Summary: Weak Convergence of Random Sums

- For each $n \in \mathbb{N}$, let $\{X_{n,k}\}_{k=1}^{n}$ be independent random variables and let

$$S_n := \sum_{k=1}^{n} X_{n,k} \text{ and} \tag{1.1}$$

$$f_{nk}(\lambda) := \mathbb{E}\left[e^{i\lambda X_{n,k}}\right] \text{ (characteristic function of } X_{n,k}). \tag{1.2}$$

- The goal of this chapter is to discuss some of the possible weak limits of such $\{S_n\}_{n=1}^{\infty}$. under various conditions.

1.1 I.I.D. Sums

Definition 1.1. *A probability distribution, μ, on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is* ***infinitely divisible*** *iff for all $n \in \mathbb{N}$ there exists i.i.d. nondegenerate random variables, $\{X_{n,k}\}_{k=1}^{n}$, such that $X_{n,1} + \cdots + X_{n,n} \overset{d}{=} \mu$.*

The previous definition may also be formulated as;

1. For all $n \in \mathbb{N}$ there should exists a non-degenerate probability measure, μ_n, on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\mu_n^{*n} = \mu$.
2. For all $n \in \mathbb{N}$, $\hat{\mu}(\lambda) = [g_n(\lambda)]^n$ for some non-constant characteristic function, g_n.

Example 1.2. The normal and Poisson distributions are infinitely divisible. Linear combination of independent infinitely divisible random variables is again infinitely divisible. All of the distributions you found in Exercise **??** are infinitely divisible. If μ is infinitely divisible, then μ

Example 1.3. If $\mu(\{-1\}) = \frac{1}{2} = \mu(\{1\})$, then $\hat{\mu}(\lambda) = \cos\lambda$. If μ were infinitely divisible there would exists $f_n(\lambda) = \hat{\mu}_n(\lambda)$ such that $\cos\lambda = f_n(\lambda)$. But in particular this would imply, $f_2(\lambda)^2 = \cos\lambda$ which would imply $f_2(\lambda)$ is smooth and differentiating this equation gives,

$$-\sin\lambda = 2f_2(\lambda) f_2'(\lambda) \implies f_2'(\lambda) = \frac{-\sin\lambda}{2f_2(\lambda)} = \frac{-\sin\lambda}{\pm 2\sqrt{\cos\lambda}}$$

which is not smooth. Similarly, if $d\mu(x) = \frac{1}{2}1_{[-1,1]}(x)\,dx$ then $\hat{\mu}(\lambda) = \frac{\sin\lambda}{\lambda}$ is not infinitely divisible.

Exercise 1.1. Suppose $n \in \mathbb{N}, \{X_j\}_{j=1}^{n}$ are i.i.d. random variables, and $Z = X_1 + \cdots + X_n$. If $\Lambda \subset [0, \infty)$ is a countable or finite set such that $P(Z \in \Lambda) = 1$ and $P(Z = 0) > 0$ (this implies $0 \in \Lambda$), show $P(X_1 \in \Lambda) = 1$.

Corollary 1.4. *Suppose $\Lambda \subset [0, \infty)$ is a finite set such that $0 \in \Lambda$ and $\#(\Lambda) \geq 2$. If $\mu \in \mathcal{P}(\mathbb{R})$ is supported on Λ, i.e. $\mu(\Lambda) = 1$ and $\mu(\{\lambda\}) > 0$ for all $\lambda \in \Lambda$, then μ is* ***not*** *infinitely divisible. Thus no finitely supported random variable is infinitely divisible.*

Proof. If μ is n-divisible, then there exists $\{X_j\}_{j=1}^{n}$ i.i.d. random variables such that $Z = X_1 + \cdots + X_n$ is distributed according to μ. By Exercise 1.1 we know that there eixsts $\Lambda_n \subset \Lambda$ such that $P(X_1 \in \Lambda_n) = 1$. Moreover, we must have $0 \in \Lambda_n$ for otherwise $P(Z > 0) = \mu(\{0\}) = 0$. We also must have

$$n\Lambda_n \subset \overbrace{\Lambda_n + \cdots + \Lambda_n}^{\text{n times}} \subset \Lambda.$$

For large enough n this is only possible if $\Lambda_n = \{0\}$ in which case $X_j = 0$ a.s. and hence $Z = 0$ a.s. But this contradicts $\#(\Lambda) \geq 2$. Hence μ can not be inifinitely divisible. ∎

Notation 1.5 *For $x, \lambda \in \mathbb{R}$, let*

$$\beta(\lambda, x) := \frac{e^{i\lambda x} - 1 - i\lambda x}{x^2} = -\frac{\lambda^2}{2}\int_0^1 e^{it\lambda x} 2(1-t)\,dt$$

where the second equality comes from Taylor's theorem applied to $e^{i\lambda x}$. We extend $\beta(\lambda, \cdot)$ to $\bar{\mathbb{R}}$ by setting $\beta(\lambda, \pm\infty) = 0$.

*Example 1.6 (Exercise **??**).* Recall from Exercise **??**, if ν is any finite measure $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, there exists a (necessarily unique) probability measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $\hat{\mu} = e^{\psi}$ where

$$\psi(\lambda) = \int_{\mathbb{R}} \frac{e^{i\lambda x} - 1 - i\lambda x}{x^2} d\nu(x) = \int_{\mathbb{R}} \beta(\lambda, x)\, d\nu(x). \tag{1.3}$$

By replacing ν by $\frac{1}{n}\nu$, there exists a probability measure, μ_n, so that $\hat{\mu}_n = e^{\frac{1}{n}\psi}$ and so $\mu = \mu_n^{*n}$ which shows μ is infinitely divisible.

Lemma 1.7. *If X is a random variable such $\mathbb{E}\left[e^{i\lambda X}\right] = f(\lambda) = e^{\psi(\lambda)}$ with ψ as in Eq. (1.3), then $\mathbb{E}\left[X^2\right] = \nu(\mathbb{R}) < \infty$ and $\mathbb{E}X = 0$.*

Proof. The function $\beta(\lambda, x)$ is a smooth function of $(\lambda, x) \in \mathbb{R}^2$ and by direct calculation (first for $x \neq 0$ and then for all x by continuity) we find,

$$\beta_\lambda(\lambda, x) = i\frac{e^{i\lambda x} - 1}{x} = -\lambda \int_0^1 e^{it\lambda x} dt \text{ and}$$
$$\beta_{\lambda,\lambda}(\lambda, x) = -e^{i\lambda x}.$$

Therefore, since ν is a finite measure and we may differentiate past the integral twice to find,

$$\psi'(\lambda) = \int_{\mathbb{R}} i\frac{e^{i\lambda x} - 1}{x} d\nu(x) \text{ and } \psi''(\lambda) = -\int_{\mathbb{R}} e^{i\lambda x} d\nu(x)$$

and the evaluating at $\lambda = 0$ shows

$$\psi'(0) = 0 \text{ and } \psi''(0) = -\nu(\mathbb{R}).$$

It now follows from Theorem **??** that $\mathbb{E}\left[X^2\right] < \infty$. Since one one hand,

$$f' = \psi' f,\ f'' = (\psi')^2 f + \psi'' f,$$
$$f'(0) = 0, \text{ and } f''(0) = -\nu(\mathbb{R}),$$

and while on the other hand,

$$f'(\lambda) = \mathbb{E}\left[iXe^{i\lambda X}\right] \text{ and } f''(\lambda) = -\mathbb{E}\left[X^2 e^{i\lambda X}\right],$$

it follows that $\mathbb{E}X = 0$ and $\mathbb{E}\left[X^2\right] = \nu(\mathbb{R})$. ■

Theorem 1.8 (Lévy Kintchine formula). *A probability measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is infinitely divisible iff $\hat{\mu}(\lambda) = e^{\psi(\lambda)}$ where*

$$\psi(\lambda) = i\lambda b - \frac{1}{2}a\lambda^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\lambda x} - 1 - i\lambda x \cdot 1_{|x|\le 1}\right) d\nu(x) \tag{1.4}$$

for some $b \in \mathbb{R}$, $a \geq 0$, and some measure ν on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} \left(x^2 \wedge 1\right) d\nu(x) < \infty. \tag{1.5}$$

[Note that the term $-\frac{1}{2}a\lambda^2$ in Eq. (1.4) the logarithm of the characteristic function of $N(0, \sqrt{a})$.]

Theorem 1.9. *Suppose that μ is a probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and* Law(X) $=$ μ*. Then μ is infinitely divisible iff there exits an array, $\{X_{n,k} : 1 \leq k \leq m_n\}$ with $\{X_{n,k}\}_{k=1}^{m_n}$ being **i.i.d.** such that $\sum_{k=1}^{m_n} X_{n,k} \implies X$ and $m_n \uparrow \infty$ as $n \to \infty$.*

Proof. The only non-trivial direction is $(\Longleftarrow)$. I will only prove the special case where $m_n = n$. [See Kallenberg [**?**, Lemma 15.13, p. 294] for the needed result involving the tail bounds needed to cover the full case.]

Fix a $k \in \mathbb{N}$. Then for $n \in \mathbb{N}$ we we decompose S_{nk} into k i.i.d. summands $\{S_n^i\}_{i=1}^k$ by

$$S_{nk} = \sum_{k=1}^{nk} X_{nk,k} = \sum_{i=1}^{k} S_n^i,$$

where

$$S_n^i = \sum_{j=k(i-1)+1}^{ki} X_{nk,j}.$$

Since $S_{nk} \implies X$ as $n \to \infty$ we know that $\{S_{nk}\}_{n=1}^\infty$ is tight and there exists $\varepsilon(r) \downarrow 0$ as $r \uparrow \infty$ such that

$$\sup_{n\in\mathbb{N}} P(|S_{nk}| > r) \leq \varepsilon(r).$$

By independence,

$$P\left(S_n^1 > r\right)^k = P\left(S_n^i > r \text{ for } 1 \leq i \leq k\right)$$
$$\leq P(S_{nk} > kr) \leq P(|S_{nk}| > kr) \leq \varepsilon(kr)$$

and similarly,

$$P\left(-S_n^1 > r\right)^k = P\left(-S_n^i > r \text{ for } 1 \leq i \leq k\right)$$
$$\leq P(-S_{nk} > kr) \leq P(|S_{nk}| > kr) \leq \varepsilon(kr).$$

Together this shows $P\left(\left|S_n^1\right| > r\right) \leq 2\varepsilon(kr) \to 0$ as $r \uparrow \infty$ which shows that $\{S_n^1\}_{n=1}^\infty$ has tight distributions as well.

Thus there exists a subsequence $\{n_l\}$ such that $S_{n_l}^1 \implies Y$ as $l \to \infty$. Let $\{Y_i\}_{i=1}^k$ be i.i.d. random variables with $Y_i \overset{d}{=} Y$. Then by Exercise **??** it follows that

$$S_{kn_l} = \sum_{i=1}^{k} S_{n_l}^i \implies Y_1 + \dots Y_k$$

from which we conclude that $X \overset{d}{=} Y_1 + \dots + Y_k$. Since k was arbitrary we have shown X is infinitely divisible. ■

1.2 Independent but not identical summands

- We now no longer assume that the $\{X_{n,k}\}_{k=1}^n$ are identically distributed.
- We do introduce normalization conditions (see Assumption 1 below).
- We will also impose conditions so that no one term, $X_{n,k}$ "dominates" the sum, $\sum_{k=1}^n X_{n,k}$.

Assumption 1 (Normalizations) *Assume* $\mathbb{E}[X_{n,k}] = 0$, $\sigma_{n,k}^2 = \mathbb{E}\left[X_{n,k}^2\right] < \infty$, *and* $\operatorname{Var}(S_n) = \sum_{k=1}^n \sigma_{n,k}^2 = 1$.

Definition 1.10 (No One Dominator Conditions). *Let* $\{X_{n,k}\}$ *be as above.*

(LiC) $\{X_{n,k}\}_{k=1}^n$ *satisfy the* ***Liapunov condition*** *(LiapC) iff*

$$\lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}|X_{n,k}|^\alpha = 0 \text{ for some } \alpha > 2. \tag{1.6}$$

(LC) $\{X_{n,k}\}$ *satisfies the* ***Lindeberg Condition (LC)*** *iff*

$$\lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > t\right] = 0 \text{ for all } t > 0. \tag{1.7}$$

[Since $\sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > t\right]$ *is a decreasing function of t it suffices to check (LC) along any sequence of* $\{t_l\}$ *with* $t_l \downarrow 0$.*]*

(M) $\{X_{n,k}\}$ *satisfies* ***condition*** *(M) if*

$$D_n := \max\left\{\sigma_{n,k}^2 = \operatorname{Var}(X_{n,k}) : k \le n\right\} \to 0 \text{ as } n \to \infty. \tag{1.8}$$

(UAN) $\{X_{n,k}\}$ *is* ***uniformly asymptotic negligibility (UAN)*** *if for all* $\varepsilon > 0$,

$$\lim_{n\to\infty} \max_{k\le n} P(|X_{n,k}| > \varepsilon) = 0. \tag{1.9}$$

**Can ignore this condition in the summary.*

Each of these conditions imposes constraints on the size of the tails of the $\{X_{n,k}\}$, see Lemma 1.12 below where it is shown $(LC) \implies (M) \implies (UAN)$. Condition (M) asserts that all of the terms in the sum $\sum_{k=1}^n \sigma_{n,k}^2 = \operatorname{Var}(S_n) = 1$ are small so that no one term is contributing by itself. Since $\sigma_{n,k}^2 = \mathbb{E}\left[X_{n,k}^2\right]$, if $t > 0$, then

$$0 \le \sigma_{n,k}^2 - \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| \le t\right] = \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > t\right]$$

and so (LC) condition is equivalent to

$$\lim_{n\to\infty} \sum_{k=1}^n \left(\sigma_{n,k}^2 - \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| \le t\right]\right) = 0. \tag{(LC')}$$

Thus the variance of $X_{n,k}$ has to be coming with high probability from small fluctuations around 0 rather than from large fluctuations happening with low probability.

Lemma 1.11. *Let* $\{X_{n,k}\}_{k=1}^n$ *for* $n \in \mathbb{N}$ *be as in Assumption 1. Then (LiapC)* $\implies$ *(LC) holds. More generally, if* $\{X_{n,k}\}$ *satisfies the Liapunov condition,*

$$\lim_{n\to\infty} \sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 \varphi(|X_{n,k}|)\right] = 0$$

where $\varphi : [0,\infty) \to [0,\infty)$ *is a non-decreasing function such that* $\varphi(t) > 0$ *for all* $t > 0$, *then* $\{X_{n,k}\}$ *satisfies (LC).*

Proof. We prove the generalization here;

$$\begin{aligned}\sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 : |X_{n,k}| > t\right] &\le \sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 \frac{\varphi(|X_{n,k}|)}{\varphi(t)} : |X_{n,k}| > t\right] \\ &\le \frac{1}{\varphi(t)} \sum_{k=1}^n \mathbb{E}\left[X_{n,k}^2 \varphi(|X_{n,k}|)\right] \to 0 \text{ as } n \to \infty.\end{aligned}$$

■

Lemma 1.12. *Let* $\{X_{n,k} : 1 \le k \le n < \infty\}$ *be as above, then (LiapC)* $\implies$ *(LC)* $\implies$ *(M)* $\implies$ *(UAN). Moreover the Lindeberg Condition (LC) implies the following strong form of (UAN),*

$$\sum_{k=1}^n P(|X_{n,k}| > \varepsilon) \le \frac{1}{\varepsilon^2} \sum_{k=1}^n \mathbb{E}\left[|X_{n,k}|^2 : |X_{n,k}| > \varepsilon\right] \to 0. \tag{1.10}$$

1.2.1 Limits under (LC)

Theorem 1.13 (Lindeberg-Feller CLT (I)). *Suppose* $\{X_{n,k}\}$ *satisfies (LC) and the hypothesis in Assumption 1, then*

$$S_n \implies N(0,1). \tag{1.11}$$

Conversely, if $\{X_{n,k}\}$ *satisfies (M) and* $S_n \implies N(0,1)$ *(i.e. the central limit theorem in Eq. (1.11) holds), then* $\{X_{n,k}\}$ *satisfies (LC). So under condition (M),* S_n *converges to a normal random variable iff (LC) holds.*

Notation 1.14 *For* $x, \lambda \in \mathbb{R}$, *let*

$$\beta(\lambda, x) := \frac{e^{i\lambda x} - 1 - i\lambda x}{x^2} = -\frac{\lambda^2}{2}\int_0^1 e^{it\lambda x} 2(1-t)\,dt \tag{1.12}$$

where the second equality comes from Taylor's theorem applied to $e^{i\lambda x}$. *We extend* $\beta(\lambda, \cdot)$ *to* $\bar{\mathbb{R}}$ *by setting* $\beta(\lambda, \pm\infty) = 0$.

1.2.2 Limits under (M)

Theorem 1.15 (Limits under (M)**).** *Suppose* $\{X_{n,k}\}_{k=1}^n$ *satisfy property* (M) *and the normalizations in Assumption 1. If* $S_n := \sum_{k=1}^n X_{n,k} \implies L$ *for some random variable* L, *then*

$$f_L(\lambda) := \mathbb{E}\left[e^{i\lambda L}\right] = \exp\left(\int_{\mathbb{R}} \beta(\lambda, x)\, d\nu(x)\right) \tag{1.13}$$

for some finite positive measure, ν, *on* $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ *with* $\nu(\mathbb{R}) \le 1$.

Proof. Let $\mu_{n,k} := P \circ X_{n,k}^{-1}$ is the law of $X_{n,k}$, $f_{n,k}(\lambda) = \mathbb{E}\left[e^{i\lambda X_{n,k}}\right]$, and use $\mathbb{E}[X_{n,k}] = 0$ to write,

$$\begin{aligned}\varphi_{n,k}(\lambda) &= f_{n,k}(\lambda) - 1 = \mathbb{E}\left[e^{i\lambda X_{n,k}} - 1\right] = \mathbb{E}\left[e^{i\lambda X_{n,k}} - 1 - i\lambda X_{n,k}\right] \\ &= \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) d\mu_{n,k}(x) \sim -\frac{\lambda^2}{2}\int_{\mathbb{R}} x^2 d\mu_{n,k}(x) = -\frac{\lambda^2}{2}\sigma_{n,k}^2.\end{aligned}$$

So under condition (M) we expect $\varphi_{n,k}(\lambda)$ to be small for large n which suggests,[1]

$$\begin{aligned}f_{S_n}(\lambda) &= \mathbb{E}\left[e^{i\lambda S_n}\right] = \prod_{k=1}^n f_{n,k}(\lambda) = \prod_{k=1}^n \left(1 + \varphi_{n,k}(\lambda)\right) \\ &\cong \prod_{k=1}^n e^{\varphi_{n,k}(\lambda)} = \exp\left(\sum_{k=1}^n \varphi_{n,k}(\lambda)\right).\end{aligned} \tag{1.15}$$

If we let $\nu_n^* := \sum_{k=1}^n \mu_{n,k}$, then

$$\begin{aligned}\sum_{k=1}^n \varphi_{n,k}(\lambda) &= \sum_{k=1}^n \int_{\mathbb{R}} \left(e^{i\lambda x} - 1 - i\lambda x\right) d\mu_{n,k}(x) \\ &= \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\lambda x} - 1 - i\lambda x\right) d\nu_n^*(x).\end{aligned}$$

The measure ν_n^* satisfies,

$$\int_{\mathbb{R}} x^2 d\nu_n^*(x) = \sum_{k=1}^n \int_{\mathbb{R}} x^2 d\mu_{n,k}(x) = \sum_{k=1}^n \sigma_{n,k}^2 = 1$$

and so if we define $d\nu_n(x) := x^2 d\nu_n^*(x)$, then ν_n is a probability measure and we find,

$$\sum_{k=1}^n \varphi_{n,k}(\lambda) = \int_{\mathbb{R}\setminus\{0\}} \beta(\lambda, x)\, d\nu_n(x) = \int_{\mathbb{R}} \beta(\lambda, x)\, d\nu_n(x)$$

and so we expect,

$$f_L(\lambda) = \lim_{n\to\infty} f_{S_n}(\lambda) = \mathbb{E}\left[e^{i\lambda S_n}\right] = \lim_{n\to\infty} \exp\left(\int_{\bar{\mathbb{R}}} \beta(\lambda, x)\, d\nu_n(x)\right).$$

The measure $\{\nu_n\}_{n=1}^\infty$ are (automatically) tight on $\bar{\mathbb{R}}$ and so by passing to a subsequence if necessary we can assert that $\nu_n \implies \bar{\nu}$ on $\bar{\mathbb{R}}$ (we may loose some mass to $\pm\infty$) and therefore this leads to

$$f_L(\lambda) = \exp\left(\int_{\bar{\mathbb{R}}} \beta(\lambda, x)\, d\bar{\nu}(x)\right) = \exp\left(\int_{\mathbb{R}} \beta(\lambda, x)\, d\bar{\nu}(x)\right).$$

In the last equality we have used $\beta(\lambda, \pm\infty) = 0$. The result is now complete by letting $\nu = \bar{\nu}|_{\mathcal{B}_{\mathbb{R}}}$. The measure ν now satisfies, $\nu(\mathbb{R}) = \bar{\nu}(\mathbb{R}) \le \bar{\nu}(\bar{\mathbb{R}}) = 1$. ■

We can replace the normalization assumption that $\mathrm{Var}(S_n) = 1$ for all n and replace it with the following property.

Definition 1.16. *We say that* $\{X_{n,k}\}_{k=1}^n$ *has* ***bounded variation*** (BV) *iff*

$$\sup_n \mathrm{Var}(S_n) = \sup_n \sum_{k=1}^n \sigma_{n,k}^2 < \infty. \tag{1.16}$$

Corollary 1.17 (Limits under (BV)**).** *Suppose* $\{X_{n,k}\}_{k=1}^n$ *are independent mean zero random variables for each* n *which satisfy properties* (M) *and* (BV). *If* $S_n := \sum_{k=1}^n X_{n,k} \implies L$ *for some random variable* L, *then*

$$f_L(\lambda) = \exp\left(\int_{\mathbb{R}} \beta(\lambda, x)\, d\nu(x)\right) \tag{1.17}$$

where ν *– is a finite positive measure on* $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. *[Note that* $f_L'(0) = 0$ *so that* $\mathbb{E}L = 0$.*]*

[1] This is in fact correct, since Lemma 1.21 indeed implies

$$\lim_{n\to\infty}\left[\mathbb{E}\left[e^{i\lambda S_n}\right] - \exp\left(\sum_{k=1}^n \left(f_{n,k}(\lambda) - 1\right)\right)\right] = 0. \tag{1.14}$$

Remark 1.18 (L is inifinitely divisible). The limit L in Corollary 1.17 is infinitely divisible. [This is obvious as

$$\hat{\mu}_k(\lambda) := \exp\left(\frac{1}{k}\int_{\mathbb{R}} \frac{e^{i\lambda x}-1-i\lambda x}{x^2} d\nu(x)\right) = \exp\left(\int_{\mathbb{R}} \frac{e^{i\lambda x}-1-i\lambda x}{x^2} d\left(\frac{\nu}{k}\right)(x)\right)$$

defines a probability measure such that $\mu_k^{*k} = \mathrm{Law}(L)$.] Here is another check though; $f_L(\lambda) = e^{\psi(\lambda)}$ where

$$\psi(\lambda) = i\lambda b - \frac{1}{2}a\lambda^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\lambda x} - 1 - i\lambda x \cdot 1_{|x|\le 1}\right) d\tilde{\nu}(x),$$

$$d\tilde{\nu}(x) = \frac{1}{x^2} 1_{|x|>0} d\nu(x), \quad a = \nu(\{0\}) \text{ and } b = -\int_{|x|\ge 1} x d\tilde{\nu}(x).$$

Theorem 1.19 (A Poisson Limit Theorem). *For each $n \in \mathbb{N}$, let $\{Y_{n,k}\}_{k=1}^n$ be independent Bernoulli random variables with $P(Y_{n,k}=1) = p_{n,k}$ and $P(Y_{n,k}=0) = q_{n,k} := 1 - p_{n,k}$. Suppose;*

1. $\lim_{n\to\infty} \sum_{k=1}^n p_{n,k} = a \in (0,\infty)$ *and*
2. $\lim_{n\to\infty} \max_{1\le k\le n} p_{n,k} = 0$. *(So no one term is dominating the sums in item 1.)*

Then $S_n = \sum_{k=1}^n Y_{n,k} \implies Z$ where Z is a Poisson random variable with mean a. (See [?, Section 2.6] for more on this theorem.)

Proof. Let $Z_n \overset{d}{=} Poi\left(\sum_{k=1}^n p_{n,k}\right)$, then by the law of rare events in Theorem **??**, we know that

$$d_{TV}(Z_n, S_n) \le \sum_{k=1}^n p_{n,k}^2 \le \max_{1\le k\le n} p_{n,k} \cdot \sum_{k=1}^n p_{n,k}.$$

From the assumptions it follows that $\lim_{n\to\infty} d_{TV}(Z_n, S_n) = 0$ and from part 3. of Exercise **??** we know that $\lim_{n\to\infty} d_{TV}(Z_n, Z) = 0$. Therefore, $\lim_{n\to\infty} d_{TV}(Z, S_n) = 0$ and this proves $S_n \implies Z$. ■

Exercise 1.2. This problem uses the **same notation and assumptions** as in Theorem 1.19 and in particular $\{Y_{n,k}\}_{k=1}^n$ be independent Bernoulli random variables with $P(Y_{n,k}=1) = p_{n,k}$ and $P(Y_{n,k}=0) = q_{n,k} := 1 - p_{n,k}$. Let $X_{n,k} := Y_{n,k} - p_{n,k}$. Show;

1. Explain why $\bar{S}_n = \sum_{k=1}^n X_{n,k} \implies L := Z - a$ where $a = \lim_{n\to\infty}\sum_{k=1}^n p_{n,k}$ and Z is a is a Poisson random variable with mean a as in Theorem 1.19
2. Show directly that $\{X_{n,k}\}_{k=1}^n$ does not satisfy the Lindeberg condition (LC).
3. Show $\{X_{n,k}\}_{k=1}^n$ satisfy condition (M), i.e. that $\sup_{1\le k\le n} \mathbb{E}X_{n,k}^2 = 0$.
4. Show $\mathrm{Var}\left(\bar{S}_n\right) = \sum_{k=1}^n \sigma_{n,k}^2 = \sum_{k=1}^n p_{n,k}(1-p_{n,k}) \to a$ as $n\to\infty$ which suffices to show condition (BV) holds.
5. Find a finite measure ν on $\mathbb{R}$ such that

$$f_L(\lambda) = \mathbb{E}e^{i\lambda L} = \exp\left(\int_{\mathbb{R}} \frac{e^{i\lambda x}-1-i\lambda x}{x^2} d\nu(x)\right).$$

1.3 Appendix (Estimates):

Lemma 1.20. *Suppose that $a_i, b_i \in \mathbb{C}$ with $|a_i|, |b_i| \le 1$ for $i = 1, 2, \dots, n$. Then*

$$\left|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i\right| \le \sum_{i=1}^n |a_i - b_i|.$$

By Taylor's thoerem, then

$$\left|e^{iy} - 1\right| \le |y| \wedge 2 \text{ for } y \in \mathbb{R} \tag{1.18}$$

and

$$|e^z - 1 - z| \le |z|^2/2 \text{ if } \mathrm{Re}\, z \le 0. \tag{1.19}$$

Lemma 1.21. *Suppose that $\{X_{n,k}\}$ satisfies property (M), i.e. $D_n := \max_{k\le n}\sigma_{n,k}^2 \to 0$. If we define,*

$$\varphi_{n,k}(\lambda) := f_{n,k}(\lambda) - 1 = \mathbb{E}\left[e^{i\lambda X_{n,k}} - 1\right],$$

then for each $\lambda \in \mathbb{R}$;

1. $\lim_{n\to\infty}\max_{k\le n}|\varphi_{n,k}(\lambda)| = 0$ *and*
2. $f_{S_n}(\lambda) - \prod_{k=1}^n e^{\varphi_{n,k}(\lambda)} \to 0$ *as $n\to\infty$, where*

$$f_{S_n}(\lambda) = \mathbb{E}\left[e^{i\lambda S_n}\right] = \prod_{k=1}^n f_{n,k}(\lambda).$$

Proof. For any $\varepsilon > 0$ we have, making use of Eq. (1.18) and Chebyschev's inequality, that

$$\begin{aligned}|\varphi_{n,k}(\lambda)| &= |f_{n,k}(\lambda) - 1| \le \mathbb{E}\left|e^{i\lambda X_{n,k}} - 1\right| \le \mathbb{E}\left[2 \wedge |\lambda X_{n,k}|\right] \\ &\le \mathbb{E}\left[2\wedge|\lambda X_{n,k}| : |X_{n,k}| \ge \varepsilon\right] + \mathbb{E}\left[2\wedge|\lambda X_{n,k}| : |X_{n,k}| < \varepsilon\right] \\ &\le 2P\left[|X_{n,k}| \ge \varepsilon\right] + |\lambda|\varepsilon \le \frac{2\sigma_{n,k}^2}{\varepsilon^2} + |\lambda|\varepsilon.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}\max_{k\le n}|\varphi_{n,k}(\lambda)|\le\limsup_{n\to\infty}\left[\frac{2D_n}{\varepsilon^2}+|\lambda|\,\varepsilon\right]=|\lambda|\,\varepsilon\to 0 \text{ as } \varepsilon\downarrow 0.$$

For the second item, observe that

$$\operatorname{Re}\varphi_{n,k}(\lambda)=\operatorname{Re}f_{n,k}(\lambda)-1\le 0\implies\left|e^{\varphi_{n,k}(\lambda)}\right|=e^{\operatorname{Re}\varphi_{n,k}(\lambda)}\le 1.$$

Therefore, by Lemma 1.20 and the estimate (1.19) we find;

$$\begin{aligned}\left|\prod_{k=1}^n e^{\varphi_{n,k}(\lambda)}-\prod_{k=1}^n f_{n,k}(\lambda)\right|&\le\sum_{k=1}^n\left|e^{\varphi_{n,k}(\lambda)}-f_{n,k}(\lambda)\right|\\&=\sum_{k=1}^n\left|e^{\varphi_{n,k}(\lambda)}-(1+\varphi_{n,k}(\lambda))\right|\\&\le\frac{1}{2}\sum_{k=1}^n|\varphi_{n,k}(\lambda)|^2\\&\le\frac{1}{2}\max_{k\le n}|\varphi_{n,k}(\lambda)|\cdot\sum_{k=1}^n|\varphi_{n,k}(\lambda)|.\end{aligned}$$

Since $\mathbb{E}X_{n,k}=0$ we may write express $\varphi_{n,k}$ as

$$\varphi_{n,k}(\lambda)=\mathbb{E}\left[e^{i\lambda X_{n,k}}-1-i\lambda X_{n,k}\right]$$

and then using estimate in Eq. (1.19) again shows

$$\begin{aligned}\sum_{k=1}^n|\varphi_{n,k}(\lambda)|&=\sum_{k=1}^n\left|\mathbb{E}\left[e^{i\lambda X_{n,k}}-1-i\lambda X_{n,k}\right]\right|\\&\le\sum_{k=1}^n\left|\mathbb{E}\left[\frac{1}{2}|\lambda X_{n,k}|^2\right]\right|\le\frac{\lambda^2}{2}\sum_{k=1}^n\sigma_{n,k}^2=\frac{\lambda^2}{2}.\end{aligned}$$

Thus we have shown,

$$\left|\prod_{k=1}^n f_{n,k}(\lambda)-\prod_{k=1}^n e^{\varphi_{n,k}(\lambda)}\right|\le\frac{\lambda^2}{4}\max_{k\le n}|\varphi_{n,k}(\lambda)|$$

and the latter expression tends to zero by item 1. ∎

1.4 Stable Distributions

Definition 1.22. *A non-degenerate distribution $\mu=\operatorname{Law}(X)$ on $\mathbb{R}$ is **stable** if whenever X_1 and X_2 are independent copies of X, then for all $a,b\in\mathbb{R}$ there exists $c,d\in\mathbb{R}$ such that $aX_1+bX_2\overset{d}{=}cX+d$ with some constants c and d.*

Example 1.23. Gaussians are stable but Poisson random variables are not.

Lemma 1.24. *If μ is a stable distribution then it is infinitely divisible.*

Theorem 1.25. *A probability measure μ on $\mathbb{R}$ is a stable distribution iff μ is Gaussian or $\hat{\mu}(\lambda)=e^{\psi(\lambda)}$ where*

$$\psi(\lambda)=i\lambda b+\int_{\mathbb{R}}\left(e^{i\lambda x}-1-\frac{i\lambda x}{1+x^2}\right)\frac{m_1 1_{x>0}-m_2 1_{x<0}}{|x|^{1+\alpha}}dx$$

for some constants, $0<\alpha<2$, $m_i\ge 0$ and $b\in\mathbb{R}$.