Math 285 Homework Problem List for S2016

Note: solutions to Lawler Problems will appear after all of the Lecture Note Solutions.

1.1 Homework 1. Due Friday, April 8, 2016
- Look at from lecture note exercises: 1.1
- Hand in lecture note exercises: 1.2, 1.3, 1.4, 1.5, 1.6
- Hand in from Lawler §5.1 on page 125.

1.2 Homework 2. Due Friday, April 15, 2016
- Look at from lecture note exercises: 1.8, 1.9, 1.11
- Hand in lecture note exercises: 1.7, 1.10, 1.12
- Hand in from Lawler §1.1, 1.4, 1.19

1.3 Homework 3. Due Friday, April 22, 2016
- Look at from Lawler §1.10,
- Hand in from Lawler §1.5, 1.8, 1.9, 1.14, 1.18*, 2.3
- *Hint: show the invariant distribution is uniform.

1.4 Homework 4. Due Friday, April 29, 2016
- Look at lecture note exercises: 1.13
- Hand in from Lawler problems: §7.6, 7.7, 7.8.
- Please use the result in 7.8 to verify your numerical approximations found in 7.7.
- *Hints for 7.8. Recall that for $n \in \mathbb{N}$ that

$$S_n = \{ k = (k_0, \ldots, k_n) \in \{0, 1\}^{n+1} : k_{i-1} + k_i \leq 1 \text{ for } 1 \leq i \leq n \}$$

and your goal is to compute

$$p_n(m) := \frac{\# \{ k \in S_n : k_m = 1 \}}{\#(S_n)} \text{ for } 0 \leq m \leq n.$$ 

As Lawler suggests, for $i, j \in \{0, 1\}$ and $m \in \mathbb{N}$, let

$$r_m(ij) = \# \{ k = (k_0, \ldots, k_m) \in S_m : k_0 = i \text{ and } k_m = j \}.$$ 

Notice that if $k = (k_0, \ldots, k_n) \in S_n$ with $k_m = 1$, then

$$(0, k_0, \ldots, k_{m-1}, 1) \in S_{m+1} \text{ and } (1, k_{m+1}, \ldots, k_n, 0) \in S_{n-m+1}$$

(and visa versa) where $k_{m-1}$ and $k_{m+1}$ must both be zero so that

$$(0, k_0, \ldots, k_{m-2}, 0) \in S_m \text{ and } (0, k_{m+2}, \ldots, k_n, 0) \in S_{n-m}.$$ 

From these considerations we find

$$\# \{ k \in S_n : k_m = 1 \} = r_{m+1}(01) \cdot r_{n-m+1}(10) = r_m(00) \cdot r_{n-m}(00).$$

Similarly $k = (k_0, \ldots, k_n) \in S_n$ then $(0, k_0, \ldots, k_n, 0) \in S_{n+2}$ and visa versa from which we learn $\#(S_n) = r_{n+2}(00)$. Combining these results shows

$$p_n(m) = \frac{r_m(00) \cdot r_{n-m}(00)}{r_{n+2}(00)}.$$ 

A little thought shows this formula is correct at $m = 0$ and $m = n - m$ provided we use the convention that $r_0(00) = 1$. Now follow the outline in the Lawler in order to find, $y_m = y(m) := r_m(00)$.

1.5 Homework 5. Due Friday, May 6, 2016
- Look at from lecture note exercises: 1.14, 1.15
- Hand in lecture note exercises: 1.16, 1.17
- Hand in from Lawler §5.4, 5.7a, 5.8a, 5.12

1 I have changed the indexing a bit since Lawler’s choices are a bit confusing.
**Exercise 1.3.** Find \( E \) observe by symmetry

**Exercise 1.4.** Apply Theorem 3. Show \( \sigma \)

**Exercise 1.2.** Use the fact that \( B \) for all \( n \in \mathbb{N} \) such that \( \{ Y = s \} = \{ X = B_x \} \).

**Exercise 1.3.** Let \( X_0 := X (\Omega) < W \). Finish the proof of Proposition ?? using the following outline:

1. Use the fact that \( \sigma (Y) < \sigma (X) \) to show for each \( s \in S \) there exists \( B_s \subset W_0 \subset W \) such that \( \{ Y = s \} = \{ X = B_x \} \).
2. Show \( B_s \cap B_{s'} = 0 \) for all \( s, s' \in S \) with \( s \neq s' \).
3. Show \( X (\Omega) = W_0 := \cup_{s \in S} B_s \).

Now fix a point \( s_* \in S \) and then define, \( f : W \to S \) by setting \( f (w) = s_* \) when \( w \in W \) \( \setminus W_0 \) and \( f (w) = s \) when \( w \in B_s \subset W_0 \).

4. Verify that \( Y = f (X) \).

**Exercise 1.2.** Suppose that \( X \) and \( Y \) are two integrable random variables such that

\[
E [X | Y] = 18 - \frac{3}{5} Y \text{ and } E [Y | X] = 10 - \frac{1}{3} X.
\]

Find \( E X \) and \( E Y \).

**Exercise 1.3.** Let \( \{ X_i \}_{i=1}^{\infty} \) be i.i.d. random variables with \( E |X_i| < \infty \) for all \( i \) and let \( S_m := X_1 + \cdots + X_m \) for \( m = 1, 2, \ldots \). Show

\[
E [S_m | S_n] = \frac{m}{n} S_n \text{ for all } m \leq n.
\]

**Hint:** observe by symmetry\(^2\) that there is a function \( h : \mathbb{R} \to \mathbb{R} \) such that

\[
E (X_i | S_n) = h (S_n) \text{ independent of } i.
\]

**Exercise 1.4.** Let \( \tau : \Omega \to \mathbb{N} \) be a function. Verify the following are equivalent:

1. \( \tau \) is a stopping time.
2. \( \{ \tau \leq n \} \in \mathcal{F}_n^\infty \) for all \( n \in \mathbb{N}_0 \).
3. \( \{ \tau > n \} \in \mathcal{F}_n^\infty \) for all \( n \in \mathbb{N}_0 \).

Also show that if \( \tau \) is a stopping time then \( \{ \tau = \infty \} \in \mathcal{F}_0^\infty \).

\(^2\) Apply Theorem ?? using \((X_1, S_n) \overset{d}{=} (X_i, S_n)\) for \( 1 \leq i \leq n \).

**Exercise 1.5.** If \( \tau \) and \( \sigma \) are two stopping times shows, \( \sigma \wedge \tau = \min \{ \sigma, \tau \} \), \( \sigma \vee \tau = \max \{ \sigma, \tau \} \), and \( \sigma + \tau \) are all stopping times.

**Exercise 1.6 (Hitting time after a stopping time).** Let \( \sigma \) be any stopping time. Show

\[
\tau_1 = \inf \{ n \geq \sigma : X_n \in B \} \quad \text{and} \quad \tau_2 = \inf \{ n > \sigma : X_n \in B \}.
\]

are both stopping times.

**Exercise 1.7 (Invariant distributions and expected return times).** Suppose that \( \{ X_n \}_{n=0}^{\infty} \) is a Markov chain on a finite state space \( S \) determined by one step transition probabilities, \( p (x, y) = P (X_{n+1} = y | X_n = x) \). For \( x \in S \), let \( R_x := \inf \{ n > 0 : X_n = x \} \) be the first passage time\(^3\) to \( x \). We will assume here that \( E_y R_x < \infty \) for all \( x, y \in S \). Use the first step analysis to show,

\[
E_x R_y = \sum_{z \neq y} p (x, z) E_z R_y + 1. \tag{1.1}
\]

Now further assume that \( \pi : S \to [0, 1] \) is an invariant distributions for \( p \), that is \( \sum_{x \in S} \pi (x) = 1 \) and \( \sum_{x \in S} \pi (x) p (x, y) = \pi (y) \) for all \( y \in S \), i.e. \( \pi P = \pi \). By multiplying Eq. (1.1) by \( \pi (x) \) and summing on \( x \in S \), show,

\[
\pi (y) E_y R_y = 1 \text{ for all } y \in S \quad \Rightarrow \quad \pi (y) = \frac{1}{E_y R_y} > 0. \tag{1.2}
\]

**Exercise 1.8 (Uniqueness of solutions to 2nd order recurrence relations).** Let \( a, b, c \) be real numbers with \( a \neq 0 \neq c \), \( \alpha, \beta \in \mathbb{Z} \cup \{ \pm \infty \} \) with \( \alpha < \beta \), and \( g : \mathbb{Z} \cap (\alpha, \beta) \to \mathbb{R} \) be a given function. Show that there is exactly one function \( u : [\alpha, \beta] \cap \mathbb{Z} \to \mathbb{R} \) with prescribed values on two consecutive points in \((\alpha, \beta) \cap \mathbb{Z}\) which satisfies the second order recurrence relation:

\[
u u (x + 1) + bu (x) + cu (x - 1) = f (x) \text{ for all } x \in \mathbb{Z} \cap (\alpha, \beta). \tag{1.3}
\]

are for \( \alpha < x < \beta \). Show; if \( u \) and \( w \) both satisfy Eq. (1.3) and \( u = w \) on two consecutive points in \((\alpha, \beta) \cap \mathbb{Z}\), then \( u (x) = w (x) \) for all \( x \in [\alpha, \beta] \cap \mathbb{Z} \).

**Exercise 1.9 (General homogeneous solutions).** Let \( a, b, c \) be real numbers with \( a \neq 0 \neq c \), \( \alpha, \beta \in \mathbb{Z} \cup \{ \pm \infty \} \) with \( \alpha < \beta \), and suppose \( \{ u (x) : x \in [\alpha, \beta] \cap \mathbb{Z} \} \) solves the second order homogeneous recurrence relation

\[
u u (x + 1) + bu (x) + cu (x - 1) = 0 \text{ for all } x \in \mathbb{Z} \cap (\alpha, \beta), \tag{1.4}
\]

i.e. Eq. (1.3) with \( f (x) \equiv 0 \). Show:

\(^3\) \( R_x \) is the first return time to \( x \) when the chain starts at \( x \).
1. Use the results of Exercises 1.8 and 1.9 to show there is a unique choice of constants, $c$, such that

$$w(x) = c \cdot (x - b) + 1$$

for all $x \in \mathbb{Z}$.

3. Explain why $c$ must be zero to again show that $\mathbb{P}_x(H_b < \infty) = 1$ for all $x \in \mathbb{Z}$.

**Exercise 1.12.** Let $H = H_{a,b}$ and $u(x) := \mathbb{E}_x H := \mathbb{E}[H|X_0 = x]$.

1. Use first step analysis to show for $a < x < b$ that

$$u(x) = \frac{1}{2} \left( u(x + 1) + u(x - 1) \right) + 1 \quad (1.10)$$

with the convention that $u(a) = 0 = u(b)$.

2. Show that

$$u(x) = A_0 + A_1 x - x^2 \quad (1.11)$$

solves Eq. (1.10) for any choice of constants $A_0$ and $A_1$.

3. Choose $A_0$ and $A_1$ so that $u(x)$ satisfies the boundary conditions, $u(a) = 0 = u(b)$. Use this to conclude that

$$\mathbb{E}_x H_{a,b} = -ab + (b + a)x - x^2 = -a(b - x) + bx - x^2. \quad (1.12)$$

**Exercise 1.13.** Let $\{X_n\}_{n=0}^\infty$ be the fair random walk on $\mathbb{Z}$ (as in Exercise 1.10) starting at $0$ and let

$$A_N := \mathbb{E} \left[ \sum_{k=0}^{2N} 1_{X_k = 0} \right]$$

denote the expected number of visits to $0$. Using Sterling’s formula and integral approximations for $\sum_{n=1}^{N} \frac{1}{\sqrt{n}}$ to argue that $A_N \sim cn/\sqrt{N}$ for some constant $c > 0$.

**Exercise 1.14.** Construct an example of a martingale, $\{M_n\}_{n=0}^\infty$ such that $\mathbb{E}[M_n] \to \infty$ as $n \to \infty$. [In particular, $\{M_n\}_{n=0}^\infty$ will be a martingale which is not of the form $M_n = \mathbb{E}_x X$ for some $X \in L^1(P)$.] Hint: try taking $M_n = \sum_{k=0}^{n} Z_k$ for a judicious choice of $\{Z_k\}_{k=0}^\infty$ which you should take to be independent, mean zero, and having $\mathbb{E}[|Z_n|]$ growing rather rapidly.

**Exercise 1.15.** Show that $M_n := 2^n 1_{(0,2^{-n}]}$ for $n \in \mathbb{N}$ as defined in Example ?? is a martingale.
Exercise 1.16. Suppose that \( \{Z_n\}_{n=0}^\infty \) are independent random variables such that \( \sigma^2 := \mathbb{E}Z_n^2 < \infty \) and \( \mathbb{E}Z_n = 0 \) for all \( n \geq 1 \). As in Example 1.7, let \( S_n := \sum_{k=0}^{\infty} Z_k \) be martingale relative to the filtration, \( \mathcal{F}_n := \sigma(Z_0, \ldots, Z_n) \). Show
\[
M_n := S_n^2 - \sigma^2 n \text{ for } n \in \mathbb{N}_0
\]
is a martingale. [Hint, use the basic properties of conditional expectations.]

Exercise 1.17. Suppose that \( \{X_n\}_{n=0}^\infty \) is a Markov chain on \( S \) with one-step transition probabilities, \( P \), and \( \mathcal{F}_n := \sigma(X_0, \ldots, X_n) \). Let \( f : S \to \mathbb{R} \) be a bounded (for simplicity) function and set \( g(x) = (Pf)(x) - f(x) \). Show
\[
M_n := f(X_n) - \sum_{0 \leq j < n} g(X_j) \text{ for } n \in \mathbb{N}_0
\]
is a martingale.

Exercise 1.18 (Quadratic Variation). Suppose \( \{M_n\}_{n=0}^\infty \) is a square integrable martingale. Show:

1. \( \mathbb{E}\left[M_{n+1}^2 - M_n^2 | \mathcal{F}_n\right] = \mathbb{E}\left[ (M_{n+1} - M_n)^2 \big| \mathcal{F}_n \right] \). Conclude from this that the Doob decomposition of \( M_n^2 \) is of the form,
\[
M_n^2 = N_n + A_n
\]
where
\[
A_n := \sum_{1 \leq k \leq n} \mathbb{E}\left[ (M_k - M_{k-1})^2 \big| \mathcal{F}_{k-1} \right].
\]
2. If we further assume that \( M_k - M_{k-1} \) is independent of \( \mathcal{F}_{k-1} \) for all \( k = 1, 2, \ldots \), explain why,
\[
A_n = \sum_{1 \leq k \leq n} \mathbb{E}(M_k - M_{k-1})^2.
\]

Exercise 1.19. Suppose Harriet has 7 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?

Exercise 1.20. Consider a contract that at time \( N \) will be worth either 100 or 0. Let \( S_n \) be its price at time \( 0 \leq n \leq N \). If \( S_n \) is a martingale, and \( S_0 = 47 \), then what is the probability that the contract will be worth 100 at time \( N \)?

Exercise 1.21. Pedro plans to buy the contract in the previous problem at time 0 and sell it the first time \( T \) at which the price goes above 55 or below 15. What is the expected value of \( S_T \)? You may assume that the value, \( S_n \), of the contract is bounded – there is only a finite amount of money in the world up to time \( N \). Also note, by assumption, \( T \leq N \).

Exercise 1.22. For \( a < 0 < b \) with \( a, b \in \mathbb{R} \), let \( \tau = \sigma_a \wedge \sigma_b \). Explain why \( \{S_n^\tau\}_{n=0}^\infty \) is a bounded martingale use this to show \( P(\tau = \infty) = 0 \). Hint: make use of the fact that \( |S_n - S_{n-1}| = |Z_n| = 1 \) for all \( n \) and hence the only way \( \lim_{n \to \infty} S_n^\tau \) can exist if it stops moving!

Exercise 1.23. Show
\[
P(\sigma_b < \sigma_a) = \frac{|a|}{b + |a|} \quad (1.13)
\]
and use this to conclude \( P(\sigma_b < \infty) = 1 \), i.e. every \( b \in \mathbb{N} \) is almost surely visited by \( S_n \).

Hint: As in Exercise 1.22 notice that \( \{S_n^\tau\}_{n=0}^\infty \) is a bounded martingale where \( \tau := \sigma_a \wedge \sigma_b \). Now compute \( \mathbb{E}[S_T] = \mathbb{E}[S_0^\tau] \) in two different ways.

Exercise 1.24. Let \( \tau := \sigma_a \wedge \sigma_b \). In this problem you are asked to show \( \mathbb{E}[\tau] = |a| |b| \) with the aid of the following outline.

1. Use Exercise 1.18 above to conclude \( N_n := S_n^\tau - n \) is a martingale.
2. Now show
\[
0 = \mathbb{E}N_0 = \mathbb{E}N_{\tau \wedge n} = \mathbb{E}S_n^\tau \wedge n - \mathbb{E}[\tau \wedge n]. \quad (1.14)
\]
3. Now use DCT and MCT along with Exercise 1.23 to compute the limit as \( n \to \infty \) in Eq. (1.14) to find
\[
\mathbb{E}[\sigma_a \wedge \sigma_b] = \mathbb{E}[\tau] = b |a|. \quad (1.15)
\]
4. By considering the limit, \( a \to -\infty \) in Eq. (1.15), show \( \mathbb{E}[\sigma_b] = \infty \).