Math 280C homeworks: Spring 2019

1.1 Homework 1. Due Wednesday, April 10, 2019
- Look at Lecture note Exercise 2.2, 2.3 (done in class), 2.4, 2.5
- Hand in Lecture note Exercise 2.1, 2.6, 2.7, 2.8, 2.9, 2.10

1.2 Homework 2. Due Wednesday, April 17, 2019
- Look at Lecture note Exercise 2.12, 2.14
- Look at Resnick Chapter 10: #14
- Hand in Lecture note Exercise 2.11, 2.13, 2.15, 2.16

1.3 Homework 3. Due Wednesday, April 24, 2019
- Look at Lecture note Exercise 2.19, 2.27, 2.25, 2.26
- Hand in Lecture note Exercise 2.20, 2.21, 2.22, 2.23, 2.24
- Hand in Resnick Chapter 10: 10.17 and 10.19*

*For Resnick 10.19, please define \( X_{n+1}/X_n = Z_{n+1} \) where

\[
Z_{n+1} = \begin{cases} 
X_{n+1}/X_n & \text{if } X_n \neq 0 \\
1 & \text{if } X_n = 0 = X_{n+1} \\
\infty \cdot X_{n+1} & \text{if } X_n = 0 \text{ and } X_{n+1} \neq 0.
\end{cases}
\]

1.4 Homework 4. Due Wednesday, May 1, 2019
- Look at Lecture note Exercise: 2.17, 2.18
- Hand in Lecture note Exercise: 2.27, 2.28, 2.29, 2.30, 2.31, 2.32
Lecture Note Problems

Exercise 2.1 (Jump - Hold Description I). Let $S$ be a countable set $(\Omega, B, \{B_n\}_{n=0}^{\infty}, P, \{Y_n\}_{n=0}^{\infty})$ be a Markov chain with transition kernel, $\{q(x,y)\}_{x,y \in S}$ and let $\nu(x) := P(Y_0 = x)$ for all $x \in S$. For simplicity let us assume there are no absorbing states[i] (i.e. $q(x,x) < 1$ for all $x \in S$) and then define,

$$\tilde{q}(x,y) := \begin{cases} q(x,y) & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

Let $j_k$ denote the time of the $k$th jump of the chain $\{Y_n\}_{n=0}^{\infty}$ so that

$$j_1 := \inf \{n > 0 : Y_n \neq Y_0 \} \quad \text{and} \quad j_{k+1} := \inf \{n > j_k : Y_n \neq Y_{j_k} \}$$

with the convention that $j_0 = 0$. Further let $\sigma_k := j_k - j_{k-1}$ denote the time spent between the $(k-1)^{st}$ and $k^{th}$ jump of the chain $\{Y_n\}_{n=0}^{\infty}$. Show;

1. For $\{x_k\}_{k=0}^{\infty} \subset S$ with $x_k \neq x_{k-1}$ for $k = 1, \ldots, n$ and $m_1, \ldots, m_k \in \mathbb{N}$, show

$$P([\cap_{k=0}^{n} Y_{j_k} = x_k]) \wedge [\cap_{k=1}^{n} \{\sigma_k = m_k \})]
= \nu(x_0) \prod_{k=1}^{n} q(x_{k-1}, x_{k-1})^{m_k-1} \left(1 - q(x_{k-1}, x_{k-1})\right) \cdot \tilde{q}(x_{k-1}, x_k).$$

(2.1)

2. Summing the previous formula on $m_1, \ldots, m_k \in \mathbb{N}$, conclude

$$P([\cap_{k=0}^{n} Y_{j_k} = x_k]) = \nu(x_0) \cdot \prod_{k=1}^{n} \tilde{q}(x_{k-1}, x_k),$$

i.e. this shows $\{Y_{j_k}\}_{k=0}^{\infty}$ is a Markov chain with transition kernel, $\tilde{q}$.

3. Conclude, relative to the conditional probability measure,

$$P([\cap_{k=0}^{n} Y_{j_k} = x_k]) \wedge [\cap_{k=1}^{n} \{\sigma_k = m_k \})],$$

that $\{\sigma_k\}_{k=1}^{n}$ are independent geometric $\sigma_k \overset{\text{d}}{=} Geo(1 - q(x_{k-1}, x_{k-1}))$ for $1 \leq k \leq n$, see Exercises 10.14 and 2.2

Exercise 2.2. Let $\sigma$ be a geometric random variable with parameter $p \in (0, 1]$, i.e. $P(\sigma = n) = (1 - p)^{n-1} p$ for all $n \in \mathbb{N}$. Show, for all $n \in \mathbb{N}$ that

$$P(\sigma > n) = (1 - p)^n \quad \text{for all } n \in \mathbb{N}$$

and then use this to conclude that

$$P(\sigma > m + n | \sigma > n) = P(\sigma > m) \quad \forall \ m, n \in \mathbb{N}.$$

[This shows that the geometric distributions are the discrete analogue of the exponential distributions.]

Exercise 2.3. Suppose that $S = \{1, 2, \ldots, n\}$ and $A$ is a matrix such that $A_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{n} A_{ij} = 0$ for all $i$. Show

$$Q_t = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

is a time homogeneous Markov kernel.

Hints: 1. To show $Q_t(i,j) \geq 0$ for all $t \geq 0$ and $i, j \in S$, write $Q_t = e^{-t\lambda} e^{(\lambda I + A)}$, where $\lambda > 0$ is chosen so that $\lambda I + A$ has only non-negative entries. 2. To show $\sum_{j \in S} Q_t(i,j) = 1$, compute $\frac{d}{dt}Q_1$.

Exercise 2.4. Let $\{T_k\}_{k=1}^{\infty}$ be i.i.d. exponential random variables with intensity $\lambda$ and $\{\sigma_t\}_{t=1}^{\infty}$ be independent geometric random variables with $\sigma_t = Geo(b_t)$ for some $b_t \in (0, 1]$. Further assume that $\{\sigma_t\}_{t=1}^{\infty}$ are independent. We also let

$$W_0 = 0, \quad W_n := T_1 + \ldots + T_n,$$

$$j_0 = 0, \quad j_k := \sigma_1 + \ldots + \sigma_k,$$

$$S_\ell := W_{j_\ell} - W_{j_{\ell-1}} \quad \text{for } 1 \leq \ell \leq n.$$

Show $\{S_\ell\}_{\ell=1}^{\infty}$ are independent exponential random variables with $S_\ell \overset{\text{d}}{=} \text{exp}(b_\ell \lambda)$ for all $1 \leq \ell \leq n$.

Exercise 2.5. Keeping the notation of Example 22.12 and 22.53, use Corollary 22.57 to show again that $P_x(T_B < \infty) = (q/p)$ for all $x > 0$ and $E_x T_0 = x/(q - p)$ for $x < 0$. You should do so without making use of the extraneous hitting times, $T_n$ for $n \neq 0$. 

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[i] A state $x$ is absorbing if $q(x,x) = 1$ since in this case there is no chance for the chain to leave $x$ once it hits $x$. 
Exercise 2.6. Let \( x \in X \). Show;

1. for all \( n \in \mathbb{N}_0 \),
   \[
P_x(\tau_x > n + 1) = \sum_{y \neq x} p(x, y) P_y(T_x > n). \tag{2.3}
   \]

2. Use Eq. (2.5) to conclude that if \( P_y(T_x = \infty) = 0 \) for all \( y \neq x \) then \( P_x(\tau_x = \infty) = 0 \), i.e. \( \{X_n\} \) will return to \( x \) when started at \( x \).

3. Sum Eq. (2.5) on \( n \in \mathbb{N}_0 \) to show
   \[
   E_x[\tau_x] = P_x(\tau_x > 0) + \sum_{y \neq x} p(x, y) E_y[T_x]. \tag{2.4}
   \]

4. Now suppose that \( S \) is a finite set and \( P_y(T_x = \infty) < 1 \) for all \( y \neq x \), i.e. there is a positive chance of hitting \( x \) from any \( y \neq x \) in \( S \). Explain how Eq. (2.6) combined with Lemma 22.42 (or see Corollary 22.59) shows that
   \[
   E_x[\tau_x] < \infty.
   \]

Exercise 2.7 (2nd order recurrence relations). Let \( a, b, c \) be real numbers with \( a \neq 0 \), \( a, b \in \mathbb{R} \), \( \alpha, \beta \in \mathbb{Z} \cup \{\pm \infty\} \), and suppose \( \{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\} \) solves the second order homogeneous recurrence relation:

\[
au(x + 1) + bu(x) + cu(x - 1) = 0 \tag{2.5}
\]

for \( a < x < \beta \). Show:

1. for any \( \lambda \in \mathbb{C} \),
   \[
a\lambda^{x+1} + b\lambda^x + c\lambda^{x-1} = \lambda^{x-1} p(\lambda) \tag{2.6}
   \]
   where \( p(\lambda) = a\lambda^2 + b\lambda + c \) is the characteristic polynomial associated to Eq. (2.7).

2. Let \( \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) be the roots of \( p(\lambda) \) and suppose for the moment that \( b^2 - 4ac \neq 0 \). From Eq. (2.7) it follows that for any choice of \( A_{\pm} \in \mathbb{R} \), the function,
   \[
w(x) := A_+ \lambda_+^x + A_- \lambda_-^x,
   \]
solves Eq. (2.7) for all \( x \in \mathbb{Z} \).

3. Show there is a unique choice of constants, \( A_{\pm} \in \mathbb{R} \), such that the function \( u(x) \) is given by
   \[
u(x) := A_+ \lambda_+^x + A_- \lambda_-^x \quad \text{for all } \alpha \leq x \leq \beta.
   \]

4. Again show that any function \( u \) solving Eq. (2.7) is of the form \( u(x) = (A_0 + A_1 x) \lambda_0^x \) for \( \alpha \leq x \leq \beta \) for some unique choice of constants \( A_0, A_1 \in \mathbb{R} \).

Exercise 2.8. Let \( w_x := P_x(X_{T_{a,b}} = b) := P(X_{T_{a,b}} = b | X_0 = x) \).

1. Use first step analysis to show for \( a < x < b \) that
   \[
w_x = \frac{1}{2} (w_{x+1} + w_{x-1}) \tag{2.7}
   \]
powered provided we define \( w_a = 0 \) and \( w_b = 1 \).

2. Use the results of Exercise 2.7 to show
   \[
P_x(X_{T_{a,b}} = b) = w_x = \frac{1}{b-a} (x-a). \tag{2.8}
   \]

3. Let
   \[
   T_b := \begin{cases} \min \{n : X_n = b\} & \text{if } \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}
   \]

be the first time \( \{X_n\} \) hits \( b \). Explain why \( \{X_{T_{a,b}} = b\} \subset \{T_b < \infty\} \) and use this along with Eq. (2.10) to conclude\footnote{The fact that \( P(T_b < \infty) = 1 \) is also follows from Example 15.82 above.} that \( P_x(T_b < \infty) = 1 \) for all \( x < b \). (By symmetry this result holds true for all \( x \in \mathbb{Z} \).)

Exercise 2.9. The goal of this exercise is to give a second proof of the fact that \( P_x(T_b < \infty) = 1 \). Here is the outline:

1. Let \( w_x := P_x(T_b < \infty) \). Again use first step analysis to show that \( w_x \) satisfies Eq. (2.9) for all \( x \) with \( w_b = 1 \).

2. Use Exercise 2.7 to show that there is a constant, \( c \), such that
   \[
w_x = c (x-b) + 1 \quad \text{for all } x \in \mathbb{Z}.
   \]

3. Explain why \( c \) must be zero to again show that \( P_x(T_b < \infty) = 1 \) for all \( x \in \mathbb{Z} \).

Exercise 2.10. Let \( T = T_{a,b} \) and \( u_x := \mathbb{E}[T] := \mathbb{E}(T | X_0 = x) \).

1. Use first step analysis to show for \( a < x < b \) that
   \[
u_x = \frac{1}{2} (u_{x+1} + u_{x-1}) + 1 \tag{2.9}
   \]
with the convention that \( u_a = 0 = u_b \).
2. Show that
\[ u_x = A_0 + A_1 x - x^2 \] (2.10)
solves Eq. (2.11) for any choice of constants \( A_0 \) and \( A_1 \).
3. Choose \( A_0 \) and \( A_1 \) so that \( u_x \) satisfies the boundary conditions, \( u_a = 0 = u_b \).

Use this to conclude that
\[ \mathbb{E}_x T_{a,b} = -ab + (b + a) x - x^2 = -a (b - x) + bx - x^2 . \] (2.11)

**Exercise 2.11.** For \( \theta \in \mathbb{R} \) let
\[ f_\theta(n, x) := Q^{-n} e^{\theta x} = (pe^{\theta} + qe^{-\theta})^{-n} e^{\theta x} \]
so that \( Q f_\theta(n + 1, \cdot) = f_\theta(n, \cdot) \) for all \( \theta \in \mathbb{R} \). Compute:

1. \( f_\theta^{(k)}(n, x) := \left( \frac{d}{dx} \right)^k f_\theta(n, x) \) for \( k = 1, 2 \).
2. Use your results to show,
\[ M_n^{(1)} := S_n - n(p - q) \]
and
\[ M_n^{(2)} := (S_n - n(p - q))^2 - 4npq \]
are martingales.

(If you are ambitious you might also find \( M_n^{(3)} \).)

**Exercise 2.12 (Very similar to above example?).** Suppose \( \{ M_n \}_{n=0}^\infty \) is a square integrable martingale. Show:

1. \( \mathbb{E} [ M_{n+1}^2 - M_n^2 | \mathcal{B}_n ] = \mathbb{E} \left( (M_{n+1} - M_n)^2 | \mathcal{B}_n \right) \). Conclude from this that the Doob decomposition of \( M_n^2 \) is of the form,
\[ M_n^2 = N_n + A_n \]
where
\[ A_n := \sum_{1 \leq k \leq n} \mathbb{E} \left[ (M_k - M_{k-1})^2 | \mathcal{B}_{k-1} \right] . \]
2. If we further assume that \( M_k - M_{k-1} \) is independent of \( \mathcal{B}_{k-1} \) for all \( k = 1, 2, \ldots \), explain why,
\[ A_n = \sum_{1 \leq k \leq n} \mathbb{E} (M_k - M_{k-1})^2 . \]

**Exercise 2.13 (Martingale problem I).** Suppose that \( \{ X_n \}_{n=0}^\infty \) is an \((\mathcal{S}, \mathcal{S})\) valued adapted process on some filtered probability space \((\Omega, \mathcal{F}, \{ \mathcal{B}_n \}_{n \in \mathbb{N}_0}, P)\) and \( Q \) is a probability kernel on \( S \). To each \( f : S \to \mathbb{R} \) which is bounded and measurable,
\[ M_n^f := f(X_n) - \sum_{k<n} (Q f(X_k) - f(X_k)) = f(X_n) - \sum_{k<n} ((Q - I) f)(X_k) . \]

Show:

1. If \( \{ X_n \}_{n \geq 0} \) is a time homogeneous Markov chain with transition kernel, \( Q \), then \( \{ M_n^f \}_{n \geq 0} \) is a martingale for each \( f \in \mathcal{S}_0 \).
2. Conversely if \( \{ M_n^f \}_{n \geq 0} \) is a martingale for each \( f \in \mathcal{S}_0 \), then \( \{ X_n \}_{n \geq 0} \) is a time homogeneous Markov chain with transition kernel, \( Q \).

**Exercise 2.14.** Suppose \( \tau \) is a stopping time, \((\mathcal{S}, \mathcal{S}) \) is a measurable space, and \( Z : \Omega \to \mathbb{R} \) is a function. Show that \( Z \) is \( \mathcal{B}_\tau / \mathcal{S} \) measurable iff \( \{ Z \}_{\{ \tau = n \}} \) is \( \mathcal{B}_n / \{ \tau = n \} \) \( / \mathcal{S} \) measurable for all \( n \in \mathbb{N}_0 \).

**Exercise 2.15.** Suppose \( \sigma \) and \( \tau \) are two stopping times. Show:

1. \( \{ \sigma < \tau \}, \{ \sigma = \tau \}, \text{ and } \{ \sigma \leq \tau \}^* \) are all in \( \mathcal{B}_\sigma \cap \mathcal{B}_\tau \).
2. \( \mathcal{B}_{\sigma \wedge \tau} = \mathcal{B}_\sigma \cap \mathcal{B}_\tau \).
3. \( \mathcal{B}_{\sigma \vee \tau} = \mathcal{B}_\sigma \vee \mathcal{B}_\tau := \sigma \mathcal{B}_\sigma \cup \mathcal{B}_\tau \] and
4. \( \mathcal{B}_\tau = \mathcal{B}_{\sigma \wedge \tau} \) on \( C \) where \( C \) is any one of the following three sets; \( \{ \sigma \leq \tau \} \), \( \{ \sigma < \tau \} \), or \( \{ \sigma = \tau \} \).

*As an example, since
\[ \{ \sigma \leq \tau \} \cap \{ \sigma \wedge \tau = n \} = \{ \sigma \leq \tau \} \cap \{ \sigma = n \} = \{ n \leq \tau \} \cap \{ \sigma = n \} = \mathcal{B}_n \]
for all \( n \in \mathbb{N}_0 \), it follows that

**Exercise 2.16.** Show, by example, that it is not necessarily true that
\[ \mathbb{E}_{\mathcal{G}_1} \mathbb{E}_{\mathcal{G}_2} = \mathbb{E}_{\mathcal{G}_1 \wedge \mathcal{G}_2} \]
for arbitrary \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) - sub-sigma algebras of \( \mathcal{B} \).

**Hint:** it suffices to take \((\Omega, \mathcal{B}, P)\) with \( \Omega = \{1, 2, 3\}, \mathcal{B} = 2^\Omega \), and \( P ((j)) = \frac{1}{2} \) for \( j = 1, 2, 3 \).

**Exercise 2.17 (Rademacher’s theorem).** Let \( \Omega := (0, 1], \mathcal{B} := \mathcal{B}_{(0,1]}, P = m \) be Lebesgue measure, and \( f \in L^1(P) \). To each partition \( \Pi := \{0 = x_0 < x_1 < x_2 < \cdots < x_n = 1\} \) of \((0, 1]\) we let \( B_{\Pi} := \sigma \{ J_i := (x_{i-1}, x_i] : 1 \leq i \leq n \} \).

\(^3\) In fact, you will likely show in your proof that every set in \( \mathcal{B}_\sigma \vee \mathcal{B}_\tau \) may be written as a disjoint union of a set from \( \mathcal{B}_\sigma \) with a set from \( \mathcal{B}_\tau \).
1. Show $\mathbb{E}[f(B_t)](x) = \sum_{i=1}^{n} \frac{1}{x_{i-1} - x_i} \left[ f(x_{i-1}, x_i) - f(x_{i-1}, x_i) \right]_{1_{(x_{i-1}, x_i)}}(x)$ for a.e. $x \in \Omega$.

2. For $f \in C([0,1], \mathbb{R})$, let

$$f_{\Pi}(x) := \sum_{i=1}^{n} \frac{\Delta_i f}{\Delta_i} 1_{\Delta_i}(x)$$

(2.12)

where $\Delta_i f := f(x_i) - f(x_{i-1})$ and $\Delta_i := x_i - x_{i-1}$. Show if $\Pi'$ is another partition of $\Omega$ which refines $\Pi$, i.e. $\Pi \subset \Pi'$, then

$$f_{\Pi} = \mathbb{E}[f_{\Pi} | B_t] \text{ a.s.}$$

3. Show for any $a, b \in \Pi$ with $a < b$ that

$$\frac{f(b) - f(a)}{b - a} = \frac{1}{b - a} \int_a^b f_{\Pi}(x) \, dx.$$  

(2.13)

**Hint:** consider the partition $\Pi_0 := \{0 < a < b < 1\}$.

Now let $B_n := B_{\Pi_n}$ and where $\Pi_n := \{\frac{k}{2^n}\}_{k=0}^{2^n}$ an observe your have now shown $g_n := f_{B_n}$ is a martingale.

4. Let us now further suppose that $|f(y) - f(x)| \leq K|y - x|$ for all $x, y \in [0,1]$, i.e. $f$ is Lipschitz. From Eq. (2.16) it follows that $|g_n| := |f_{B_n}| \leq K$ so that $(g_n)_{n=1}^{\infty}$ is a bounded martingale. Use this along with Eq. (2.17) and Theorem 23.70 to conclude there exists $g \in L^\infty(P)$ such that

$$f(b) - f(a) = \int_a^b g(x) \, dx \text{ for all } 0 \leq a < b \leq 1.$$  

[You may be interested to know that under these hypothesis, $f'(x)$ exists a.e. and $g(x) = f'(x)$ a.e.. Thus this a version of the fundamental theorem of calculus.]

**Exercise 2.18.** Suppose that $\{M_n\}_{n=1}^{\infty}$ is a decreasing sequence of closed subspaces of a Hilbert space, $H$. Let $M_{\infty} := \cap_{n=1}^{\infty} M_n$. Show $\lim_{n \to \infty} P_{M_n x} = P_{M_{\infty} x}$ for all $x \in H$. [**Hint:** you might make use of Exercise 18.5]

**Exercise 2.19.** Let $(M_n)_{n=0}^{\infty}$ be a martingale with $M_0 = 0$ and $E[M_n^2] < \infty$ for all $n$. Show that for all $\lambda > 0$,

$$P \left( \max_{1 \leq m \leq n} M_m \geq \lambda \right) \leq \frac{E[M_n^2]}{E[M_n^2] + \lambda^2}.$$  

**Hints:** First show that for any $c > 0$ that $\{X_n := (M_n + c)^2\}_{n=0}^{\infty}$ is a submartingale and then observe,

$$\left\{ \max_{1 \leq m \leq n} M_m \geq \lambda \right\} \subset \left\{ \max_{1 \leq m \leq n} X_n \geq (\lambda + c)^2 \right\}.$$  

Now use Doob’ Maximal inequality (Proposition 23.46) to estimate the probability of the last set and then choose $c$ so as to optimize the resulting estimate you get for $P(\max_{1 \leq m \leq n} M_m \geq \lambda)$. (Notice that this result applies to $-M_n$ as well so it also holds that;

$$P \left( \min_{1 \leq m \leq n} M_m \leq -\lambda \right) \leq \frac{E[M_n^2]}{E[M_n^2] + \lambda^2} \text{ for all } \lambda > 0.$$  

**Exercise 2.20.** Let $(Z_n)_{n=1}^{\infty}$ be independent random variables, $S_0 = 0$ and $S_n := Z_1 + \cdots + Z_n$, and $f_n(\lambda) := \mathbb{E} [e^{\lambda Z_n}]$. Suppose $\mathbb{E} e^{\pi S_n} = \prod_{n=1}^{N} f_n(\lambda)$ converges to a continuous function, $F(\lambda)$, as $N \to \infty$. Show for each $\lambda \in \mathbb{R}$ that

$$P \left( \lim_{n \to \infty} e^{\pi S_n} \text{ exists} \right) = 1.$$  

(2.14)

**Hints:**

1. Show it is enough to find an $\varepsilon > 0$ such that Eq. (2.18) holds for $|\lambda| \leq \varepsilon$.

2. Choose $\varepsilon > 0$ such that $|F(\lambda) - 1| < 1/2$ for $|\lambda| \leq \varepsilon$. Show $M_n(\lambda) := e^{\pi S_n}$ is a bounded complex martingale relative to the filtration, $\mathcal{B}_n = \sigma(Z_1, \ldots, Z_n)$.

**Exercise 2.21.** For $a < b \in \mathbb{N}$ with $a, b \in \mathbb{Z}$, let $\tau = \sigma_a \wedge \sigma_b$. Explain why $\tau$ is regular for $S$. Use this to show $P(\tau = \infty) = 0$. [**Hint:** make use of Remark 23.76 and the fact that $|S_n - S_{n-1}| = |Z_n| = 1$ for all $n$.

**Exercise 2.22.** In this exercise, you are asked to use the central limit Theorem 15.50 to prove again that $P(\tau = \infty) = 0$, Exercise 2.21

**Hints:** Use the central limit theorem to show

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-x^2/2} \, dx \geq f(0) \mathbb{P}(\tau = \infty)$$  

(2.15)

for all $f \in C^3(\mathbb{R} \to [0, \infty))$ with $M := \sup_{x \in \mathbb{R}} |f^{(3)}(x)| < \infty$. Use this inequality to conclude that $P(\tau = \infty) = 0$. [**Hint:** consider $\mathbb{E} f \left( \frac{S_n}{\sqrt{n}} \right)$.

**Exercise 2.23.** Show

$$P(\sigma_b < \sigma_a) = \frac{|a|}{b + |a|}$$  

(2.16)

Please use the obvious generalization of a martingale for complex valued processes.

It will be useful to observe that the real and imaginary parts of a complex martingales are real martingales.
and use this to conclude \( P(\sigma_b < \infty) = 1 \), i.e. every \( b \in \mathbb{N} \) is almost surely visited by \( S_n \). (This last result also follows by the Hewitt-Savage Zero-One Law, see Example 15.82 where it is shown \( b \) is visited infinitely often.)

**Hint:** Using properties of martingales and Exercise 2.21 compute \( \lim_{n \to \infty} E[S_n^{\sigma_a \land \sigma_b}] \) in two different ways.

**Exercise 2.24.** Let \( \tau := \sigma_a \land \sigma_b \). In this problem you are asked to show \( E[\tau] = |a| b \) with the aid of the following outline.

1. Use Exercise 2.12 above to conclude \( N_n := S_n^2 - n \) is a martingale.
2. Now show
   \[
   0 = E[N_0] = E[N_{\tau \land n}] = ES_{\tau \land n}^2 - E[\tau \land n].
   \] (2.17)
3. Now use DCT and MCT along with Exercise 2.23 to compute the limit as \( n \to \infty \) in Eq. (2.22) to find
   \[
   E[\sigma_a \land \sigma_b] = E[\tau] = b |a|.
   \] (2.18)
4. By considering the limit, \( a \to -\infty \) in Eq. (2.23), show \( E[\sigma_b] = \infty \).

**Exercise 2.25.** Verify,

\[
M_n := S_n - n (p - q)
\]

and

\[
N_n := M_n^2 - \sigma^2 n
\]

are martingales, where \( \sigma^2 = 1 - (p - q)^2 \). (This should be simple; see either Exercise 2.12 or Exercise 2.11)

**Exercise 2.26.** Using exercise 2.25 show

\[
E(\sigma_a \land \sigma_b) = \left( \frac{b [1 - (q/p)^a] + a [(q/p)^b - 1]}{(q/p)^b - (q/p)^a} \right) (p - q)^{-1}.
\] (2.19)

By considering the limit of this equation as \( a \to -\infty \), show

\[
E[\sigma_b] = \frac{b}{p - q}
\]

and by considering the limit as \( b \to \infty \), show \( E[\sigma_a] = \infty \).

**Exercise 2.27.** Let \( S_n \) be the total assets of an insurance company in year \( n \in \mathbb{N}_0 \). Assume \( S_0 > 0 \) is a constant and that for all \( n \geq 1 \) that \( S_n = S_{n-1} + \xi_n \), where \( \xi_n = c - Z_n \) and \( \{Z_n\}_{n=1}^\infty \) are i.i.d. random variables having the normal distribution with mean \( \mu < c \) and variance \( \sigma^2 \). (The number \( c \) is to be interpreted as the yearly premium.) Let \( R = \{S_n \leq 0 \text{ for some } n\} \) be the event that the company eventually becomes bankrupt, i.e. is Ruined. Show

\[
P(\text{Ruin}) = P(R) \leq e^{-2(c-\mu)S_0/\sigma^2}.
\]

**Outline:**

1. Show that \( \lambda = -2 (c-\mu)/\sigma^2 \geq 0 \) satisfies, \( \mathbb{E} e^{\lambda \xi_n} = 1 \).
2. With this \( \lambda \) show
   \[
   Y_n := \exp (\lambda S_n) = e^{\lambda S_0} \prod_{j=1}^n e^{\lambda \xi_j}
   \] (2.20)
   is a non-negative \( B_n = \sigma(Z_1, \ldots, Z_n) \) martingale.
3. Use a martingale convergence theorem to argue that \( \lim_{n \to \infty} Y_n = Y_{\infty} \) exists a.s. and then use Fatou’s lemma to show \( \mathbb{E} Y_{\infty} \leq e^{\lambda S_0} \).
4. Finally conclude that
   \[
P(R) \leq \mathbb{E}[Y_{\infty} : \tau < \infty] \leq \mathbb{E}Y_{\infty} \leq e^{\lambda S_0} = e^{-2(c-\mu)S_0/\sigma^2}.
   \]

**Exercise 2.28.** Suppose that \( Z \) is exponentially integrable and \( \psi(\theta) := \ln M(\theta) = \ln \mathbb{E} e^{\theta Z} \).

[Use Proposition 10.59 in order to give a short solution to this problem.]

**Exercise 2.29.** Let \( Z \overset{d}{=} N(0, \sigma^2) \) and \( t > 0 \). By Lemma 10.47 we know that

\[
P(Z \geq t) = P(\sigma N \geq t) = P(N \geq t/\sigma) \leq ce^{-\frac{t^2}{2\sigma^2}}
\] (2.21)

where \( c = 1/2 \). The goal of this exercise is to use Proposition 25.4 to prove this same bound above but with \( c = 1 \). In more detail show:

1. Recall that Gaussian integration formulas implies,
   \[
   M(\theta) = \mathbb{E} e^{\theta^2} = e^{\frac{1}{2} \theta^2 \sigma^2} \text{ and so } \psi(\theta) = \frac{1}{2} \theta^2 \sigma^2.
   \]
2. Show
   \[
   \theta \to \theta t - \psi(\theta) = \theta t - \frac{1}{2} \theta^2 \sigma^2
   \]
   is maximized at \( \theta_t = t/\sigma^2 \) and that
   \[
   \psi^*(t) = \sup_{\theta \in \mathbb{R}} \left( \theta t - \frac{1}{2} \theta^2 \sigma^2 \right) = \frac{t^2}{2\sigma^2}.
   \]

This assertion along with Proposition 25.4 verifies the tail bound in Eq. 2.26 with \( c = 1 \).
3. Show Law \( P_\theta (Z) \overset{d}{=} N(\sigma^2, \theta \sigma^2) \) a normal random variable with variance \( \sigma^2 \) and mean \( \theta \sigma^2 \). Hence when \( \theta = \theta_t = t/\sigma^2 \),

\[
\text{Law}_{\theta_t} (Z) \overset{d}{=} N(\sigma^2, \theta_t \sigma^2).
\] (2.22)

4. Conclude that

\[
\frac{1}{2} = P_{\theta_t} (Z \geq t) = \frac{1}{M(\theta_t)} E [1_{Z \geq t} e^{\theta_t Z}]
\]

and explain that this inequality then implies Eq. (2.26) with \( c = 1/2 \).

**Exercise 2.30.** Suppose \( -\infty < a < b < \infty \) and \( Z \) is a random variable such that \( a \leq Z \leq b \). Let \( \mu = E Z \) and \( \psi(\theta) = \ln E [e^{\theta Z}] \).

1. Use Taylor’s theorem along with Exercise 2.28 to show for any \( \theta \in \mathbb{R} \), there exists \( \theta^* \) between 0 and \( \theta \) such that

\[
\psi(\theta) = \theta \mu + \frac{1}{2} \theta^2 \text{Var}_{\theta^*}(Z).
\]

2. Use item 1. to show

\[
\psi(\theta) \leq \theta \mu + \frac{(b - a)^2}{8} \theta^2
\]

by showing \( \text{Var}_{\theta^*}(Z) \leq (b - a)^2 / 4 \). **Hint:** this variance inequality holds no matter the distribution of \( Z \) as long as \( a \leq Z \leq b \) a.s.

3. Use items 1. and 2. to prove **Hoeffding’s inequality**, i.e.

\[
E \left[ e^{\theta (Z - \mu)} \right] \leq \exp \left( \frac{(b - a)^2}{8} \theta^2 \right) \quad \forall \theta \in \mathbb{R}.
\] (2.23)

4. Then use this and Lemma 25.7 to prove the Chernoff type bound,

\[
P(Z - \mu \geq t) \leq e^{-\frac{t^2}{2(b-a)^2}} \quad \forall t > 0
\]

5. Show, by applying the previous inequality with \( Z \) replaced by \( -Z \), that

\[
P(Z - \mu \leq -t) \leq e^{-\frac{t^2}{2(b-a)^2}} \quad \forall t > 0
\]

By adding the two previous bounds it follows that

\[
P(|Z - \mu| \geq t) \leq 2e^{-\frac{t^2}{2(b-a)^2}} \quad \forall t > 0.
\]

**Exercise 2.31.** Suppose that \( -\infty < a_j < b_j < \infty \) and \( \{Z_j\}_{j=1}^n \) are independent random variables with \( a_j \leq Z_j \leq b_j \) for \( 1 \leq j \leq n \). If \( S = \sum_{j=1}^n Z_j \), \( \mu = E S \), and \( v = \sum_{j=1}^n (b_j - a_j)^2 \), show

\[
E \left[ e^{\theta (S_n - \mu)} \right] \leq e^{\frac{v}{2} \theta^2} \quad \text{and} \quad (2.24)
\]

and

\[
P(S_n - \mu \geq t) \leq e^{-\frac{t^2}{2v}} \quad \text{for all } t \geq 0.
\]

Use this result to conclude, if \( L = \max_j (b_j - a_j) \), then \( v \leq nL^2 \) and

\[
P \left( \frac{S_n - \mu}{n} \geq t \right) \leq e^{-\frac{2nt^2}{L^2}} \quad \forall t \geq 0.
\]

**Exercise 2.32.** Prove Theorem 26.1.
Resnick Problems