Math 285 Homework Problem List for S2016

Note: solutions to Lawler Problems will appear after all of the Lecture Note Solutions.

1.1 Homework 1. Due Friday, April 8, 2016

- **Look at** from lecture note exercises: 1.1
- **Hand in** lecture note exercises: 1.2, 1.3, 1.4, 1.5, 1.6
- **Hand in** from **Lawler** §5.1 on page 125.

1.2 Homework 2. Due Friday, April 15, 2016

- **Look at** from lecture note exercises: 1.8, 1.9, 1.11
- **Hand in** lecture note exercises: 1.7, 1.10, 1.12
- **Hand in** from **Lawler** §1.1, 1.4, 1.19

1.3 Homework 3. Due Friday, April 22, 2016

- **Look at** from **Lawler** §1.10,
- **Hand in** from **Lawler** §1.5, 1.8, 1.9, 1.14, 1.18*, 2.3

*Hint: show the invariant distribuiton is uniform.

1.4 Homework 4. Due Friday, April 29, 2016

- **Look at** lecture note exercises: 1.13
- **Hand in** from **Lawler** problems: §7.6, 7.7, 7.8.
- Please use the result in 7.8 to verify your numerical approximations found in 7.7.

***Hints for 7.8.** Recall that for $n \in \mathbb{N}$ that

$$S_n = \left\{ \mathbf{k} = (k_0, \ldots, k_n) \in \{0,1\}^{n+1} : \; k_{i-1} + k_i \le 1 \text{ for } 1 \le i \le n \right\}$$

and your goal is to compute

$$p_n(m) := \frac{\#\{\mathbf{k} \in S_n : k_m = 1\}}{\#(S_n)} \text{ for } 0 \le m \le n.$$

As Lawler suggests, for $i, j \in \{0,1\}$ and $m \in \mathbb{N}$, let[1]

$$r_m(ij) = \#\{\mathbf{k} = (k_0, \ldots, k_m) \in S_m : k_0 = i \text{ and } k_m = j\}.$$

Notice that if $\mathbf{k} = (k_0, \ldots, k_n) \in S_n$ with $k_m = 1$, then

$$(0, k_0, \ldots, k_{m-1}, 1) \in S_{m+1} \text{ and } (1, k_{m+1}, \ldots, k_n, 0) \in S_{n-m+1}$$

(and visa versa) where k_{m-1} and k_{m+1} must both be zero so that

$$(0, k_0, \ldots, k_{m-2}, 0) \in S_m \text{ and } (0, k_{m+2}, \ldots, k_n, 0) \in S_{n-m}.$$

From these considerations we find

$$\#\{\mathbf{k} \in S_n : k_m = 1\} = r_{m+1}(01) \cdot r_{n-m+1}(10) = r_m(00) \cdot r_{n-m}(00).$$

Similarly $\mathbf{k} = (k_0, \ldots, k_m) \in S_m$ then $(0, k_0, \ldots, k_n, 0) \in S_{n+2}$ and visa versa from which we learn $\#(S_n) = r_{n+2}(00)$. Combining these results shows

$$p_n(m) = \frac{r_m(00) \cdot r_{n-m}(00)}{r_{n+2}(00)}.$$

A little thought shows this formula is correct at $m = 0$ and $m = n - m$ provided we use the convention that $r_0(00) = 1$. Now follow the outline in the Lawler in order to find, $y_m = y(m) := r_m(00)$.

1.5 Homework 5. Due Friday, May 6, 2016

- **Look at** from lecture note exercises: 1.14, 1.15
- **Hand in** lecture note exercises: 1.16, 1.18
- **Hand in** from **Lawler** § 5.4, 5.7a, 5.8a, 5.12

[1] I have changed the indexing a bit since Lawler's choices are a bit confusing.

1.6 Homework 6. Due Friday, May 13, 2016

- **Look at** from lecture note exercises: 1.19, 1.20, 1.21, 1.24
- **Look at from Lawler** § 5.13
- **Hand in** lecture note exercises: 1.22, 1.23
- **Hand in** from **Lawler** § 5.7b, 5.9*, 5.14

***Correction to 5.9** The condition, $\mathbf{P}f(x) = g(x)$ for $x \in S \setminus A$, should read $\mathbf{P}f(x) - f(x) = g(x)$ for $x \in S \setminus A$.

1.7 Homework 7. Due Friday, May 20, 2016

- **Look at** from lecture note exercises: 1.28, 1.30
- **Hand in** lecture note exercises: 1.25, 1.26, 1.27, 1.29

Exercise 1.1 (Optional). Let $W_0 := X(\Omega) \subset W$. Finish the proof of Proposition **??** using the following outline;

1. Use the fact that $\sigma(Y) \subset \sigma(X)$ to show for each $s \in S$ there exists $B_s \subset W_0 \subset W$ such that $\{Y = s\} = \{X \in B_s\}$.
2. Show $B_s \cap B_{s'} = \emptyset$ for all $s, s' \in S$ with $s \neq s'$.
3. Show $X(\Omega) = W_0 := \cup_{s \in S} B_s$.
   Now fix a point $s_* \in S$ and then define, $f : W \to S$ by setting $f(w) = s_*$ when $w \in W \setminus W_0$ and $f(w) = s$ when $w \in B_s \subset W_0$.
4. Verify that $Y = f(X)$.

Exercise 1.2. Suppose that X and Y are two integrable random variables such that

$$\mathbb{E}[X|Y] = 18 - \frac{3}{5}Y \text{ and } \mathbb{E}[Y|X] = 10 - \frac{1}{3}X.$$

Find $\mathbb{E}X$ and $\mathbb{E}Y$.

Exercise 1.3. Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. random variables with $\mathbb{E}|X_i| < \infty$ for all i and let $S_m := X_1 + \cdots + X_m$ for $m = 1, 2, \ldots$. Show

$$\mathbb{E}[S_m|S_n] = \frac{m}{n}S_n \text{ for all } m \leq n.$$

Hint: observe by symmetry[2] that there is a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{E}(X_i|S_n) = h(S_n) \text{ independent of } i.$$

[2] Apply Theorem **??** using $(X_1, S_n) \overset{d}{=} (X_i, S_n)$ for $1 \leq i \leq n$.

Exercise 1.4. Let $\tau : \Omega \to \bar{\mathbb{N}}$ be a function. Verify the following are equivalent;

1. τ is a stopping time.
2. $\{\tau \leq n\} \in \mathcal{F}_n^{\mathbb{X}}$ for all $n \in \mathbb{N}_0$.
3. $\{\tau > n\} \in \mathcal{F}_n^{\mathbb{X}}$ for all $n \in \mathbb{N}_0$.

Also show that if τ is a stopping time then $\{\tau = \infty\} \in \mathcal{F}_\infty^{\mathbb{X}}$.

Exercise 1.5. If τ and σ are two stopping times shows, $\sigma \wedge \tau = \min\{\sigma, \tau\}$, $\sigma \vee \tau = \max\{\sigma, \tau\}$, and $\sigma + \tau$ are all stopping times.

Exercise 1.6 (Hitting time after a stopping time). Let σ be any stopping time. Show

$$\tau_1 = \inf\{n \geq \sigma : X_n \in B\} \text{ and}$$
$$\tau_2 = \inf\{n > \sigma : X_n \in B\}.$$

are both stopping times.

Exercise 1.7 (Invariant distributions and expected return times). Suppose that $\{X_n\}_{n=0}^{\infty}$ is a Markov chain on a finite state space S determined by one step transition probabilities, $p(x, y) = \mathbb{P}(X_{n+1} = y | X_n = x)$. For $x \in S$, let $R_x := \inf\{n > 0 : X_n = x\}$ be the first **passage time**[3] to x. [We will assume here that $\mathbb{E}_y R_x < \infty$ for all $x, y \in S$.] Use the first step analysis to show,

$$\mathbb{E}_x R_y = \sum_{z : z \neq y} p(x, z)\, \mathbb{E}_z R_y + 1. \tag{1.1}$$

Now further assume that $\pi : S \to [0, 1]$ is an invariant distributions for p, that is $\sum_{x \in S} \pi(x) = 1$ and $\sum_{x \in S} \pi(x)\, p(x, y) = \pi(y)$ for all $y \in S$, i.e. $\pi\mathbf{P} = \pi$. By multiplying Eq. (1.1) by $\pi(x)$ and summing on $x \in S$, show,

$$\pi(y)\, \mathbb{E}_y R_y = 1 \text{ for all } y \in S \implies \pi(y) = \frac{1}{\mathbb{E}_y R_y} > 0. \tag{1.2}$$

Exercise 1.8 (Uniqueness of solutions to 2nd order recurrence relations). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and $g : \mathbb{Z} \cap (\alpha, \beta) \to \mathbb{R}$ be a given function. Show that there is exactly one function $u : [\alpha, \beta] \cap \mathbb{Z} \to \mathbb{R}$ with prescribed values on two two consecutive points in $[\alpha, \beta] \cap \mathbb{Z}$ which satisfies the second order recurrence relation:

$$au(x+1) + bu(x) + cu(x-1) = f(x) \text{ for all } x \in \mathbb{Z} \cap (\alpha, \beta). \tag{1.3}$$

are for $\alpha < x < \beta$. Show; if u and w both satisfy Eq. (1.3) and $u = w$ on two consecutive points in $(\alpha, \beta) \cap \mathbb{Z}$, then $u(x) = w(x)$ for all $x \in [\alpha, \beta] \cap \mathbb{Z}$.

[3] R_x is the first **return time** to x when the chain starts at x.

Exercise 1.9 (General homogeneous solutions). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and suppose $\{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\}$ solves the second order homogeneous recurrence relation

$$au(x+1) + bu(x) + cu(x-1) = 0 \text{ for all } x \in \mathbb{Z} \cap (\alpha, \beta), \tag{1.4}$$

i.e. Eq. (1.3) with $f(x) \equiv 0$. Show:

1. for any $\lambda \in \mathbb{C}$,
$$a\lambda^{x+1} + b\lambda^x + c\lambda^{x-1} = \lambda^{x-1} p(\lambda) \tag{1.5}$$
where $p(\lambda) = a\lambda^2 + b\lambda + c$ is the **characteristic polynomial** associated to Eq. (1.3).
Let $\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ be the roots of $p(\lambda)$ and suppose for the moment that $b^2 - 4ac \neq 0$. From Eq. (1.3) it follows that for any choice of $A_\pm \in \mathbb{R}$, the function,
$$w(x) := A_+ \lambda_+^x + A_- \lambda_-^x, \tag{1.6}$$
solves Eq. (1.3) for all $x \in \mathbb{Z}$.
2. Show there is a unique choice of constants, $A_\pm \in \mathbb{R}$, such that the function $u(x)$ is given by
$$u(x) := A_+ \lambda_+^x + A_- \lambda_-^x \text{ for all } \alpha \le x \le \beta.$$
3. Now suppose that $b^2 = 4ac$ and $\lambda_0 := -b/(2a)$ is the double root of $p(\lambda)$. Show for any choice of A_0 and A_1 in $\mathbb{R}$ that
$$w(x) := (A_0 + A_1 x)\lambda_0^x \tag{1.7}$$
solves Eq. (1.3) for all $x \in \mathbb{Z}$. **Hint:** Differentiate Eq. (1.5) with respect to λ and then set $\lambda = \lambda_0$.
4. Again show that any function u solving Eq. (1.3) is of the form $u(x) = (A_0 + A_1 x)\lambda_0^x$ for $\alpha \le x \le \beta$ for some unique choice of constants $A_0, A_1 \in \mathbb{R}$.

Exercise 1.10. Let $w(x) := \mathbb{P}_x\left(X_{H_{a,b}} = b\right) := \mathbb{P}\left(X_{H_{a,b}} = b | X_0 = x\right)$.

1. Use first step analysis to show for $a < x < b$ that
$$w(x) = \frac{1}{2}\left(w(x+1) + w(x-1)\right) \tag{1.8}$$
provided we define $w(a) = 0$ and $w(b) = 1$.
2. Use the results of Exercises 1.8 and 1.9 to show
$$\mathbb{P}_x\left(X_{H_{a,b}} = b\right) = w(x) = \frac{1}{b-a}(x-a). \tag{1.9}$$
3. Let
$$H_b := \begin{cases} \min\{n : X_n = b\} & \text{if } \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}$$
be the first time $\{X_n\}$ hits b. Explain why, $\left\{X_{H_{a,b}} = b\right\} \subset \{H_b < \infty\}$ and use this along with Eq. (1.9) to conclude that $\mathbb{P}_x(H_b < \infty) = 1$ for all $x < b$. (By symmetry this result holds true for all $x \in \mathbb{Z}$.)

Exercise 1.11. The goal of this exercise is to give a second proof of the fact that $\mathbb{P}_x(H_b < \infty) = 1$. Here is the outline:

1. Let $w(x) := \mathbb{P}_x(H_b < \infty)$. Again use first step analysis to show that $w(x)$ satisfies Eq. (1.8) for all x with $w(b) = 1$.
2. Use Exercises 1.8 and 1.9 to show that there is a constant, c, such that
$$w(x) = c \cdot (x - b) + 1 \text{ for all } x \in \mathbb{Z}.$$
3. Explain why c must be zero to again show that $\mathbb{P}_x(H_b < \infty) = 1$ for all $x \in \mathbb{Z}$.

Exercise 1.12. Let $H = H_{a,b}$ and $u(x) := \mathbb{E}_x H := \mathbb{E}[H | X_0 = x]$.

1. Use first step analysis to show for $a < x < b$ that
$$u(x) = \frac{1}{2}\left(u(x+1) + u(x-1)\right) + 1 \tag{1.10}$$
with the convention that $u(a) = 0 = u(b)$.
2. Show that
$$u(x) = A_0 + A_1 x - x^2 \tag{1.11}$$
solves Eq. (1.10) for any choice of constants A_0 and A_1.
3. Choose A_0 and A_1 so that $u(x)$ satisfies the boundary conditions, $u(a) = 0 = u(b)$. Use this to conclude that
$$\mathbb{E}_x H_{a,b} = -ab + (b+a)x - x^2 = -a(b-x) + bx - x^2. \tag{1.12}$$

Exercise 1.13. Let $\{X_n\}_{n=0}^\infty$ be the fair random walk on $\mathbb{Z}$ (as in Exercise 1.10) starting at 0 and let

$$A_N := \mathbb{E}\left[\sum_{k=0}^{2N} 1_{X_k = 0}\right]$$

denote the expected number of visits to 0. Using Sterling's formula and integral approximations for $\sum_{n=1}^N \frac{1}{\sqrt{n}}$ to argue that $A_N \sim c\sqrt{N}$ for some constant $c > 0$.

Exercise 1.14. Construct an example of a martingale, $\{M_n\}_{n=0}^\infty$ such that $\mathbb{E}|M_n| \to \infty$ as $n \to \infty$. [In particular, $\{M_n\}_{n=1}^\infty$ will be a martingale which is not of the form $M_n = \mathbb{E}_{\mathcal{F}_n} X$ for some $X \in L^1(P)$.] **Hint:** try taking $M_n = \sum_{k=0}^n Z_k$ for a judicious choice of $\{Z_k\}_{k=0}^\infty$ which you should take to be independent, mean zero, and having $\mathbb{E}|Z_n|$ growing rather rapidly.

Exercise 1.15. Show that $M_n := 2^n 1_{(0,2^{-n}]}$ for $n \in \mathbb{N}_0$ as defined in Example **??** is a martingale.

Exercise 1.16. Suppose that $\{Z_n\}_{n=0}^\infty$ are independent random variables such that $\sigma^2 := \mathbb{E}Z_n^2 < \infty$ and $\mathbb{E}Z_n = 0$ for all $n \geq 1$. As in Example **??**, let $S_n := \sum_{k=0}^n Z_k$ be the martingale relative to the filtration, $\mathcal{F}_n^Z := \sigma(Z_0, \dots, Z_n)$. Show

$$M_n := S_n^2 - \sigma^2 n \text{ for } n \in \mathbb{N}_0$$

is a martingale. [Hint, make use of the basic properties of conditional expectations.]

Exercise 1.17 (Quadratic Variation). Suppose $\{M_n\}_{n=0}^\infty$ is a square integrable martingale. Show;

1. $\mathbb{E}\left[M_{n+1}^2 - M_n^2 | \mathcal{F}_n\right] = \mathbb{E}\left[(M_{n+1} - M_n)^2 | \mathcal{F}_n\right]$. Conclude from this that the Doob decomposition of M_n^2 is of the form,

$$M_n^2 = N_n + A_n$$

where

$$A_n := \sum_{1 \leq k \leq n} \mathbb{E}\left[(\Delta_k M)^2 | \mathcal{F}_{k-1}\right] = \sum_{1 \leq k \leq n} \mathbb{E}\left[(M_k - M_{k-1})^2 | \mathcal{F}_{k-1}\right].$$

In particular we see from this that M_n^2 is a sub-martingale.

2. Show that N_n above may be expressed as, $N_0 = M_0^2$ and for $n \geq 1$,

$$N_n = M_0^2 + 2\sum_{k=1}^n M_{k-1}\Delta_k M + \sum_{k=1}^n \left((\Delta_k M)^2 - \mathbb{E}\left[(\Delta_k M)^2 | \mathcal{F}_{k-1}\right]\right).$$

[From Example **??** and Proposition **??** below one may see directly that the expression above is a martingale.]

3. If we further assume that $M_k - M_{k-1}$ is independent of $\mathcal{F}_{k-1}$ for all $k = 1, 2, \dots$, explain why,

$$A_n = \sum_{1 \leq k \leq n} \mathbb{E}(M_k - M_{k-1})^2.$$

Exercise 1.18. Suppose that $\{X_n\}_{n=0}^\infty$ is a Markov chain on S with one step transition probabilities, $\mathbf{P}$, and $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$. Let $f : S \to \mathbb{R}$ be a bounded (for simplicity) function and set $g(x) = (\mathbf{P}f)(x) - f(x)$. Show

$$M_n := f(X_n) - \sum_{0 \leq j < n} g(X_j) \text{ for } n \in \mathbb{N}_0$$

is a martingale.

Exercise 1.19. Suppose Harriet has 7 dollars. Her plan is to make one dollar bets on fair coin tosses until her wealth reaches either 0 or 50, and then to go home. What is the expected amount of money that Harriet will have when she goes home? What is the probability that she will have 50 when she goes home?

Exercise 1.20. Consider a contract that at time N will be worth either 100 or 0. Let S_n be its price of the contract at time $0 \leq n \leq N$. If S_n is a martingale, and $S_0 = 47$, then what is the probability that the contract will be worth 100 at time N?

Exercise 1.21. Pedro plans to buy the contract in the previous problem at time 0 and sell it the first time T at which the price goes above 55 or below 15. What is the expected value of S_T? You may assume that the value, S_n, of the contract is bounded – there is only a finite amount of money in the world up to time N. Also note, by assumption, $T \leq N$.

Exercise 1.22. For $a < 0 < b$ with $a, b \in \mathbb{Z}$, let $\tau = \sigma_a \wedge \sigma_b$. Explain why $\{S_n^\tau\}_{n=0}^\infty$ is a bounded martingale use this to show $\mathbb{P}(\tau = \infty) = 0$. **Hint:** make use of the fact that $|S_n - S_{n-1}| = |Z_n| = 1$ for all n and hence the only way $\lim_{n\to\infty} S_n^\tau$ can exist is if it stops moving!

Exercise 1.23. Show

$$\mathbb{P}(\sigma_b < \sigma_a) = \frac{|a|}{b + |a|} \tag{1.13}$$

and use this to conclude $\mathbb{P}(\sigma_b < \infty) = 1$, i.e. every $b \in \mathbb{N}$ is almost surely visited by S_n.

Hint: As in Exercise 1.22 notice that $\{S_n^\tau\}_{n=0}^\infty$ is a bounded martingale where $\tau := \sigma_a \wedge \sigma_b$. Now compute $\mathbb{E}[S_\tau] = \mathbb{E}[S_\tau^\tau]$ in two different ways.

Exercise 1.24. Let $\tau := \sigma_a \wedge \sigma_b$. In this problem you are asked to show $\mathbb{E}[\tau] = |a|\, b$ with the aid of the following outline.

1. Use Exercise 1.17 above to conclude $N_n := S_n^2 - n$ is a martingale.
2. Now show

$$0 = \mathbb{E}N_0 = \mathbb{E}N_{\tau\wedge n} = \mathbb{E}S_{\tau\wedge n}^2 - \mathbb{E}[\tau \wedge n]. \tag{1.14}$$

3. Now use DCT and MCT along with Exercise 1.23 to compute the limit as $n \to \infty$ in Eq. (1.14) to find

$$\mathbb{E}\left[\sigma_a \wedge \sigma_b\right] = \mathbb{E}\left[\tau\right] = b\left|a\right|. \tag{1.15}$$

4. By considering the limit, $a \to -\infty$ in Eq. (1.15), show $\mathbb{E}\left[\sigma_b\right] = \infty$.

Exercise 1.25 (Some Discrete Distributions). Let $p \in (0,1]$ and $\lambda > 0$. In the two parts below, the distribution of N will be described. In each case find the generating function (see Proposition **??**) and use this to verify the stated values for $\mathbb{E}N$ and $\mathrm{Var}\left(N\right)$.

1. Geometric(p) : $\mathbb{P}\left(N = k\right) = p\left(1-p\right)^{k-1}$ for $k \in \mathbb{N}$. ($\mathbb{P}\left(N = k\right)$ is the probability that the kth – trial is the first time of success out a sequence of independent trials with probability of success being p.) You should find $\mathbb{E}N = 1/p$ and $\mathrm{Var}\left(N\right) = \frac{1-p}{p^2}$.
2. Poisson(λ) : $\mathbb{P}\left(N = k\right) = \frac{\lambda^k}{k!}e^{-\lambda}$ for all $k \in \mathbb{N}_0$. You should find $\mathbb{E}N = \lambda = \mathrm{Var}\left(N\right)$.

Exercise 1.26. Let $T \overset{d}{=} E\left(\lambda\right)$ be as in Definition **??**. Show;

1. $\mathbb{E}T = \lambda^{-1}$ and
2. $\mathrm{Var}\left(T\right) = \lambda^{-2}$.

Exercise 1.27 (Discrete time M.C. jump-hold description). Let S be a countable or finite set $(\Omega, \mathbb{P}, \{Y_n\}_{n=0}^{\infty})$ be a Markov chain with transition kernel, $\mathbf{P} := \{p\left(x,y\right)\}_{x,y\in S}$ and let $\nu\left(x\right) := \mathbb{P}\left(Y_0 = x\right)$ for all $x \in S$. For simplicity let us assume there are no **absorbing states**,[4] (i.e. $p\left(x,x\right) < 1$ for all $x \in S$) and then define,

$$\tilde{p}\left(x,y\right) := \begin{cases} \frac{p(x,y)}{1-p(x,x)} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $\mathbf{j}_k$ denote the time of the kth – jump of the chain $\{Y_n\}_{n=0}^{\infty}$ so that

$$\mathbf{j}_1 := \inf\left\{n > 0 : Y_n \neq Y_0\right\} \text{ and}$$
$$\mathbf{j}_{k+1} := \inf\left\{n > \mathbf{j}_k : Y_n \neq Y_{\mathbf{j}_k}\right\}$$

with the convention that $\mathbf{j}_0 = 0$. Further let $\sigma_k := \mathbf{j}_k - \mathbf{j}_{k-1}$ denote the time spent between the $(k-1)^{\text{st}}$ and k^{th} jump of the chain $\{Y_n\}_{n=0}^{\infty}$. Show;

[4] A state x is absorbing if $p\left(x,x\right) = 1$ since in this case there is no chance for the chain to leave x once it hits x.

1. For $\{x_k\}_{k=0}^{n} \subset S$ with $x_k \neq x_{k-1}$ for $k = 1, \ldots, n$ and $m_1, \ldots, m_k \in \mathbb{N}$, show

$$\begin{aligned} &\mathbb{P}\left(\left[\cap_{k=0}^{n}\left\{Y_{\mathbf{j}_k} = x_k\right\}\right] \cap \left[\cap_{k=1}^{n}\left\{\sigma_k = m_k\right\}\right]\right) \\ &\quad = \nu\left(x_0\right)\prod_{k=1}^{n} p\left(x_{k-1}, x_{k-1}\right)^{m_k - 1}\left(1 - p\left(x_{k-1}, x_{k-1}\right)\right)\cdot \tilde{p}\left(x_{k-1}, x_k\right). \end{aligned} \tag{1.16}$$

2. Summing the previous formula on $m_1, \ldots, m_k \in \mathbb{N}$, conclude

$$\mathbb{P}\left(\left[\cap_{k=0}^{n}\left\{Y_{\mathbf{j}_k} = x_k\right\}\right]\right) = \nu\left(x_0\right)\cdot\prod_{k=1}^{n}\tilde{p}\left(x_{k-1}, x_k\right),$$

   i.e. this shows $\{Y_{\mathbf{j}_k}\}_{k=0}^{\infty}$ is a Markov chain with transition kernel, $\tilde{p}$.
3. Conclude, relative to the conditional probability measure, $\mathbb{P}\left(\cdot \middle| \left[\cap_{k=0}^{n}\left\{Y_{\mathbf{j}_k} = x_k\right\}\right]\right)$, that $\{\sigma_k\}_{k=1}^{n}$ are independent geometric $\sigma_k \overset{d}{=} Geo\left(1 - p\left(x_{k-1}, x_{k-1}\right)\right)$ for $1 \le k \le n$.

Exercise 1.28. Show; if $\{a_n\}_{n=1}^{\infty} \subset [0,1)$, then

$$\prod_{n=1}^{\infty}\left(1 - a_n\right) = 0 \iff \sum_{n=1}^{\infty} a_n = \infty.$$

The implication, $\Longleftarrow$, holds even if $a_n = 1$ is allowed.

Exercise 1.29. Suppose that $S = \{1, 2, \ldots, n\}$ and $\mathbf{A}$ is a matrix such that $\mathbf{A}\left(i,j\right) \ge 0$ for $i \neq j$ and $\sum_{j=1}^{n}\mathbf{A}\left(i,j\right) = 0$ for all i. Show

$$\mathbf{P}_t = e^{t\mathbf{A}} := \sum_{n=0}^{\infty}\frac{t^n}{n!}\mathbf{A}^n \tag{1.17}$$

is a time homogeneous Markov kernel.

Hints: 1. To show $\mathbf{P}_t\left(i,j\right) \ge 0$ for all $t \ge 0$ and $i,j \in S$, write $\mathbf{P}_t = e^{-t\lambda}e^{t(\lambda I + \mathbf{A})}$ where $\lambda > 0$ is chosen so that $\lambda I + \mathbf{A}$ has only non-negative entries. 2. To show $\sum_{j\in S}P_t\left(i,j\right) = 1$, compute $\frac{d}{dt}P_t\mathbf{1}$.

Exercise 1.30. For $0 \le s < t < \infty$, show by induction on n that

$$a_n\left(s,t\right) := \int_{s\le w_1\le w_2\le\cdots\le w_n\le t} dw_1\ldots dw_n = \frac{\left(t-s\right)^n}{n!}.$$