Math 280C homeworks: Spring 2019

1.1 Homework 1. Due Wednesday, April 10, 2019

- **Look at** Lecture note Exercise 2.2, 2.3(done in class), 2.4, 2.5
- **Hand in** Lecture note Exercise 2.1, 2.6, 2.7, 2.8, 2.9, 2.10

1.2 Homework 2. Due Wednesday, April 17, 2019

- **Look at** Lecture note Exercise 2.12, 2.14
- **Look at** Resnick Chapter 10: #14
- **Hand in** Lecture note Exercise 2.11, 2.13, 2.15, 2.16
- Resnick Chapter 10: **Hand in** 15, 16, 17, 33.

1.3 Homework 3. Due Wednesday, April 24, 2019

- **Look at** Lecture note Exercise 2.19, 23.27, 2.25, 2.26
- **Hand in** Lecture note Exercise 2.20, 2.21, 2.22, 2.23, 2.24
- **Hand in** Resnick Chapter 10: 10.17 and 10.19*

*For Resnick 10.19, please define $X_{n+1}/X_n = Z_{n+1}$ where

$$Z_{n+1} = \begin{cases} X_{n+1}/X_n & \text{if} \quad X_n \neq 0 \\ 1 & \text{if} \quad X_n = 0 = X_{n+1} \\ \infty \cdot X_{n+1} & \text{if } X_n = 0 \text{ and } X_{n+1} \neq 0. \end{cases}$$

1.4 Homework 4. Due Wednesday, May 1, 2019

- **Look at** Lecture note Exercise: 2.17, 2.18
- **Hand in** Lecture note Exercise: 2.27, 2.28, 2.29, 2.30, 2.31, 2.32

1.5 Homework 5. Due Wednesday, May 8, 2019

- **Look at** Lecture note Exercise: 2.39.
- **Hand in** Lecture note Exercise: 2.33, 2.34, 2.35, 2.36, 2.37, 2.38, 2.40.

1.6 Homework 6. Due Wednesday, May 15, 2019

- **Look at** Lecture note Exercise: 2.43, 2.46, 29.2
- **Hand in Resnick Chapter 9**: #5, #6, #9 a-e., #11 (Exercise 2.40 may be useful here.)
- **Hand in** Lecture note Exercise: 2.41, 2.42, 2.44, 2.45, 2.47

1.7 Homework 7. Due Wednesday, May 29, 2019 (was May 22)

- **Look at Resnick Chapter 9:** 28, 34 (assume $\sum_n \sigma_n^2 > 0$), 35 (hint: show $P[\xi_n \neq 0 \text{ i.o. }] = 0$.)
- **Hand in Resnick Chapter 9**: #10, 22, 38 (Hint: make use $\{X_k\}$ in Proposition 15.88 after appropriate translation and scalings.)
- **Hand in** Lecture note Exercise: 2.48, 2.51, 2.49, 2.50

The last two problems were added to what was given originally.

1.8 Homework 8. Due Wednesday, June 5, 2019

- **Look at** Lecture note Exercise: 2.58, 2.52 – 2.56, 32.2
- **Hand in** Lecture note Exercise: 2.53, 2.54, 2.57, 2.59, 2.60

Lecture Note Problems

Exercise 2.1 (Jump - Hold Description I). Let S be a countable set $\left(\Omega,\mathcal{B},\{\mathcal{B}_n\}_{n=0}^{\infty},P,\{Y_n\}_{n=0}^{\infty}\right)$ be a Markov chain with transition kernel, $\{q(x,y)\}_{x,y\in S}$ and let $\nu(x) := P(Y_0 = x)$ for all $x \in S$. For simplicity let us assume there are no **absorbing states**,[1] (i.e. $q(x,x) < 1$ for all $x \in S$) and then define,

$$\tilde{q}(x,y) := \begin{cases} \frac{q(x,y)}{1-q(x,x)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $\mathbf{j}_k$ denote the time of the k^{th} – jump of the chain $\{Y_n\}_{n=0}^{\infty}$ so that

$$\begin{aligned} \mathbf{j}_1 &:= \inf\{n > 0 : Y_n \neq Y_0\} \text{ and} \\ \mathbf{j}_{k+1} &:= \inf\{n > \mathbf{j}_k : Y_n \neq Y_{\mathbf{j}_k}\} \end{aligned}$$

with the convention that $\mathbf{j}_0 = 0$. Further let $\sigma_k := \mathbf{j}_k - \mathbf{j}_{k-1}$ denote the time spent between the $(k-1)^{\text{st}}$ and k^{th} jump of the chain $\{Y_n\}_{n=0}^{\infty}$. Show;

1. For $\{x_k\}_{k=0}^{n} \subset S$ with $x_k \neq x_{k-1}$ for $k = 1,\dots,n$ and $m_1,\dots,m_k \in \mathbb{N}$, show

$$\begin{aligned} &P\left([\cap_{k=0}^{n}\{Y_{\mathbf{j}_k} = x_k\}] \cap [\cap_{k=1}^{n}\{\sigma_k = m_k\}]\right) \\ &= \nu(x_0)\prod_{k=1}^{n} q(x_{k-1},x_{k-1})^{m_k-1}\left(1-q(x_{k-1},x_{k-1})\right)\cdot\tilde{q}(x_{k-1},x_k). \end{aligned} \tag{2.1}$$

2. Summing the previous formula on $m_1,\dots,m_k \in \mathbb{N}$, conclude

$$P\left([\cap_{k=0}^{n}\{Y_{\mathbf{j}_k} = x_k\}]\right) = \nu(x_0)\cdot\prod_{k=1}^{n}\tilde{q}(x_{k-1},x_k),$$

   i.e. this shows $\{Y_{\mathbf{j}_k}\}_{k=0}^{\infty}$ is a Markov chain with transition kernel, $\tilde{q}$.
3. Conclude, relative to the conditional probability measure, $P\left(\cdot|\,[\cap_{k=0}^{n}\{Y_{\mathbf{j}_k} = x_k\}]\right)$, that $\{\sigma_k\}_{k=1}^{n}$ are independent geometric $\sigma_k \overset{d}{=} Geo\left(1-q(x_{k-1},x_{k-1})\right)$ for $1 \le k \le n$, see Exercises 10.14 and 2.2.

[1] A state x is absorbing if $q(x,x) = 1$ since in this case there is no chance for the chain to leave x once it hits x.

Exercise 2.2. Let σ be a geometric random variable with parameter $p \in (0,1]$, i.e. $P(\sigma = n) = (1-p)^{n-1}p$ for all $n \in \mathbb{N}$. Show, for all $n \in \mathbb{N}$ that

$$P(\sigma > n) = (1-p)^n \text{ for all } n \in \mathbb{N}$$

and then use this to conclude that

$$P(\sigma > m+n|\sigma > n) = P(\sigma > m)\ \forall\, m,n \in \mathbb{N}.$$

[This shows that the geometric distributions are the discrete analogue of the exponential distributions.]

Exercise 2.3. Suppose that $S = \{1,2,\dots,n\}$ and A is a matrix such that $A_{ij} \ge 0$ for $i \neq j$ and $\sum_{j=1}^{n} A_{ij} = 0$ for all i. Show

$$Q_t = e^{tA} := \sum_{n=0}^{\infty}\frac{t^n}{n!}A^n \tag{2.2}$$

is a time homogeneous Markov kernel.

Hints: 1. To show $Q_t(i,j) \ge 0$ for all $t \ge 0$ and $i,j \in S$, write $Q_t = e^{-t\lambda}e^{t(\lambda I + A)}$ where $\lambda > 0$ is chosen so that $\lambda I + A$ has only non-negative entries. 2. To show $\sum_{j\in S} Q_t(i,j) = 1$, compute $\frac{d}{dt}Q_t\mathbf{1}$.

Exercise 2.4. Let $\{T_k\}_{k=1}^{\infty}$ be i.i.d. exponential random variables with intensity λ and $\{\sigma_\ell\}_{\ell=1}^{n}$ be independent geometric random variables with $\sigma_\ell = Geo(b_\ell)$ for some $b_\ell \in (0,1]$. Further assume that $\{\sigma_\ell\}_{\ell=1}^{n} \cup \{T_k\}_{k=1}^{\infty}$ are independent. We also let

$$\begin{aligned} W_0 = 0,&\quad W_n := T_1 + \ldots + T_n, \\ \mathbf{j}_0 = 0,&\quad \mathbf{j}_\ell := \sigma_1 + \cdots + \sigma_\ell, \\ S_\ell &:= W_{\mathbf{j}_\ell} - W_{\mathbf{j}_{\ell-1}} \text{ for } 1 \le \ell \le n. \end{aligned}$$

Show $\{S_\ell\}_{\ell=1}^{n}$ are independent exponential random variables with $S_\ell \overset{d}{=} \exp(b_\ell\lambda)$ for all $1 \le \ell \le n$.

Exercise 2.5. Keeping the notation of Example 22.52 and 22.53. Use Corollary 22.57 to show again that $P_x(T_B < \infty) = (q/p)^x$ for all $x > 0$ and $\mathbb{E}_x T_0 = x/(q-p)$ for $x < 0$. You should do so without making use of the extraneous hitting times, T_n for $n \neq 0$.

Exercise 2.6. Let $x \in X$. Show;

1. for all $n \in \mathbb{N}_0$,
$$P_x(\tau_x > n+1) = \sum_{y \neq x} p(x,y) P_y(T_x > n). \quad (2.3)$$
2. Use Eq. (2.5) to conclude that if $P_y(T_x = \infty) = 0$ for all $y \neq x$ then $P_x(\tau_x = \infty) = 0$, i.e. $\{X_n\}$ will return to x when started at x.
3. Sum Eq. (2.5) on $n \in \mathbb{N}_0$ to show
$$\mathbb{E}_x[\tau_x] = P_x(\tau_x > 0) + \sum_{y \neq x} p(x,y) \mathbb{E}_y[T_x]. \quad (2.4)$$
4. Now suppose that S is a finite set and $P_y(T_x = \infty) < 1$ for all $y \neq x$, i.e. there is a positive chance of hitting x from any $y \neq x$ in S. Explain how Eq. (2.6) combined with Lemma 22.42 (or see Corollary 22.59) shows that $\mathbb{E}_x[\tau_x] < \infty$.

Exercise 2.7 (2nd order recurrence relations). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and suppose $\{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\}$ solves the second order homogeneous recurrence relation:
$$au(x+1) + bu(x) + cu(x-1) = 0 \quad (2.5)$$
for $\alpha < x < \beta$. Show:

1. for any $\lambda \in \mathbb{C}$,
$$a\lambda^{x+1} + b\lambda^x + c\lambda^{x-1} = \lambda^{x-1} p(\lambda) \quad (2.6)$$
where $p(\lambda) = a\lambda^2 + b\lambda + c$ is the **characteristic polynomial** associated to Eq. (2.7).
Let $\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ be the roots of $p(\lambda)$ and suppose for the moment that $b^2 - 4ac \neq 0$. From Eq. (2.7) it follows that for any choice of $A_\pm \in \mathbb{R}$, the function,
$$w(x) := A_+ \lambda_+^x + A_- \lambda_-^x,$$
solves Eq. (2.7) for all $x \in \mathbb{Z}$.
2. Show there is a unique choice of constants, $A_\pm \in \mathbb{R}$, such that the function $u(x)$ is given by
$$u(x) := A_+ \lambda_+^x + A_- \lambda_-^x \text{ for all } \alpha \leq x \leq \beta.$$
3. Now suppose that $b^2 = 4ac$ and $\lambda_0 := -b/(2a)$ is the double root of $p(\lambda)$. Show for any choice of A_0 and A_1 in $\mathbb{R}$ that
$$w(x) := (A_0 + A_1 x) \lambda_0^x$$
solves Eq. (2.7) for all $x \in \mathbb{Z}$. **Hint:** Differentiate Eq. (2.8) with respect to λ and then set $\lambda = \lambda_0$.
4. Again show that any function u solving Eq. (2.7) is of the form $u(x) = (A_0 + A_1 x) \lambda_0^x$ for $\alpha \leq x \leq \beta$ for some unique choice of constants $A_0, A_1 \in \mathbb{R}$.

Exercise 2.8. Let $w_x := P_x\left(X_{T_{a,b}} = b\right) := P\left(X_{T_{a,b}} = b | X_0 = x\right)$.

1. Use first step analysis to show for $a < x < b$ that
$$w_x = \frac{1}{2}(w_{x+1} + w_{x-1}) \quad (2.7)$$
provided we define $w_a = 0$ and $w_b = 1$.
2. Use the results of Exercise 2.7 to show
$$P_x\left(X_{T_{a,b}} = b\right) = w_x = \frac{1}{b-a}(x-a). \quad (2.8)$$
3. Let
$$T_b := \begin{cases} \min\{n : X_n = b\} & \text{if} \quad \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}$$
be the first time $\{X_n\}$ hits b. Explain why, $\left\{X_{T_{a,b}} = b\right\} \subset \{T_b < \infty\}$ and use this along with Eq. (2.10) to conclude[2] that $P_x(T_b < \infty) = 1$ for all $x < b$. (By symmetry this result holds true for all $x \in \mathbb{Z}$.)

Exercise 2.9. The goal of this exercise is to give a second proof of the fact that $P_x(T_b < \infty) = 1$. Here is the outline:

1. Let $w_x := P_x(T_b < \infty)$. Again use first step analysis to show that w_x satisfies Eq. (2.9) for all x with $w_b = 1$.
2. Use Exercise 2.7 to show that there is a constant, c, such that
$$w_x = c(x-b) + 1 \text{ for all } x \in \mathbb{Z}.$$
3. Explain why c must be zero to again show that $P_x(T_b < \infty) = 1$ for all $x \in \mathbb{Z}$.

Exercise 2.10. Let $T = T_{a,b}$ and $u_x := \mathbb{E}_x T := \mathbb{E}[T | X_0 = x]$.

1. Use first step analysis to show for $a < x < b$ that
$$u_x = \frac{1}{2}(u_{x+1} + u_{x-1}) + 1 \quad (2.9)$$
with the convention that $u_a = 0 = u_b$.

[2] The fact that $P_j(T_b < \infty) = 1$ is also follows from Example 15.82 above.

2. Show that
$$u_x = A_0 + A_1 x - x^2 \tag{2.10}$$
solves Eq. (2.11) for any choice of constants A_0 and A_1.
3. Choose A_0 and A_1 so that u_x satisfies the boundary conditions, $u_a = 0 = u_b$. Use this to conclude that
$$\mathbb{E}_x T_{a,b} = -ab + (b+a)\,x - x^2 = -a\,(b-x) + bx - x^2. \tag{2.11}$$

Exercise 2.11. For $\theta \in \mathbb{R}$ let
$$f_\theta(n,x) := Q^{-n} e^{\theta x} = \left(pe^\theta + qe^{-\theta}\right)^{-n} e^{\theta x}$$
so that $Qf_\theta(n+1,\cdot) = f_\theta(n,\cdot)$ for all $\theta \in \mathbb{R}$. Compute;

1. $f_\theta^{(k)}(n,x) := \left(\frac{d}{d\theta}\right)^k f_\theta(n,x)$ for $k = 1,2$.
2. Use your results to show,
$$M_n^{(1)} := S_n - n\,(p-q)$$
and
$$M_n^{(2)} := (S_n - n\,(p-q))^2 - 4npq$$
are martingales.

(If you are ambitious you might also find $M_n^{(3)}$.)

Exercise 2.12 (Very similar to above example?). Suppose $\{M_n\}_{n=0}^\infty$ is a square integrable martingale. Show;

1. $\mathbb{E}\left[M_{n+1}^2 - M_n^2 | \mathcal{B}_n\right] = \mathbb{E}\left[(M_{n+1} - M_n)^2 | \mathcal{B}_n\right]$. Conclude from this that the Doob decomposition of M_n^2 is of the form,
$$M_n^2 = N_n + A_n$$
where
$$A_n := \sum_{1 \le k \le n} \mathbb{E}\left[(M_k - M_{k-1})^2 | \mathcal{B}_{k-1}\right].$$
2. If we further assume that $M_k - M_{k-1}$ is independent of $\mathcal{B}_{k-1}$ for all $k = 1,2,\dots$, explain why,
$$A_n = \sum_{1 \le k \le n} \mathbb{E}\,(M_k - M_{k-1})^2.$$

Exercise 2.13 (Martingale problem I). Suppose that $\{X_n\}_{n=0}^\infty$ is an $(S,\mathcal{S})$ – valued adapted process on some filtered probability space $\left(\Omega, \mathcal{B}, \{\mathcal{B}_n\}_{n\in\mathbb{N}_0}, P\right)$ and Q is a probability kernel on S. To each $f : S \to \mathbb{R}$ which is bounded and measurable, let
$$M_n^f := f(X_n) - \sum_{k<n} \left(Qf(X_k) - f(X_k)\right) = f(X_n) - \sum_{k<n} \left((Q-I)\,f\right)(X_k).$$
Show;

1. If $\{X_n\}_{n\ge0}$ is a time homogeneous Markov chain with transition kernel, Q, then $\left\{M_n^f\right\}_{n\ge0}$ is a martingale for each $f \in \mathcal{S}_b$.
2. Conversely if $\left\{M_n^f\right\}_{n\ge0}$ is a martingale for each $f \in \mathcal{S}_b$, then $\{X_n\}_{n\ge0}$ is a time homogeneous Markov chain with transition kernel, Q.

Exercise 2.14. Suppose τ is a stopping time, $(S,\mathcal{S})$ is a measurable space, and $Z : \Omega \to S$ is a function. Show that Z is $\mathcal{B}_\tau/\mathcal{S}$ measurable iff $Z|_{\{\tau=n\}}$ is $(\mathcal{B}_n)_{\{\tau=n\}}/\mathcal{S}$ – measurable for all $n \in \bar{\mathbb{N}}_0$.

Exercise 2.15. Suppose σ and τ are two stopping times. Show;

1. $\{\sigma < \tau\}$, $\{\sigma = \tau\}$, and $\{\sigma \le \tau\}^*$ are all in $\mathcal{B}_\sigma \cap \mathcal{B}_\tau$,
2. $\mathcal{B}_{\sigma\wedge\tau} = \mathcal{B}_\sigma \cap \mathcal{B}_\tau$,
3. $\mathcal{B}_{\sigma\vee\tau} = \mathcal{B}_\sigma \vee \mathcal{B}_\tau := \sigma(\mathcal{B}_\sigma \cup \mathcal{B}_\tau)$,[3] and
4. $\mathcal{B}_\sigma = \mathcal{B}_{\sigma\wedge\tau}$ on C where C is any one of the following three sets; $\{\sigma \le \tau\}$, $\{\sigma < \tau\}$, or $\{\sigma = \tau\}$.

*As an example, since
$$\{\sigma \le \tau\} \cap \{\sigma\wedge\tau = n\} = \{\sigma \le \tau\} \cap \{\sigma = n\} = \{n \le \tau\} \cap \{\sigma = n\} \in \mathcal{B}_n$$
for all $n \in \bar{\mathbb{N}}_0$, it follows that

Exercise 2.16. Show, by example, that it is not necessarily true that
$$\mathbb{E}_{\mathcal{G}_1}\mathbb{E}_{\mathcal{G}_2} = \mathbb{E}_{\mathcal{G}_1\wedge\mathcal{G}_2}$$
for arbitrary $\mathcal{G}_1$ and $\mathcal{G}_2$ – sub-sigma algebras of $\mathcal{B}$.

Hint: it suffices to take $(\Omega,\mathcal{B},P)$ with $\Omega = \{1,2,3\}$, $\mathcal{B} = 2^\Omega$, and $P(\{j\}) = \frac{1}{3}$ for $j = 1,2,3$.

Exercise 2.17 (Rademacher's theorem). Let $\Omega := (0,1]$, $\mathcal{B} := \mathcal{B}_{(0,1]}$, $P = m$ be Lebesgue measure, and $f \in L^1(P)$. To each partition $\Pi := \{0 = x_0 < x_1 < x_2 < \dots < x_n = 1\}$ of $(0,1]$ we let $\mathcal{B}_\Pi := \sigma\left(J_i := (x_{i-1}, x_i] : 1 \le i \le n\right)$.

[3] In fact, you will likely show in your proof that every set in $\mathcal{B}_\sigma \vee \mathcal{B}_\tau$ may be written as a disjoint union of a set from $\mathcal{B}_\sigma$ with a set from $\mathcal{B}_\tau$.

1. Show $\mathbb{E}[f|\mathcal{B}_\Pi](x) = \sum_{i=1}^n \frac{1}{x_i - x_{i-1}} \left[\int_{x_{i-1}}^{x_i} f(s)\,ds\right] \cdot 1_{(x_{i-1},x_i]}(x)$ for a.e. $x \in \Omega$.
2. For $f \in C([0,1],\mathbb{R})$, let
$$f_\Pi(x) := \sum_{i=1}^n \frac{\Delta_i f}{\Delta_i} 1_{J_i}(x) \tag{2.12}$$
where $\Delta_i f := f(x_i) - f(x_{i-1})$ and $\Delta_i := x_i - x_{i-1}$. Show if Π' is another partition of Ω which refines Π, i.e. $\Pi \subset \Pi'$, then
$$f_\Pi = \mathbb{E}[f_{\Pi'}|\mathcal{B}_\Pi] \text{ a.s.}$$
3. Show for any $a, b \in \Pi$ with $a < b$ that
$$\frac{f(b) - f(a)}{b - a} = \frac{1}{b-a}\int_a^b f_\Pi(x)\,dx. \tag{2.13}$$
Hint: consider the partition $\Pi_0 := \{0 < a < b < 1\}$.
Now let $\mathcal{B}_n := \mathcal{B}_{\Pi_n}$ and where $\Pi_n := \left\{\frac{k}{2^n}\right\}_{k=0}^{2^n}$ an observe your have now shown $g_n := f_{\Pi_n}$ is a martingale.
4. Let us now further suppose that $|f(y) - f(x)| \le K|y - x|$ for all $x, y \in [0,1]$, i.e. f is Lipschitz. From Eq. (2.16) it follows that $|g_n| := |f_{\Pi_n}| \le K$ so that $\{g_n\}_{n=1}^\infty$ is a bounded martingale. Use this along with Eq. (2.17) and Theorem 23.70 to conclude there exists $g \in L^\infty(P)$ such that
$$f(b) - f(a) = \int_a^b g(x)\,dx \text{ for all } 0 \le a < b \le 1.$$
[You may be interested to know that under these hypothesis, $f'(x)$ exists a.e. and $g(x) = f'(x)$ a.e.. Thus this a version of the fundamental theorem of calculus.]

Exercise 2.18. Suppose that $\{M_n\}_{n=1}^\infty$ is a decreasing sequence of closed subspaces of a Hilbert space, H. Let $M_\infty := \cap_{n=1}^\infty M_n$. Show $\lim_{n\to\infty} P_{M_n} x = P_{M_\infty} x$ for all $x \in H$. [**Hint:** you might make use of Exercise 18.5.]

Exercise 2.19. Let $(M_n)_{n=0}^\infty$ be a martingale with $M_0 = 0$ and $E[M_n^2] < \infty$ for all n. Show that for all $\lambda > 0$,
$$P\left(\max_{1\le m\le n} M_m \ge \lambda\right) \le \frac{E[M_n^2]}{E[M_n^2] + \lambda^2}.$$

Hints: First show that for any $c > 0$ that $\left\{X_n := (M_n + c)^2\right\}_{n=0}^\infty$ is a submartingale and then observe,
$$\left\{\max_{1\le m\le n} M_m \ge \lambda\right\} \subset \left\{\max_{1\le m\le n} X_n \ge (\lambda + c)^2\right\}.$$
Now use Doob' Maximal inequality (Proposition 23.46) to estimate the probability of the last set and then choose c so as to optimize the resulting estimate you get for $P(\max_{1\le m\le n} M_m \ge \lambda)$. (Notice that this result applies to $-M_n$ as well so it also holds that;
$$P\left(\min_{1\le m\le n} M_m \le -\lambda\right) \le \frac{E[M_n^2]}{E[M_n^2] + \lambda^2} \text{ for all } \lambda > 0.$$

Exercise 2.20. Let $\{Z_n\}_{n=1}^\infty$ be independent random variables, $S_0 = 0$ and $S_n := Z_1 + \dots + Z_n$, and $f_n(\lambda) := \mathbb{E}\left[e^{i\lambda Z_n}\right]$. Suppose $\mathbb{E}e^{i\lambda S_n} = \prod_{n=1}^N f_n(\lambda)$ converges to a continuous function, $F(\lambda)$, as $N \to \infty$. Show for each $\lambda \in \mathbb{R}$ that
$$P\left(\lim_{n\to\infty} e^{i\lambda S_n} \text{ exists}\right) = 1. \tag{2.14}$$

Hints:

1. Show it is enough to find an $\varepsilon > 0$ such that Eq. (2.18) holds for $|\lambda| \le \varepsilon$.
2. Choose $\varepsilon > 0$ such that $|F(\lambda) - 1| < 1/2$ for $|\lambda| \le \varepsilon$. For $|\lambda| \le \varepsilon$, show $M_n(\lambda) := \frac{e^{i\lambda S_n}}{\mathbb{E}e^{i\lambda S_n}}$ is a bounded complex[4] martingale relative to the filtration, $\mathcal{B}_n = \sigma(Z_1, \dots, Z_n)$.

Exercise 2.21. For $a < 0 < b$ with $a, b \in \mathbb{Z}$, let $\tau = \sigma_a \wedge \sigma_b$. Explain why τ is regular for S. Use this to show $P(\tau = \infty) = 0$. **Hint:** make use of Remark 23.76 and the fact that $|S_n - S_{n-1}| = |Z_n| = 1$ for all n.

Exercise 2.22. In this exercise, you are asked to use the central limit Theorem 15.50 to prove again that $P(\tau = \infty) = 0$, Exercise 2.21. **Hints:** Use the central limit theorem to show
$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,e^{-x^2/2}dx \ge f(0)\,P(\tau = \infty) \tag{2.15}$$
for all $f \in C^3(\mathbb{R} \to [0,\infty))$ with $M := \sup_{x\in\mathbb{R}} \left|f^{(3)}(x)\right| < \infty$. Use this inequality to conclude that $P(\tau = \infty) = 0$. **Hint:** consider $\mathbb{E}\left[f\left(\frac{S_n}{\sqrt{n}}\right)\right]$.

Exercise 2.23. Show
$$P(\sigma_b < \sigma_a) = \frac{|a|}{b + |a|} \tag{2.16}$$

[4] Please use the obvious generalization of a martingale for complex valued processes. It will be useful to observe that the real and imaginary parts of a complex martingales are real martingales.

and use this to conclude $P(\sigma_b < \infty) = 1$, i.e. every $b \in \mathbb{N}$ is almost surely visited by S_n. (This last result also follows by the Hewitt-Savage Zero-One Law, see Example 15.82 where it is shown b is visited infinitely often.)

Hint: Using properties of martingales and Exercise 2.21, compute $\lim_{n\to\infty} \mathbb{E}\left[S_n^{\sigma_a \wedge \sigma_b}\right]$ in two different ways.

Exercise 2.24. Let $\tau := \sigma_a \wedge \sigma_b$. In this problem you are asked to show $\mathbb{E}[\tau] = |a|\, b$ with the aid of the following outline.

1. Use Exercise 2.12 above to conclude $N_n := S_n^2 - n$ is a martingale.
2. Now show
$$0 = \mathbb{E}N_0 = \mathbb{E}N_{\tau\wedge n} = \mathbb{E}S_{\tau\wedge n}^2 - \mathbb{E}[\tau \wedge n]. \tag{2.17}$$
3. Now use DCT and MCT along with Exercise 2.23 to compute the limit as $n \to \infty$ in Eq. (2.22) to find
$$\mathbb{E}[\sigma_a \wedge \sigma_b] = \mathbb{E}[\tau] = b\,|a|. \tag{2.18}$$
4. By considering the limit, $a \to -\infty$ in Eq. (2.23), show $\mathbb{E}[\sigma_b] = \infty$.

Exercise 2.25. Verify,
$$M_n := S_n - n(p-q)$$
and
$$N_n := M_n^2 - \sigma^2 n$$
are martingales, where $\sigma^2 = 1 - (p-q)^2$. (This should be simple; see either Exercise 2.12 or Exercise 2.11.)

Exercise 2.26. Using exercise 2.25, show
$$\mathbb{E}(\sigma_a \wedge \sigma_b) = \left(\frac{b\left[1 - (q/p)^a\right] + a\left[(q/p)^b - 1\right]}{(q/p)^b - (q/p)^a}\right)(p-q)^{-1}. \tag{2.19}$$
By considering the limit of this equation as $a \to -\infty$, show
$$\mathbb{E}[\sigma_b] = \frac{b}{p-q}$$
and by considering the limit as $b \to \infty$, show $\mathbb{E}[\sigma_a] = \infty$.

Exercise 2.27. Let S_n be the total assets of an insurance company in year $n \in \mathbb{N}_0$. Assume $S_0 > 0$ is a constant and that for all $n \geq 1$ that $S_n = S_{n-1} + \xi_n$, where $\xi_n = c - Z_n$ and $\{Z_n\}_{n=1}^\infty$ are i.i.d. random variables having the normal distribution with mean $\mu < c$ and variance σ^2. (The number c is to be interpreted as the yearly premium.) Let $R = \{S_n \leq 0 \text{ for some } n\}$ be the event that the company eventually becomes bankrupt, i.e. is **R**uined. Show
$$P(\text{Ruin}) = P(R) \leq e^{-2(c-\mu)S_0/\sigma^2}.$$

Outline:

1. Show that $\lambda = -2(c-\mu)/\sigma^2 < 0$ satisfies, $\mathbb{E}\left[e^{\lambda \xi_n}\right] = 1$.
2. With this λ show
$$Y_n := \exp(\lambda S_n) = e^{\lambda S_0} \prod_{j=1}^n e^{\lambda \xi_j} \tag{2.20}$$
is a non-negative $\mathcal{B}_n = \sigma(Z_1, \dots, Z_n)$ – martingale.
3. Use a martingale convergence theorem to argue that $\lim_{n\to\infty} Y_n = Y_\infty$ exists a.s. and then use Fatou's lemma to show $\mathbb{E}Y_\tau \leq e^{\lambda S_0}$, where
$$\tau = \inf\{n : S_n \leq 0\}$$
is the time of the companies ruin.
4. Finally conclude that
$$P(R) \leq \mathbb{E}[Y_\tau : \tau < \infty] \leq \mathbb{E}Y_\tau \leq e^{\lambda S_0} = e^{-2(c-\mu)S_0/\sigma^2}.$$

Exercise 2.28. Suppose that Z is exponentially integrable and $\psi(\theta) := \ln M(\theta) = \ln \mathbb{E}\left[e^{\theta Z}\right]$. Show
$$\psi'(\theta) = \mathbb{E}_\theta Z \text{ and } \psi''(\theta) = \mathrm{Var}_\theta(Z).$$
[Use Proposition 10.59 in order to give a short solution to this problem.]

Exercise 2.29. Let $Z \overset{d}{=} N(0, \sigma^2)$ and $t > 0$. By Lemma 10.47, we know that
$$P(Z \geq t) = P(\sigma N \geq t) = P(N \geq t/\sigma) \leq c e^{-\frac{t^2}{2\sigma^2}} \tag{2.21}$$
where $c = 1/2$. The goal of this exercise is to use Proposition 25.4 to prove this same bound above but with $c = 1$. In more detail show;

1. Recall that Gaussian integration formulas implies,
$$M(\theta) = \mathbb{E}e^{\theta Z} = e^{\frac{1}{2}\theta^2\sigma^2} \text{ and so } \psi(\theta) = \frac{1}{2}\theta^2\sigma^2.$$
2. Show
$$\theta \to \theta t - \psi(\theta) = \theta t - \frac{1}{2}\theta^2\sigma^2$$
is maximized at $\theta_t = t/\sigma^2$ and that
$$\psi^*(t) = \sup_{\theta\in\mathbb{R}}\left(\theta t - \frac{1}{2}\theta^2\sigma^2\right) = \frac{t^2}{2\sigma^2}.$$
This assertion along with Proposition 25.4 verifies the tail bound in Eq. (2.26) with $c = 1$.

3. Show $\mathrm{Law}_{P_\theta}(Z) \overset{d}{=} N(\sigma^2, \theta\sigma^2)$ – a normal random variable with variance σ^2 and mean $\theta\sigma^2$. Hence when $\theta = \theta_t = t/\sigma^2$,
$$\mathrm{Law}_{P_{\theta_t}}(Z) \overset{d}{=} N(\sigma^2, t). \tag{2.22}$$
4. Conclude that
$$\frac{1}{2} = P_{\theta_t}(Z \ge t) = \frac{1}{M(\theta_t)} \mathbb{E}\left[1_{Z\ge t} e^{\theta_t Z}\right]$$
and explain (using Eq. (25.3)) that this inequality then implies Eq. (2.26) with $c = 1/2$.

Exercise 2.30. Suppose $-\infty < a < b < \infty$ and Z is a random variable such that $a \le Z \le b$. Let $\mu = \mathbb{E}Z$ and $\psi(\theta) = \ln \mathbb{E}\left[e^{\theta Z}\right]$.

1. Use Taylor's theorem along with Exercise 2.28 to show for any $\theta \in \mathbb{R}$, there exists θ^* between 0 and θ such that
$$\psi(\theta) = \theta\mu + \frac{1}{2}\theta^2 \operatorname{Var}_{\theta^*}(Z).$$
2. Use item 1. to show
$$\psi(\theta) \le \theta\mu + \frac{(b-a)^2}{8}\theta^2$$
by showing $\operatorname{Var}_{\theta^*}(Z) \le (b-a)^2/4$. **Hint:** this variance inequality holds no matter the distribution of Z as long as $a \le Z \le b$ a.s.
3. Use items 1. and 2. to prove **Hoeffding's inequality**, i.e.
$$\mathbb{E}\left[e^{\theta(Z-\mu)}\right] \le \exp\left(\frac{(b-a)^2}{8}\theta^2\right) \ \forall\, \theta \in \mathbb{R}. \tag{2.23}$$
4. Then use this and Lemma 25.7 to prove the Chernoff type bound,
$$P(Z - \mu \ge t) \le e^{-2\frac{t^2}{(b-a)^2}} \quad \forall\, t > 0$$
5. Show, by applying the previous inequality with Z replaced by $-Z$, that
$$P(Z - \mu \le -t) \le e^{-2\frac{t^2}{(b-a)^2}} \quad \forall\, t > 0$$
By adding the two previous bounds it follows that
$$P(|Z - \mu| \ge t) \le 2e^{-2\frac{t^2}{(b-a)^2}} \quad \forall\, t > 0.$$

Exercise 2.31. Suppose that $-\infty < a_j < b_j < \infty$ and $\{Z_j\}_{j=1}^n$ are independent random variables with $a_j \le Z_j \le b_j$ for $1 \le j \le n$. If $S = \sum_{j=1}^n Z_j$, $\mu = \mathbb{E}S$, and $v = \sum_{j=1}^n (b_j - a_j)^2$, show
$$\mathbb{E}\left[e^{\theta(S_n-\mu)}\right] \le e^{\frac{v}{8}\theta^2} \text{ and} \tag{2.24}$$
and
$$P(S_n - \mu \ge t) \le e^{-\frac{2}{v}t^2} \text{ for all } t \ge 0.$$
Use this result to conclude, if $L = \max_j(b_j - a_j)$, then $v \le nL^2$ and
$$P\left(\frac{S_n - \mu}{n} \ge t\right) \le e^{-2nt^2/L^2} \ \forall\, t \ge 0.$$

Exercise 2.32. Prove Theorem 26.1.

Exercise 2.33 (Resnik 7.1). Does $\sum_n 1/n$ converge? Does $\sum_n (-1)^n/n$ converge? Let $\{X_n\}$ be iid with $P[X_n = \pm 1] = 1/2$ Does $\sum_n X_n/n$ converge? [See Example 26.42 below for a more thorough investigation of this sort.]

Exercise 2.34 (Two Series Theorem – Resnik 7.15). Prove that the three series theorem reduces to a two series theorem when the random variables are positive. That is, if $X_n \ge 0$ are independent, then $\sum_n X_n < \infty$ a.s. iff for any $c > 0$ we have
$$\sum_n P(X_n > c) < \infty \text{ and} \tag{2.25}$$
$$\sum_n \mathbb{E}[X_n 1_{X_n \le c}] < \infty, \tag{2.26}$$
that is it is unnecessary to verify the convergence of the second series in Theorem 26.43 involving the variances.

Exercise 2.35. Let $\mathcal{P}$ denote the set of probability measures on $(\Omega, \mathcal{B})$. Show d_{TV} is a complete metric on $\mathcal{P}$.

Exercise 2.36. Suppose that μ, ν, and γ are probability measures on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. Show $d_{TV}(\mu * \nu, \mu * \gamma) \le d_{TV}(\nu, \gamma)$. Use this fact along with Exercise 2.35 to show,
$$d_{TV}(\mu_1 * \mu_2 * \cdots * \mu_n, \nu_1 * \nu_2 * \cdots * \nu_n) \le \sum_{i=1}^n d_{TV}(\mu_i, \nu_i)$$
for all choices probability measures, μ_i and ν_i on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$.

Exercise 2.37. Suppose that Ω is a (at most) countable set, $\mathcal{B} := 2^{\Omega}$, and $\{\mu_n\}_{n=0}^{\infty}$ are probability measures on $(\Omega, \mathcal{B})$. Show

$$d_{TV}(\mu_n, \mu_0) = \frac{1}{2} \sum_{\omega \in \Omega} |\mu_n(\{\omega\}) - \mu_0(\{\omega\})|$$

and $\lim_{n\to\infty} d_{TV}(\mu_n, \mu_0) = 0$ iff $\lim_{n\to\infty} \mu_n(\{\omega\}) = \mu_0(\{\omega\})$ for all $\omega \in \Omega$.

Exercise 2.38. Let $\mu_p(\{1\}) = p$ and $\mu_p(\{0\}) = 1 - p$ and $\nu_\lambda(\{n\}) := e^{-\lambda} \frac{\lambda^n}{n!}$ for all $n \in \mathbb{N}_0$.

1. Find $d_{TV}(\mu_p, \mu_q)$ for all $0 \le p, q \le 1$.
2. Show $d_{TV}(\mu_p, \nu_p) = p(1 - e^{-p})$ for all $0 \le p \le 1$. From this estimate and the estimate,
$$1 - e^{-p} = \int_0^p e^{-x} dx \le \int_0^p 1 dx = p, \tag{2.27}$$
it follows that $d_{TV}(\mu_p, \nu_p) \le p^2$ for all $0 \le p \le 1$.
3. Show
$$d_{TV}(\nu_\lambda, \nu_\gamma) \le |\lambda - \gamma| \text{ for all } \lambda, \gamma \in \mathbb{R}_+. \tag{2.28}$$
Hints: (**Andy Parrish's method – a former 280 student.**)
a) Observe that for any $n \in \mathbb{N}$ we have ν_λ and ν_γ are equal to the n – fold convolutions of $\nu_{\lambda/n}$ and $\nu_{\gamma/n}$ and use this to conclude
$$d_{TV}(\nu_\lambda, \nu_\gamma) \le n d_{TV}(\nu_{\lambda/n}, \nu_{\gamma/n}). \tag{2.29}$$
b) Using item 2. of this exercise, show
$$\left| d_{TV}(\nu_{\lambda/n}, \nu_{\gamma/n}) - d_{TV}(\mu_{\lambda/n}, \mu_{\gamma/n}) \right| \le C n^{-2}.$$
c) Finally make use of your results in item 1. part b. in order to let $n \to \infty$ in Eq. (2.35).

Exercise 2.39. Let (S_1, ρ_1) and (S_2, ρ_2) be separable metric spaces and $\mathcal{B}_{S_1}$ and $\mathcal{B}_{S_2}$ be the Borel σ – algebras on S_1 and S_2 respectively. Prove the analogue of Lemma 9.29, namely show $\mathcal{B}_{S_1 \times S_2} = \mathcal{B}_{S_1} \otimes \mathcal{B}_{S_2}$. Hint: you may find Exercise 9.10 helpful.

Exercise 2.40. Let (S_1, ρ_1) and (S_2, ρ_2) be separable metric spaces and $\mathcal{B}_{S_1}$ and $\mathcal{B}_{S_2}$ be the Borel σ – algebras on S_1 and S_2 respectively. Further suppose that $\{\mu_n\} \cup \{\mu\} \subset \mathcal{P}(S_1)$ and $\{\nu_n\} \cup \{\nu\} \subset \mathcal{P}(S_2)$. Show; if $\mu_n \implies \mu$ and $\nu_n \implies \nu$, then $\mu_n \otimes \nu_n \implies \mu \otimes \nu$. **Hint:** You may find it useful to use Skorohod's Theorem 28.8.

Exercise 2.41. To each finite and compactly supported measure, ν, on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ show there exists a sequence $\{\nu_n\}_{n=1}^\infty$ of finitely supported finite measures on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that $\nu_n \implies \nu$. Here we say ν is compactly supported if there exists $M < \infty$ such that $\nu(\{x : |x| \ge M\}) = 0$ and we say ν is finitely supported if there exists a finite subset, $\Lambda \subset \mathbb{R}$ such that $\nu(\mathbb{R} \setminus \Lambda) = 0$.

Exercise 2.42. Use Theorem 28.20 to give a proof of half of Theorem 28.16 when $S = \mathbb{R}^d$ and $\Lambda \subset \mathcal{P}(S)$, i.e. show; if Λ is weakly sequentially compact then Λ is tight. **Hint:** start by showing that if Λ were not tight, then there would exist an $\varepsilon > 0$ and $\mu_n \in \Lambda$ so that $\mu_n(C_n) < 1 - \varepsilon$ for all $n \in \mathbb{N}$.

Exercise 2.43. Let (S, ρ) be a separable metric space, $S_0 \subset S$ be a countable dense set, and $\{x_n\}_{n=1}^\infty \cup \{x\} \subset S$. Show $\lim_{n\to\infty} \rho(x_n, x) = 0$ iff $\lim_{n\to\infty} \rho(x_n, y) = \rho(x, y)$ for all $y \in S_0$.

Exercise 2.44 (Continuous Mapping Theorem II). Let (S_1, ρ_1) and (S_2, ρ_2) be separable metric spaces and $\mathcal{B}_{S_1}$ and $\mathcal{B}_{S_2}$ be the Borel σ – algebras on S_1 and S_2 respectively. Let Further suppose that $\{\mu_n\} \cup \{\mu\}$ are probability measures on $(S_1, \mathcal{B}_{S_1})$ such that $\mu_n \implies \mu$. If $f : S_1 \to S_2$ is a Borel measurable function such that $\mu(\mathcal{D}(f)) = 0$ (see Notation 28.22), then $f_*\mu_n \implies f_*\mu$ where $f_*\mu := \mu \circ f^{-1}$.

Exercise 2.45. Let $\{X_n\}_{n=1}^\infty$ be an i.i.d. sequence of random variables with zero mean and $\operatorname{Var}(X_n) = 1$ and $\mathbb{E}|X_n|^3 < \infty$ (so that Corollary 15.43 applies). For $t \ge 0$, let $W_n(t) := \frac{1}{\sqrt{n}} S_{[nt]}$ where $[nt]$ is the nearest integer to nt less than or equal to nt and $S_m := \sum_{k \le m} X_k$ where $S_0 = 0$ by definition. Show that $W_n \overset{\text{f.d.}}{\implies} B$ where $\{B(t) : t \ge 0\}$ is a Brownian motion as defined in Definition 22.26. You might use the following outline.

1. For any $0 \le s < t < \infty$, explain why $W_n(t) - W_n(s) \implies N(0, (t - s))$. (You may find Slutzky's Theorem 28.25 useful here.)
2. Given $\Lambda := \{0 = t_0 < t_1 < t_2 < \cdots < t_K\} \subset \mathbb{R}_+$ argue that $\{W_n(t_i) - W_n(t_{i-1})\}_{i=1}^K$ are independent and then show
$$\{W_n(t_i) - W_n(t_{i-1})\}_{i=1}^K \implies \{B(t_i) - B(t_{i-1})\}_{i=1}^K \text{ as } n \to \infty.$$
3. Now show that $\{W_n(t_i)\}_{i=1}^K \implies \{B(t_i)\}_{i=1}^K$ as $n \to \infty$. Hint; use Exercise 2.44.

Exercise 2.46 (Lemma 29.22 generalization). Suppose now $X : (\Omega, \mathcal{B}, P) \to \mathbb{R}^d$ is a random vector and $f_X(\lambda) := \mathbb{E}\left[e^{i\lambda \cdot X}\right]$ is its characteristic function. Show for $a > 0$,

$$P(|X|_\infty \geq a) \leq 2\left(\frac{a}{4}\right)^d \int_{[-2/a,2/a]^d} (1 - f_X(\lambda))\, d\lambda$$
$$= 2\left(\frac{a}{4}\right)^d \int_{[-2/a,2/a]^d} (1 - \mathrm{Re}\, f_X(\lambda))\, d\lambda \qquad (2.30)$$

where $|X|_\infty = \max_i |X_i|$ and $d\lambda = d\lambda_1, \dots, d\lambda_d$.

Exercise 2.47. For $x, \lambda \in \mathbb{R}$, let (also see Eq. (2.41))

$$\varphi(\lambda, x) := \begin{cases} \frac{e^{i\lambda x} - 1 - i\lambda x}{x^2} & \text{if } x \neq 0 \\ -\frac{1}{2}\lambda^2 & \text{if } x = 0. \end{cases} \qquad (2.31)$$

Let $\{x_k\}_{k=1}^n \subset \mathbb{R} \setminus \{0\}$, $\{Z_k\}_{k=1}^n \cup \{N\}$ be independent random variables with $N \overset{d}{=} N(0,1)$ and Z_k being Poisson random variables with mean $a_k > 0$, i.e. $P(Z_k = n) = e^{-a_k} \frac{a_k^n}{n!}$ for $n = 0, 1, 2 \dots$. With $Y := \sum_{k=1}^n x_k (Z_k - a_k) + \alpha N$, show

$$f_Y(\lambda) := \mathbb{E}\left[e^{i\lambda Y}\right] = \exp\left(\int_{\mathbb{R}} \varphi(\lambda, x)\, d\nu(x)\right)$$

where ν is the discrete measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ given by

$$\nu = \alpha^2 \delta_0 + \sum_{k=1}^n a_k x_k^2 \delta_{x_k}. \qquad (2.32)$$

[**Remark:** It is easy to see that $\varphi(\lambda, 0) = \lim_{x \to 0} \varphi(\lambda, x)$. In fact by Taylor's theorem with integral remainder we have

$$\varphi(\lambda, x) = -\lambda^2 \int_0^1 e^{it\lambda x} (1 - t)\, dt. \qquad (2.33)$$

From this formula it is clear that φ is a smooth function of (λ, x).]

Exercise 2.48. Show that if ν is a finite measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, then

$$f(\lambda) := \exp\left(\int_{\mathbb{R}} \varphi(\lambda, x)\, d\nu(x)\right) \qquad (2.34)$$

is the characteristic function of a probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Here is an outline to follow. (You may find the calculus estimates in Section 29.8 to be of help.)

1. Show $f(\lambda)$ is continuous.
2. Now suppose that ν is compactly supported. Show, using Exercises 2.47, 2.41, and the continuity Theorem 29.25 that $\exp\left(\int_{\mathbb{R}} \varphi(\lambda, x)\, d\nu(x)\right)$ is the characteristic function of a probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.
3. For the general case, approximate ν by a sequence of finite measures with compact support as in item 2.

Exercise 2.49. Suppose X and Y are independent random variables such that $Z = X + Y$ is discrete, i.e. there exists an at most countable set, $\Lambda \subset \mathbb{R}$, such that $P(Z \in \Lambda) = 1$. Show that X and Y must also be discrete.

*__Hint:__ let $\mu = \mathrm{Law}\, X$, $\nu = \mathrm{Law}\, Y$, and $\rho(y) := \sum_{z \in \Lambda} \mu(\{z - y\})$, then show $\rho(y) < 1$ for all y if μ is not a discrete measure and also show $\int_{\mathbb{R}} \rho(y)\, d\nu(y) = 1$.

Exercise 2.50. Suppose $n \in \mathbb{N}$, $\{X_j\}_{j=1}^n$ are i.i.d. random variables, and $Z = X_1 + \cdots + X_n$. If $\Lambda \subset [0, \infty)$ is a countable or finite set such that $P(Z \in \Lambda) = 1$ and $P(Z = 0) > 0$ (this implies $0 \in \Lambda$), show $P(X_1 \in \Lambda) = 1$.

Exercise 2.51. This problem uses the **same notation and assumptions** as in Theorem 30.26 and in particular $\{Y_{n,k}\}_{k=1}^n$ be independent Bernoulli random variables with $P(Y_{n,k} = 1) = p_{n,k}$ and $P(Y_{n,k} = 0) = q_{n,k} := 1 - p_{n,k}$. Let $X_{n,k} := Y_{n,k} - p_{n,k}$.

1. Explain why $\bar{S}_n = \sum_{k=1}^n X_{n,k} \implies L := Z - a$ where $a = \lim_{n\to\infty} \sum_{k=1}^n p_{n,k}$ and Z is a is a Poisson random variable with mean a as in Theorem 30.26
2. Show directly that $\{X_{n,k}\}_{k=1}^n$ does not satisfy the Lindeberg condition (LC).
3. Show $\{X_{n,k}\}_{k=1}^n$ satisfy condition (M), i.e. that $\sup_{1 \leq k \leq n} \mathbb{E} X_{n,k}^2 = 0$.
4. Show $\mathrm{Var}\left(\bar{S}_n\right) = \sum_{k=1}^n \sigma_{n,k}^2 = \sum_{k=1}^n p_{n,k}(1 - p_{n,k}) \to a$ as $n \to \infty$ which suffices to show condition (BV) holds.
5. Find a finite measure ν on $\mathbb{R}$ such that

$$f_L(\lambda) = \mathbb{E} e^{i\lambda L} = \exp\left(\int_{\mathbb{R}} \frac{e^{i\lambda x} - 1 - i\lambda x}{x^2} d\nu(x)\right).$$

Exercise 2.52. Suppose $T = [0, \infty)$ and $\{X_t : t \in T\}$ is a mean zero Gaussian random field (process). Show that $\mathcal{B}_{[0,\sigma]} \overset{X_\sigma}{\perp\!\!\!\perp} \mathcal{B}_{[\sigma,\infty)}$ for all $0 \leq \sigma < \infty$ iff

$$Q(s, \sigma)\, Q(\sigma, t) = Q(\sigma, \sigma)\, Q(s, t) \;\; \forall\, 0 \leq s \leq \sigma \leq t < \infty. \qquad (2.35)$$

Hint: see use Exercises 19.12 and 19.11.

Exercise 2.53 (Independent increments). Let

$$\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_n = T\}$$

be a partition of $[0, T]$, $\Delta_i B := B_{t_i} - B_{t_{i-1}}$ and $\Delta_i t := t_i - t_{i-1}$. Show $\{\Delta_i B\}_{i=1}^n$ are independent mean zero normal random variables with $\mathrm{Var}(\Delta_i B) = \Delta_i t$.

Exercise 2.54 (Increments independent of the past). Let $\mathcal{B}_t := \sigma(B_s : s \le t)$. For each $s \in (0,\infty)$ and $t > s$, show;

1. $B_t - B_s$ is independent of $\mathcal{B}_s$ and
2. more generally show, $B_t - B_s$ is independent of $\mathcal{B}_{s+} := \cap_{\sigma > s}\mathcal{B}_\sigma$.

Exercise 2.55 (The simple Markov property). Show $B_t - B_s$ is independent of $\mathcal{B}_s$ for all $t \ge s$. Use this to show, for any bounded measurable function, $f : \mathbb{R} \to \mathbb{R}$ that

$$\begin{aligned}\mathbb{E}\left[f(B_t)|\mathcal{B}_{s+}\right] &= \mathbb{E}\left[f(B_t)|\mathcal{B}_s\right] = \mathbb{E}\left[f(B_t)|B_s\right]\\ &= (p_{t-s} * f)(B_s) =: \left(e^{(t-s)\Delta/2}f\right)(B_s) \text{ a.s.,}\end{aligned}$$

where

$$p_t(x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}x^2}$$

so that $p_t * f = Q_t(\cdot, f)$. This problem verifies that $\{B_t\}_{t\ge0}$ is a "**Markov process**" with transition kernels $\{Q_t\}_{t\ge0}$ which have $\frac{1}{2}\Delta = \frac{1}{2}\frac{d^2}{dx^2}$ as there "**infinitesimal generator.**"

Exercise 2.56. Let

$$\mathcal{P} := \{0 = t_0 < t_1 < \dots < t_n = T\}$$

and $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function. Show

$$\mathbb{E}\left[f(B_{t_1},\dots,B_{t_n})\right] = \int_{\mathbb{R}^n} f(x_1,\dots,x_n)\, q_{\mathcal{P}}(x)\, dx$$

where

$$q_{\mathcal{P}}(x) := p_{t_1}(x_1)\, p_{t_2-t_1}(x_2 - x_1)\dots p_{t_n - t_{n-1}}(x_n - x_{n-1}).$$

Hint: Either use Exercise 2.53 by writing

$$f(x_1,\dots,x_n) = g(x_1, x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1})$$

for some function, g or use Exercise 2.55 first for functions, f of the form,

$$f(x_1,\dots,x_n) = \prod_{j=1}^n \varphi_j(x_j).$$

Better yet, do it by both methods!

Exercise 2.57. Suppose $\{Y_t\}_{t\ge0}$ is a version of a process, $\{X_t\}_{t\ge0}$. Further suppose that $t \to Y_t(\omega)$ and $t \to X_t(\omega)$ are both right continuous everywhere. Show $E := \{Y_\cdot \ne X_\cdot\}$ is a measurable set such that $P(E) = 0$ and hence X and Y are indistinguishable. **Hint:** replace the union in Eq. (32.1) by an appropriate countable union.

Exercise 2.58. Show $(C([0,1],S), \rho_\infty)$ is separable. **Hints:**

1. Choose a countable dense subset, Λ, of S and then choose finite subset $\Lambda_n \subset \Lambda$ such that $\Lambda_n \uparrow \Lambda$.
2. Let $\mathbb{D}_n := \left\{\frac{k}{2^n} : 0 \le k \le 2^n\right\}$ and $\mathbb{D} = \cup_{n=0}^\infty \mathbb{D}_n$. Further let $\mathbb{F}_n := \{x : [0,1] \to \Lambda_n\}$ such that $x|_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}$ is constant for all $1 \le k \le 2^n$ and further suppose that $x|_{[0,2^{-n}]}$ is constant.
3. Given $y \in C([0,1],S)$ and $\varepsilon > 0$, show there exists $n \in \mathbb{N}$ and an $x \in \mathbb{F}_n$ such that $\rho_\infty(y,x) \le \varepsilon$.
4. For $k,n \in \mathbb{N}$ let
$$\mathcal{F}_n^k := \left\{y \in C([0,1],S) : \min_{x\in\mathbb{F}_n} \rho_\infty(y,x) \le \frac{1}{k}\right\}$$
and let $\Gamma := \left\{(k,n) \in \mathbb{N}^2 : \mathcal{F}_n^k \ne \emptyset\right\}$. For each $(k,n) \in \Gamma$, choose a function, $y_{k,n} \in \mathcal{F}_n^k$.
5. Now show that $\{y_{k,n} : (k,n) \in \Gamma\}$ is a countable dense subset of $(C([0,1],S), \rho_\infty)$.

Exercise 2.59. Provide a proof of Proposition 33.6. **Hints:** Use the results of Exercise 15.7, namely that

$$\mathbb{E}|S_l|^4 = l\gamma + 3l(l-1), \tag{2.36}$$

to verify that Eq. (33.4) holds for $s,t \in D_n := \frac{1}{n}\mathbb{N}_0$. Take care of the case where $s,t \ge 0$ with $|t-s| < 1/n$ by hand and finish up using these results along with Minkowski's inequality.

Exercise 2.60 (Quadratic Variation). Let

$$\mathcal{P}_m := \left\{0 = t_0^m < t_1^m < \dots < t_{n_m}^m = T\right\}$$

be a sequence of partitions such that $\text{mesh}(\mathcal{P}_m) \to 0$ as $m \to \infty$. Further let

$$Q_m := \sum_{i=1}^{n_m} (\Delta_i^m B)^2 := \sum_{i=1}^{n_m}\left(B_{t_i^m} - B_{t_{i-1}^m}\right)^2. \tag{2.37}$$

Show

$$\lim_{m\to\infty} \mathbb{E}\left[(Q_m - T)^2\right] = 0$$

and $\lim_{m\to\infty} Q_m = T$ a.s. if $\sum_{m=1}^{\infty} \text{mesh}(\mathcal{P}_m) < \infty$. This result is often abbreviated by the writing, $dB_t^2 = dt$. **Hint:** it is useful to observe; 1)

$$Q_m - T = \sum_{i=1}^{n_m} \left[(\Delta_i^m B)^2 - \Delta_i t \right]$$

and 2) using Eq. (33.2) there is a constant, $c < \infty$ such that

$$\mathbb{E}\left[(\Delta_i^m B)^2 - \Delta_i t \right]^2 = c\,(\Delta_i t)^2 .$$

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Solutions for selected problems from Resnick