \( a < x < b \) that

\[
w_x = \frac{1}{2} (w_{x+1} + w_{x-1})
\]

(2.7)

provided we define \( w_a = 0 \) and \( w_b = 1 \).

2. Use the results of Exercise 2.7 to show

\[
P_x (X_{T_{a,b}} = b) = w_x = \frac{1}{b - a} (x - a).
\]

(2.8)

3. Let

\[
T_b := \begin{cases} \min \{ n : X_n = b \} & \text{if } \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}
\]

be the first time \( \{X_n\} \) hits \( b \). Explain why \( \{X_{T_{a,b}} = b\} \subset\{T_b < \infty\} \) and use this along with Eq. (2.10) to conclude \( P_x (T_b < \infty) = 1 \) for all \( x < b \). (By symmetry this result holds true for all \( x \in \mathbb{Z} \).)

Exercise 2.9. The goal of this exercise is to give a second proof of the fact that \( P_x (T_b < \infty) = 1 \). Here is the outline:

1. Let \( w_x := P_x (T_b < \infty) \). Again use first step analysis to show that \( w_x \) satisfies Eq. (2.9) for all \( x \) with \( w_b = 1 \).

2. Use Exercise 2.7 to show that there is a constant, \( c \), such that \( w_x = c (x - b) + 1 \) for all \( x \in \mathbb{Z} \).

3. Explain why \( c \) must be zero to again show that \( P_x (T_b < \infty) = 1 \) for all \( x \in \mathbb{Z} \).

Exercise 2.10. Let \( T = T_{a,b} \) and \( u_x := E_x T := E [T | X_0 = x] \).

1. Use first step analysis to show for \( a < x < b \) that

\[
u_x = \frac{1}{2} (u_{x+1} + u_{x-1}) + 1
\]

(2.9)

with the convention that \( u_a = 0 = u_b \).

\footnote{The fact that \( P_x (T_b < \infty) = 1 \) is also follows from Example 15.82 above.}
2. Show that

\[ u_x = A_0 + A_1 x - x^2 \]  \hspace{1cm} \text{(2.10)}

solves Eq. (2.11) for any choice of constants \( A_0 \) and \( A_1 \).

3. Choose \( A_0 \) and \( A_1 \) so that \( u_x \) satisfies the boundary conditions, \( u_a = 0 = u_b \).

Use this to conclude that

\[ \mathbb{E} T_{a,b} = -ab + (b + a) x - x^2 = -a (b - x) + bx - x^2. \] \hspace{1cm} \text{(2.11)}
Resnick Problems