Math 285 Homework Problem List for S2016

Note: solutions to Lawler Problems will appear after all of the Lecture Note Solutions.

1.1 Homework 1. Due Friday, April 8, 2016

- Look at from lecture note exercises: 1.1
- Hand in lecture note exercises: 1.2, 1.3, 1.4, 1.5, 1.6
- Hand in from Lawler §5.1 on page 125.

1.2 Homework 2. Due Friday, April 15, 2016

- Look at from lecture note exercises: 1.8, 1.9, 1.11
- Hand in lecture note exercises: 1.7, 1.10, 1.12
- Hand in from Lawler §1.1, 1.4, 1.19

Exercise 1.1 (Optional). Let $W_0 := X$ ($\Omega$) \subset W. Finish the proof of Proposition ?? using the following outline:

1. Use the fact that $\sigma (Y) \subset \sigma (X)$ to show for each $s \in S$ there exists $B_s \subset W_0 \subset W$ such that \{ $Y = s$ \} = \{ $X \in B_s$ \}.
2. Show $B_s \cap B_{s'} = \emptyset$ for all $s, s' \in S$ with $s \neq s'$.
3. Show $X (\Omega) = W_0 := \cup_{s \in S} B_s$.
   Now fix a point $s_* \in S$ and then define, $f : W \rightarrow S$ by setting $f (w) = s_*$ when $w \in W \setminus W_0$ and $f (w) = s$ when $w \in B_s \subset W_0$.
4. Verify that $Y = f (X)$.

Exercise 1.2. Suppose that $X$ and $Y$ are two integrable random variables such that
\[ \mathbb{E} [X | Y] = 18 - \frac{3}{5} Y \text{ and } \mathbb{E} [Y | X] = 10 - \frac{1}{3} X. \]
Find $\mathbb{E} X$ and $\mathbb{E} Y$.

Exercise 1.3. Let $\{ X_i \}_{i=1}^\infty$ be i.i.d. random variables with $\mathbb{E} |X_i| < \infty$ for all $i$ and let $S_m := X_1 + \cdots + X_m$ for $m = 1, 2, \ldots$. Show
\[ \mathbb{E} [S_m | S_n] = \frac{m}{n} S_n \text{ for all } m \leq n. \]

Hint: observe by symmetry$^1$ that there is a function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that
\[ \mathbb{E} (X_i | S_n) = h (S_n) \text{ independent of } i. \]

Exercise 1.4. Let $\tau : \Omega \rightarrow \mathbb{N}$ be a function. Verify the following are equivalent:

1. $\tau$ is a stopping time.
2. \{ $\tau \leq n$ \} $\in \mathcal{F}_n^X$ for all $n \in \mathbb{N}_0$.
3. \{ $\tau > n$ \} $\in \mathcal{F}_n^X$ for all $n \in \mathbb{N}_0$.

Also show that if $\tau$ is a stopping time then \{ $\tau = \infty$ \} $\in \mathcal{F}_\infty^X$.

Exercise 1.5. If $\tau$ and $\sigma$ are two stopping times shows, $\sigma \wedge \tau = \min \{ \sigma, \tau \}$, $\sigma \vee \tau = \max \{ \sigma, \tau \}$, and $\sigma + \tau$ are all stopping times.

Exercise 1.6 (Hitting time after a stopping time). Let $\sigma$ be any stopping time. Show
\[ \tau_1 = \inf \{ n \geq \sigma : X_n \in B \} \quad \text{and} \quad \tau_2 = \inf \{ n > \sigma : X_n \in B \}. \]
are both stopping times.

Exercise 1.7 (Invariant distributions and expected return times). Suppose that $\{ X_n \}_{n=0}^\infty$ is a Markov chain on a finite state space $S$ determined by one step transition probabilities, $p (x, y) = \mathbb{P} (X_{n+1} = y | X_n = x)$. For $x \in S$, let $R_x := \inf \{ n > 0 : X_n = x \}$ be the first passage time$^2$ to $x$. We will assume here that $\mathbb{E}_x R_x < \infty$ for all $x, y \in S$. Use the first step analysis to show,
\[ \mathbb{E}_x R_y = \sum_{z : z \neq y} p (x, z) \mathbb{E}_z R_y + 1. \hspace{1cm} (1.1) \]

Now further assume that $\pi : S \rightarrow [0, 1]$ is an invariant distributions for $p$, that is $\sum_{x \in S} \pi (x) = 1$ and $\sum_{x \in S} \pi (x) p (x, y) = \pi (y)$ for all $y \in S$, i.e. $\pi \mathbb{P} = \pi$. By multiplying Eq. (1.1) by $\pi (x)$ and summing on $x \in S$, show,

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$^1$ Apply Theorem ?? using $(X_1, S_n) \overset{d}{=} (X_i, S_n)$ for $1 \leq i \leq n$.

$^2$ $R_x$ is the first return time to $x$ when the chain starts at $x$. 

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Exercise 1.8 (Uniqueness of solutions to 2nd order recurrence relations). Let \( a, b, c \) be real numbers with \( a \neq 0 \neq \pm 1 \), \( \alpha, \beta \in \mathbb{Z} \cup \{-\infty, +\infty\} \) with \( \alpha < \beta \), and \( g: \mathbb{Z} \setminus (\alpha, \beta) \rightarrow \mathbb{R} \) be a given function. Show that there is exactly one function \( u: [\alpha, \beta] \cap \mathbb{Z} \rightarrow \mathbb{R} \) with prescribed values on two consecutive points in \( [\alpha, \beta] \cap \mathbb{Z} \) which satisfies the second order recurrence relation:

\[
au(x + 1) + bu(x) + cu(x - 1) = f(x) \quad \text{for all } x \in \mathbb{Z} \cap (\alpha, \beta).
\]

(1.3)

are for \( \alpha < x < \beta \). Show: if \( u \) and \( w \) both satisfy Eq. (1.3) and \( u = w \) on two consecutive points in \((\alpha, \beta) \cap \mathbb{Z}\), then \( u(x) = w(x) \) for all \( x \in [\alpha, \beta] \cap \mathbb{Z} \).

Exercise 1.9 (General homogeneous solutions). Let \( a, b, c \) be real numbers with \( a \neq 0 \neq \pm 1 \), \( \alpha, \beta \in \mathbb{Z} \cup \{-\infty, +\infty\} \) with \( \alpha < \beta \), and suppose \( \{u(x): x \in [\alpha, \beta] \cap \mathbb{Z}\} \) solves the second order homogeneous recurrence relation

\[
aux(x + 1) + bu(x) + cu(x - 1) = 0 \quad \text{for all } x \in \mathbb{Z} \cap (\alpha, \beta),
\]

(1.4)

i.e. Eq. (1.3) with \( f(x) \equiv 0 \). Show:

1. for any \( \lambda \in \mathbb{C} \),

\[
a\lambda^{x+1} + b\lambda^x + c\lambda^{x-1} = \lambda^x - 1 p(\lambda)
\]

where \( p(\lambda) = a\lambda^2 + b\lambda + c \) is the characteristic polynomial associated to Eq. (1.3).

Let \( \lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) be the roots of \( p(\lambda) \) and suppose for the moment that \( b^2 - 4ac \neq 0 \). From Eq. (1.3) it follows that for any choice of \( A_{\pm} \in \mathbb{R} \), the function,

\[
w(x) : = A_+\lambda_+^x + A_-\lambda_-^x,
\]

(1.6)

solves Eq. (1.3) for all \( x \in \mathbb{Z} \).

2. Show there is a unique choice of constants, \( A_{\pm} \in \mathbb{R} \), such that the function \( u(x) \) is given by

\[
u(x) := A_+\lambda_+^x + A_-\lambda_-^x \quad \text{for all } \alpha \leq x \leq \beta.
\]

3. Now suppose that \( b^2 = 4ac \) and \( \lambda_0 := -b/(2a) \) is the double root of \( p(\lambda) \). Show for any choice of \( A_0 \) and \( A_1 \) in \( \mathbb{R} \) that

\[
w(x) := (A_0 + A_1x)\lambda_0^x
\]

(1.7)

solves Eq. (1.3) for all \( x \in \mathbb{Z} \). \textbf{Hint:} Differentiate Eq. (1.5) with respect to \( \lambda \) and then set \( \lambda = \lambda_0 \).

4. Again show that any function \( u \) solving Eq. (1.3) is of the form \( u(x) = (A_0 + A_1x)\lambda_0^x \) for \( \alpha \leq x \leq \beta \) for some unique choice of constants \( A_0, A_1 \in \mathbb{R} \).

Exercise 1.10. Let \( w(x) := \mathbb{P}_x(X_{T_{a,b}} = b) := \mathbb{P}(X_{T_{a,b}} = b|X_0 = x) \).

1. Use first step analysis to show for \( a < x < b \) that

\[
w(x) = \frac{1}{2} \left( w(x + 1) + w(x - 1) \right)
\]

provided we define \( w(a) = 0 \) and \( w(b) = 1 \).

2. Use the results of Exercises 1.8 and 1.9 to show

\[
\mathbb{P}_x(X_{T_{a,b}} = b) = w(x) = \frac{1}{b - a} (x - a).
\]

3. Let

\[
T_b := \begin{cases} \min \{ n : X_n = b \} & \text{if } \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}
\]

be the first time \( \{X_n\} \) hits \( b \). Explain why, \( \{X_{T_{a,b}} = b\} \subset \{T_b < \infty\} \) and use this along with Eq. (1.9) to conclude that \( \mathbb{P}_x(T_b < \infty) = 1 \) for all \( x < b \).

(By symmetry this result holds true for all \( x \in \mathbb{Z} \).)

Exercise 1.11. The goal of this exercise is to give a second proof of the fact that \( \mathbb{P}_x(T_b < \infty) = 1 \). Here is the outline:

1. Let \( w(x) := \mathbb{P}_x(T_b < \infty) \). Again use first step analysis to show that \( w(x) \) satisfies Eq. (1.8) for all \( x \) with \( w(b) = 1 \).

2. Use Exercises 1.8 and 1.9 to show that there is a constant, \( c \), such that

\[
w(x) = c \cdot (x - b) + 1 \quad \text{for all } x \in \mathbb{Z}.
\]

3. Explain why \( c \) must be zero to again show that \( \mathbb{P}_x(T_b < \infty) = 1 \) for all \( x \in \mathbb{Z} \).

Exercise 1.12. Let \( T = T_{a,b} \) and \( u(x) := E_xT := E[T|X_0 = x] \).

1. Use first step analysis to show for \( a < x < b \) that

\[
u(x) = \frac{1}{2} (u(x + 1) + u(x - 1)) + 1
\]

with the convention that \( u(a) = 0 = u(b) \).

2. Show that

\[
u(x) = A_0 + A_1x - x^2
\]

(1.11)

solves Eq. (1.10) for any choice of constants \( A_0 \) and \( A_1 \).

3. Choose \( A_0 \) and \( A_1 \) so that \( u(x) \) satisfies the boundary conditions, \( u(a) = 0 = u(b) \). Use this to conclude that

\[
E_xT_{a,b} = -ab + (b + a)x - x^2 = -a(b - x) + bx - x^2.
\]

(1.12)