Math 285 Homework Problem List for S2016

Note: solutions to Lawler Problems will appear after all of the Lecture Note Solutions.

1.1 Homework 1. Due Friday, April 8, 2016

- **Look at** from lecture note exercises: 1.1
- **Hand in** lecture note exercises: 1.2, 1.3, 1.4, 1.5, 1.6
- **Hand in** from **Lawler** §5.1 on page 125.

1.2 Homework 2. Due Friday, April 15, 2016

- **Look at** from lecture note exercises: 1.8, 1.9, 1.11
- **Hand in** lecture note exercises: 1.7, 1.10, 1.12
- **Hand in** from **Lawler** §1.1, 1.4, 1.19

Exercise 1.1 (Optional). Let $W_0 := X(\Omega) \subset W$. Finish the proof of Proposition **??** using the following outline;

1. Use the fact that $\sigma(Y) \subset \sigma(X)$ to show for each $s \in S$ there exists $B_s \subset W_0 \subset W$ such that $\{Y = s\} = \{X \in B_s\}$.
2. Show $B_s \cap B_{s'} = \emptyset$ for all $s, s' \in S$ with $s \neq s'$.
3. Show $X(\Omega) = W_0 := \cup_{s \in S} B_s$.
   Now fix a point $s_* \in S$ and then define, $f : W \to S$ by setting $f(w) = s_*$ when $w \in W \setminus W_0$ and $f(w) = s$ when $w \in B_s \subset W_0$.
4. Verify that $Y = f(X)$.

Exercise 1.2. Suppose that X and Y are two integrable random variables such that

$$\mathbb{E}[X|Y] = 18 - \frac{3}{5}Y \text{ and } \mathbb{E}[Y|X] = 10 - \frac{1}{3}X.$$

Find $\mathbb{E}X$ and $\mathbb{E}Y$.

Exercise 1.3. Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables with $\mathbb{E}|X_i| < \infty$ for all i and let $S_m := X_1 + \cdots + X_m$ for $m = 1, 2, \ldots$. Show

$$\mathbb{E}[S_m|S_n] = \frac{m}{n}S_n \text{ for all } m \leq n.$$

Hint: observe by symmetry[1] that there is a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}(X_i|S_n) = h(S_n) \text{ independent of } i.$$

Exercise 1.4. Let $\tau : \Omega \to \bar{\mathbb{N}}$ be a function. Verify the following are equivalent;

1. τ is a stopping time.
2. $\{\tau \leq n\} \in \mathcal{F}_n^{\mathbb{X}}$ for all $n \in \mathbb{N}_0$.
3. $\{\tau > n\} \in \mathcal{F}_n^{\mathbb{X}}$ for all $n \in \mathbb{N}_0$.

Also show that if τ is a stopping time then $\{\tau = \infty\} \in \mathcal{F}_\infty^{\mathbb{X}}$.

Exercise 1.5. If τ and σ are two stopping times shows, $\sigma \wedge \tau = \min\{\sigma, \tau\}$, $\sigma \vee \tau = \max\{\sigma, \tau\}$, and $\sigma + \tau$ are all stopping times.

Exercise 1.6 (Hitting time after a stopping time). Let σ be any stopping time. Show

$$\tau_1 = \inf\{n \geq \sigma : X_n \in B\} \text{ and}$$
$$\tau_2 = \inf\{n > \sigma : X_n \in B\}.$$

are both stopping times.

Exercise 1.7 (Invariant distributions and expected return times). Suppose that $\{X_n\}_{n=0}^{\infty}$ is a Markov chain on a finite state space S determined by one step transition probabilities, $p(x, y) = \mathbb{P}(X_{n+1} = y | X_n = x)$. For $x \in S$, let $R_x := \inf\{n > 0 : X_n = x\}$ be the first **passage time**[2] to x. [We will assume here that $\mathbb{E}_y R_x < \infty$ for all $x, y \in S$.] Use the first step analysis to show,

$$\mathbb{E}_x R_y = \sum_{z : z \neq y} p(x, z)\, \mathbb{E}_z R_y + 1. \tag{1.1}$$

Now further assume that $\pi : S \to [0, 1]$ is an invariant distributions for p, that is $\sum_{x \in S} \pi(x) = 1$ and $\sum_{x \in S} \pi(x)\, p(x, y) = \pi(y)$ for all $y \in S$, i.e. $\pi \mathbf{P} = \pi$. By multiplying Eq. (1.1) by $\pi(x)$ and summing on $x \in S$, show,

[1] Apply Theorem **??** using $(X_1, S_n) \overset{d}{=} (X_i, S_n)$ for $1 \leq i \leq n$.
[2] R_x is the first **return time** to x when the chain starts at x.

$$\pi(y)\,\mathbb{E}_y R_y = 1 \text{ for all } y \in S \implies \pi(y) = \frac{1}{\mathbb{E}_y R_y} > 0. \tag{1.2}$$

Exercise 1.8 (Uniqueness of solutions to 2nd order recurrence relations). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and $g : \mathbb{Z} \cap (\alpha, \beta) \to \mathbb{R}$ be a given function. Show that there is exactly one function $u : [\alpha, \beta] \cap \mathbb{Z} \to \mathbb{R}$ with prescribed values on two two consecutive points in $[\alpha, \beta] \cap \mathbb{Z}$ which satisfies the second order recurrence relation:

$$au(x+1) + bu(x) + cu(x-1) = f(x) \text{ for all } x \in \mathbb{Z} \cap (\alpha, \beta). \tag{1.3}$$

are for $\alpha < x < \beta$. Show; if u and w both satisfy Eq. (1.3) and $u = w$ on two consecutive points in $(\alpha, \beta) \cap \mathbb{Z}$, then $u(x) = w(x)$ for all $x \in [\alpha, \beta] \cap \mathbb{Z}$.

Exercise 1.9 (General homogeneous solutions). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and suppose $\{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\}$ solves the second order homogeneous recurrence relation

$$au(x+1) + bu(x) + cu(x-1) = 0 \text{ for all } x \in \mathbb{Z} \cap (\alpha, \beta), \tag{1.4}$$

i.e. Eq. (1.3) with $f(x) \equiv 0$. Show:

1. for any $\lambda \in \mathbb{C}$,
$$a\lambda^{x+1} + b\lambda^x + c\lambda^{x-1} = \lambda^{x-1} p(\lambda) \tag{1.5}$$
where $p(\lambda) = a\lambda^2 + b\lambda + c$ is the **characteristic polynomial** associated to Eq. (1.3).
Let $\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ be the roots of $p(\lambda)$ and suppose for the moment that $b^2 - 4ac \neq 0$. From Eq. (1.3) it follows that for any choice of $A_\pm \in \mathbb{R}$, the function,
$$w(x) := A_+ \lambda_+^x + A_- \lambda_-^x, \tag{1.6}$$
solves Eq. (1.3) for all $x \in \mathbb{Z}$.
2. Show there is a unique choice of constants, $A_\pm \in \mathbb{R}$, such that the function $u(x)$ is given by
$$u(x) := A_+ \lambda_+^x + A_- \lambda_-^x \text{ for all } \alpha \le x \le \beta.$$
3. Now suppose that $b^2 = 4ac$ and $\lambda_0 := -b/(2a)$ is the double root of $p(\lambda)$. Show for any choice of A_0 and A_1 in $\mathbb{R}$ that
$$w(x) := (A_0 + A_1 x)\lambda_0^x \tag{1.7}$$
solves Eq. (1.3) for all $x \in \mathbb{Z}$. **Hint:** Differentiate Eq. (1.5) with respect to λ and then set $\lambda = \lambda_0$.
4. Again show that any function u solving Eq. (1.3) is of the form $u(x) = (A_0 + A_1 x)\lambda_0^x$ for $\alpha \le x \le \beta$ for some unique choice of constants $A_0, A_1 \in \mathbb{R}$.

Exercise 1.10. Let $w(x) := \mathbb{P}_x\left(X_{T_{a,b}} = b\right) := \mathbb{P}\left(X_{T_{a,b}} = b | X_0 = x\right)$.

1. Use first step analysis to show for $a < x < b$ that
$$w(x) = \frac{1}{2}\left(w(x+1) + w(x-1)\right) \tag{1.8}$$
provided we define $w(a) = 0$ and $w(b) = 1$.
2. Use the results of Exercises 1.8 and 1.9 to show
$$\mathbb{P}_x\left(X_{T_{a,b}} = b\right) = w(x) = \frac{1}{b-a}(x-a). \tag{1.9}$$
3. Let
$$T_b := \begin{cases} \min\{n : X_n = b\} & \text{if } \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}$$
be the first time $\{X_n\}$ hits b. Explain why, $\left\{X_{T_{a,b}} = b\right\} \subset \{T_b < \infty\}$ and use this along with Eq. (1.9) to conclude that $\mathbb{P}_x(T_b < \infty) = 1$ for all $x < b$. (By symmetry this result holds true for all $x \in \mathbb{Z}$.)

Exercise 1.11. The goal of this exercise is to give a second proof of the fact that $\mathbb{P}_x(T_b < \infty) = 1$. Here is the outline:

1. Let $w(x) := \mathbb{P}_x(T_b < \infty)$. Again use first step analysis to show that $w(x)$ satisfies Eq. (1.8) for all x with $w(b) = 1$.
2. Use Exercises 1.8 and 1.9 to show that there is a constant, c, such that
$$w(x) = c \cdot (x - b) + 1 \text{ for all } x \in \mathbb{Z}.$$
3. Explain why c must be zero to again show that $\mathbb{P}_x(T_b < \infty) = 1$ for all $x \in \mathbb{Z}$.

Exercise 1.12. Let $T = T_{a,b}$ and $u(x) := \mathbb{E}_x T := \mathbb{E}[T | X_0 = x]$.

1. Use first step analysis to show for $a < x < b$ that
$$u(x) = \frac{1}{2}\left(u(x+1) + u(x-1)\right) + 1 \tag{1.10}$$
with the convention that $u(a) = 0 = u(b)$.
2. Show that
$$u(x) = A_0 + A_1 x - x^2 \tag{1.11}$$
solves Eq. (1.10) for any choice of constants A_0 and A_1.
3. Choose A_0 and A_1 so that $u(x)$ satisfies the boundary conditions, $u(a) = 0 = u(b)$. Use this to conclude that
$$\mathbb{E}_x T_{a,b} = -ab + (b+a)x - x^2 = -a(b-x) + bx - x^2. \tag{1.12}$$