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Heat kernel weighted L^2 – spaces

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Part I

Preliminaries

1

Where we are going

These lectures are devoted to the following two vague questions.

Question 1.1. Given a real manifold W equipped with a measure μ. Does there exist a complexification $W_{\mathbb{C}} = W + iW$ of W, a measure $\mu_{\mathbb{C}}$ on $W_{\mathbb{C}}$, and a unitary map $U : L^2(W, \mu) \to \mathcal{H}L^2(W_{\mathbb{C}}, \mu_{\mathbb{C}})$ where $\mathcal{H}L^2(W_{\mathbb{C}}, \mu_{\mathbb{C}})$ denotes the holomorphic L^2 – functions on $W_{\mathbb{C}}$.

Question 1.2. Given a pointed complex manifold (G, o) equipped with a measure λ. Let

$$\mathcal{D} := \{\text{“derivatives” of } f \text{ at } o : f \in \mathcal{H}(G)\}$$

be the **derivative space** associated to $\mathcal{H}(G)$ (the holomorphic functions on G) and let $T : \mathcal{H}(G) \to \mathcal{D}$ be the “**Taylor map;**”

$$Tf := \{\text{“derivatives” of } f \text{ at } o\}.$$

Can we;

1. characterize the derivative space, $\mathcal{D}$?
2. Find the norm, $\|\cdot\|_{\mathcal{D}}$, on $\mathcal{D}$ such that

$$\int_G |f|^2 \, d\lambda = \|Tf\|_{\mathcal{D}}^2 \text{ for all } f \in \mathcal{H}(G).$$

We will only be able to provide some partial answers to these questions by way of a few examples where W and G are Lie groups or homogenous spaces equipped with “heat kernel measures.” The prototypical example we have in mind here goes under the names of the Segal-Bargmann transform and the Itô chaos expansion. The original context of this theorem was for the case where W is a Banach space equipped with a Gaussian measure – for example $W = C([0, T], \mathbb{R})$ equipped with Wiener measure μ, i.e. the law of a Brownian motion. In this classical case everyone knows that Brownian motion is intimately related to the heat equation on $\mathbb{R}$. However are point of view will be to exploit the less widely appreciated fact that μ is related to a heat equation on W.

Here is the outline of these lectures;

1. Preliminary measure theoretic and holomorphic function theory for Banach spaces.
2. Gaussian measures an Banach spaces including;
   a) The Cameron-Martin space and theorem and
   b) heat equation interpretations.
3. Gaussian processes as Gaussian measures along with the informal path integral description of Brownian motion and how to interpret it rigorously.
4. A detour into path integral quantization;
   a) basic idea in for finite dimensional configurations spaces
   b) Hilbert space support properties of Gaussian measures
   c) applications to (not) understanding quantum field theories.
5. Heat kernel smoothing, the Segal - Bargmann transforms, and the Kakutani-Itô-Fock space isomorphisms
6. Applications to canonical quantization of Yang-Mills in 1+1 dimensions leading to Gross' generalization of the Kakutani-Itô-Fock space isomorphisms and Hall's generalization of the Segal Bargmann transform.
7. Topics that might of been covered but alas were not;
   a) The Taylor isomorphism for arbitrary finite dimensional Lie groups equipped with subelliptic heat kernel measures.
   b) Extensions of parts of this theory to path and loop groups, Hilbert – Schmidt groups (see Masha Gordina's lectures), and infinite dimensional nilpotent Lie group – see the work of Matt Cecil and Tai Melcher.
   c) Interpretation of path integrals on manifolds.

Banach Space Preliminaries

Given a real Banach space W, we let W^* denote the continuous dual of W and $\mathcal{B}_W$ be the Borel σ – algebra on W. Given a probability measure, μ, on $\mathcal{B}_W$ we let $\hat{\mu} : W^* \to \mathbb{C}$ be its Fourier transform defined by

$$\hat{\mu}(\alpha) = \int_W e^{i\alpha(x)} d\mu(x).$$

2.1 Measurability and Density Facts

Definition 2.1 (Cylinder functions). *To any non-empty subset, $\mathcal{L}$, of W^* we let $\mathcal{F}C_c^\infty(\mathcal{L})$ denote those $f : W \to \mathbb{R}$ of the form*

$$f = F(\alpha_1, \alpha_2, \dots, \alpha_n) \tag{2.1}$$

for some $n \in \mathbb{N}$, $F \in C_c^\infty(\mathbb{R}^n)$, and $\{\alpha_i\}_{i=1}^n \subset \mathcal{L}$. Similarly let $\mathcal{FP}(\mathcal{L})$ denote the polynomial cylinder functions f as in Eq. (2.1) where now $F : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function on $\mathbb{R}^n$.

We will use the following standard results through out these notes.

Theorem 2.2. *If W and V be real separable Banach spaces, $1 \le p < \infty$, and μ and ν be probability measures on $(W, \mathcal{B}_W)$ and $(V, \mathcal{B}_V)$ respectively, then;*

1. $\mathcal{B}_W = \sigma(W^*)$.
2. $\mathcal{B}_W \otimes \mathcal{B}_V = \mathcal{B}_{(W \times V)}$.
3. *The vector space operations are measurable on W.*
4. $L^p(W, \mathcal{B}_W, \mu)$ *is separable.*
5. *If $\sigma(\mathcal{L}) = \mathcal{B}_W$ then $\mathcal{F}C_c^\infty(\mathcal{L})$ is dense in $L^p(W, \mathcal{B}_W, \mu)$.*
6. *If $\mathcal{L}$ is a subspace of W^* such that $\sigma(\mathcal{L}) = \mathcal{B}_W$ then $\left\{e^{i\alpha} : \alpha \in \mathcal{L}\right\}$ is total in $L^p(W, \mathcal{B}_W, \mu)$.*
7. *If $\mathcal{L}$ is a subspace of W^* such that $\sigma(\mathcal{L}) = \mathcal{B}_W$ and $\hat{\mu}|_{\mathcal{L}} = \hat{\nu}|_{\mathcal{L}}$, then $\mu = \nu$.*
8. *If $\mathcal{L}$ is a subspace of W^* such that for all $\alpha \in \mathcal{L}$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that $e^{\varepsilon|\alpha|} \in L^1(W, \mathcal{B}_W, \mu)$, then $\mathcal{FP}(\mathcal{L})$ is a dense subspace of $L^p(W, \mathcal{B}_W, \mu)$.*

Proof. For a full proof of these results of [6, Chapter **??**]. Here I will just give a brief hint at the proofs.

1. By the separability of W along with the Hahn - Banach theorem one may find $\{\alpha_n\}_{n=1}^\infty \subset W^*$ such that $\|x\|_W = \sup_n |\alpha_n(x)|$. The result now follows from Exercise 2.1 below.
2. Let p_W and p_V denote the projections of $W \times V$ to W and V respectively. Then
$$\begin{aligned}\mathcal{B}_W \otimes \mathcal{B}_V &= \sigma(\alpha : \alpha \in W^*) \otimes \sigma(\beta : \beta \in V^*) \\ &= \sigma(\{\alpha \circ p_V : \alpha \in W^*\} \cup \{\beta \circ p_W : \beta \in V^*\}) \\ &= \sigma(\{\alpha \circ p_V + \beta \circ p_W : (\alpha, \beta) \in W^* \times V^*\}) \\ &= \sigma\left(\left\{\psi : \psi \in (W \times V)^*\right\}\right) = \mathcal{B}_{W \times V}.\end{aligned}$$
3. The vector space operations are continuous and hence measurable by item 2.
4. This is a consequence of the fact that $\mathcal{B}_W$ is countably generated say by $\{B(x_n, r) : n \in \mathbb{N} \text{ and } r \in \mathbb{Q}_+\}$ where $\{x_n\}_{n=1}^\infty$ is a countable dense subset of W.

5-7. These are applications of the multiplicative systems theorem.

8. If $\overline{\mathcal{FP}(W)}^{L^p} \not\subseteq L^p$ then by the Hahn Banach theorem there exists $\lambda \in (L^p)^*$ such that $\lambda \neq 0$ while $\lambda(\mathcal{FP}(W)) = \{0\}$. Under these assumptions it can be shown that $\lambda\left(e^{i\alpha}\right) = 0$ for all $\alpha \in \mathcal{L}$. From item 6. it now follows that $\lambda \equiv 0$ which is a contradiction.

■

Exercise 2.1. Suppose that $\mathcal{L} \subset W^*$. Show that $\sigma(\mathcal{L}) = \mathcal{B}_W$ iff $\|\cdot\|_W$ is $\sigma(\mathcal{L})$ – measurable.

The following is a typical example for W and $\mathcal{L}$.

Example 2.3 (Canonical Continuous Stochastic Processes). Suppose that $T \in (0, \infty)$ is given and let $W := C([0, T], \mathbb{R})$ and

$$\|x\|_W := \max_{t \in [0,T]} |x(t)|.$$

By the Stone – Weierstrass theorem we know that $(W, \|\cdot\|_W)$ is a separable Banach space. For $t \in [0, T]$ let $\alpha_t \in W^*$ be the evaluation maps, $\alpha_t(x) = x(t)$. Since

$$\|\cdot\|_W = \sup_{t\in\mathbb{Q}\cap[0,T]} |\alpha_t|$$

it follows from Exercise 2.1 that $\sigma(\mathcal{L}) = \mathcal{B}_W$ when $\mathcal{L} = \{\alpha_t : 0 \le t \le T\}$ or $\mathcal{L} = \operatorname{span}_{\mathbb{R}}\{\alpha_t : 0 \le t \le T\}$. In particular if μ is a probability measure on $(W, \mathcal{B}_W)$, then μ is completely determined by its finite dimensional distributions or equivalently by $\hat{\mu}$ restricted to $\mathcal{L} = \operatorname{span}_{\mathbb{R}}\{\alpha_t : 0 \le t \le T\}$.

Assumption 1 *Through out these notes W will be a separable Banach space which is taken to be real unless otherwise specified.*

2.2 Holomorphic Functions

The following material is taken directly from [8, Section 5.1]. Let X and Y be two complex Banach space and for $a \in X$ and $\delta > 0$ let

$$B_X(a,\delta) := \{x \in X : \|x - a\|_X < \delta\}$$

be the open ball in X with center a and radius δ.

Definition 2.4. *Let $\mathcal{D}$ be an open subset of X. A function $u : \mathcal{D} \to Y$ is said to be* ***holomorphic (or analytic)*** *if the following two conditions hold.*

1. *u is locally bounded, namely for all $a \in \mathcal{D}$ there exists an $r_a > 0$ such that*

$$M_a := \sup\{\|u(x)\|_Y : x \in B_X(a, r_a)\} < \infty.$$

2. *The function u is complex Gâteaux differentiable on $\mathcal{D}$, i.e. for each $a \in \mathcal{D}$ and $h \in X$, the function $\lambda \to u(a + \lambda h)$ is complex differentiable at $\lambda = 0 \in \mathbb{C}$.*

(Holomorphic and analytic will be considered to be synonymous terms for the purposes of this paper.)

Typically the easiest way to check that $\lambda \to u(a + \lambda h)$ is holomorphic in a neighborhood of zero is to use Morera's theorem which I recall for the reader's convenience.

Theorem 2.5 (Morera's Theorem). *Suppose that Ω is an open subset of $\mathbb{C}$ and $f \in C(\Omega)$ is a complex function such that*

$$\int_{\partial T} f(z)dz = 0 \text{ for all solid triangles } T \subset \Omega, \tag{2.2}$$

then f is holomorphic on Ω.

The notion of holomorphic given in Definition 2.4 looks rather weak but it is equivalent to a much stronger notion as described in Theorem 2.6. This theorem gathers together a number of basic properties of holomorphic functions which may be found in [22]. (Also see [21].) One of the key ingredients to all of these results is Hartog's theorem, see [22, Theorem 3.15.1] or Theorem 1.6 in [30, Theorem 1.6 on p. 7] and Exercise 2.2 below.

Theorem 2.6. *If $u : \mathcal{D} \to Y$ is holomorphic, then there exists a function $u' : \mathcal{D} \to \operatorname{Hom}(X,Y)$, the space of bounded* ***complex*** *linear operators from X to Y, satisfying*

1. *If $a \in \mathcal{D}$, $x \in B_X(a, r_a/2)$, and $h \in B_X(0, r_a/2)$, then*

$$\|u(x+h) - u(x) - u'(x)h\|_Y \leqslant \frac{4M_a}{r_a(r_a - 2\|h\|_X)}\|h\|_X^2. \tag{2.3}$$

In particular, u is continuous and Frechét differentiable on $\mathcal{D}$.

2. *The function $u' : \mathcal{D} \to \operatorname{Hom}(X,Y)$ is holomorphic.*

Remark 2.7. By applying Theorem 2.6 repeatedly, it follows that any holomorphic function, $u : \mathcal{D} \to Y$ is Frechét differentiable to all orders and each of the Frechét differentials are again holomorphic functions on $\mathcal{D}$.

Proof. By [22, Theorem 26.3.2 on p. 766.], for each $a \in \mathcal{D}$ there is a linear operator, $u'(a) : X \to Y$ such that $du(a + \lambda h)/d\lambda|_{\lambda=0} = u'(a)h$. The Cauchy estimate in Theorem 3.16.3 (with $n = 1$) of [22] implies that if $a \in \mathcal{D}$, $x \in B_X(a, r_a/2)$ and $h \in B_X(0, r_a/2)$ (so that $x + h \in B_X(a, r_a)$), then $\|u'(x)h\|_Y \leqslant M_a$. It follows from this estimate that

$$\sup\left\{\|u'(x)\|_{\operatorname{Hom}(X,Y)} : x \in B_X(a, r_a/2)\right\} \leqslant 2M_a/r_a. \tag{2.4}$$

and hence that $u' : \mathcal{D} \to \operatorname{Hom}(X,Y)$ is a locally bounded function. The estimate in Eq. (2.3) appears in the proof of the Theorem 3.17.1 in [22] which completes the proof of item 1.

To prove item 2. we must show u' is Gâteaux differentiable on $\mathcal{D}$. We will in fact show more, namely, that u' is Frechét differentiable on $\mathcal{D}$. Given $h \in X$, let $F_h : \mathcal{D} \to Y$ be defined by $F_h(x) := u'(x)h$. According to [22, Theorem 26.3.6], F_h is holomorphic on $\mathcal{D}$ as well. Moreover, if $a \in \mathcal{D}$ and $x \in B(a, r_a/2)$ we have by Eq. (2.4) that

$$\|F_h(x)\|_Y \leqslant 2M_a\|h\|_X / r_a.$$

So applying the estimate in Eq. (2.3) to F_h, we learn that

$$\|F_h(x+k) - F_h(x) - F_h'(x)k\|_Y \leqslant \frac{4(2M_a\|h\|_X/r_a)}{\frac{r_a}{2}\left(\frac{r_a}{2} - 2\|k\|_X\right)} \cdot \|k\|_X^2 \tag{2.5}$$

for $x \in B(a, r_a/4)$ and $\|k\|_X < r_a/4$, where

$$F_h'(x)k = \frac{d}{d\lambda}|_0 F_h(x+\lambda k) = \frac{d}{d\lambda}|_0 u'(x+\lambda k)h =: \left(\delta^2 u\right)(x;h,k).$$

Again by [22, Theorem 26.3.6], for each fixed $x \in \mathcal{D}$, $\left(\delta^2 u\right)(x;h,k)$ is a continuous symmetric bilinear form in $(h,k) \in X \times X$. Taking the supremum of Eq. (2.5) over those $h \in X$ with $\|h\|_X = 1$, we may conclude that

$$\begin{aligned}\left\|u'(x+k) - u'(x) - \delta^2 u(x;\cdot,k)\right\|_{\mathrm{Hom}(X,Y)} &= \sup_{\|h\|_X=1} \|F_h(x+k) - F_h(x) - F_h'(x)k\|_Y \\ &\leqslant \frac{4(2M_a/r_a)}{\frac{r_a}{2}\left(\frac{r_a}{2} - 2\|k\|_X\right)}\|k\|_X^2.\end{aligned}$$

This estimate shows u' is Frechét differentiable with $u''(x) \in \mathrm{Hom}(X, \mathrm{Hom}(X,Y))$ being given by $u''(x)k = \left(\delta^2 u\right)(x;\cdot,k) \in \mathrm{Hom}(X,Y)$ for all $k \in X$ and $x \in \mathcal{D}$. ■

Here is an exercise to give you a feel for this theorem.

Exercise 2.2 (Baby Hartog's theorem). Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a function such that $z \to f(z,w)$ is holomorphic for each $w \in \mathbb{C}$ and $w \to f(z,w)$ is holomorphic for each $z \in \mathbb{C}$. Further assume that f is locally bounded. Show that f is a smooth function on $\mathbb{C}^2$ using the following outline.

1. Show $(z,w) \to f(z,w)$ is jointly measurable. **Hint:** approximate f by functions of the form $\sum 1_Q(z) f(z_Q, w)$ where the sum is over a countable partition $\{Q\}$ of $\mathbb{C}$ and $z_Q \in Q$ for each Q.
2. Use the one dimensional Cauchy integral formula twice to find,
$$\begin{aligned}f(z,w) &= \left(\frac{1}{2\pi i}\right)^2 \oint_{C_1}\oint_{C_2} \frac{f(z',w')}{(z'-z)(w'-w)} dz'dw' \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{[0,1]^2} \frac{f(C_1(s), C_2(t))}{(C_1(s)-z)(C_2(t)-w)} C_1'(s)\dot{C}_2(t)\,dsdt \qquad (2.6)\end{aligned}$$
where C_1 and C_2 denote appropriately chosen contours surrounding z and w respectively.
3. Conclude from items 1. and 2. that you may differentiate f in both z and w as many times as you please.

Lemma 2.8 (Holomorphic L^2). *Let M be a finite dimensional complex analytic manifold and ρ be a smooth positive measure on M. Let $\mathcal{H}(M)$ denote the holomorphic functions on M. Then $\mathcal{H}(M) \cap \mathrm{L}^2(\rho)$ is a closed subspace of $\mathrm{L}^2(\rho)$ and for each compact subset $K \subset M$ there exists $C_K < \infty$ such that*

$$\max_{m \in K} |f(m)| \le C_K \|f\|_{L^2(\rho)} \ \forall\ f \in \mathcal{H}(M) \qquad (2.7)$$

Moreover, if $f_n \to f$ in $\mathrm{L}^2(\rho)$ as $n \to \infty$, then f_n and df_n converges to f and df respectively uniformly on compact subsets of M.

Proof. Since the property that a function on M is holomorphic is local, it suffices to prove the lemma in the case that $M \doteq D_1$ and $\rho = 1$, where for any $R > 0$,

$$D_R := \{z \in \mathbb{C}^d : |z_i| < R\,\forall i = 1,2,\dots d\}. \qquad (2.8)$$

Let f be a holomorphic function on D_1, $0 < \alpha < 1$, and $z \in D_\alpha$. By the mean value theorem for holomorphic functions;

$$f(z) = (2\pi)^{-d} \int_{[0,2\pi]^d} f_n(\{z_j + r_j e^{\sqrt{-1}\theta_j}\}_{j=1}^d) \prod_{j=1}^d d\theta_j, \qquad (2.9)$$

where $r = (r_1, r_2, \dots, r_d) \in \mathbb{R}^d$ such that $0 \le r_i < \epsilon \doteq 1-\alpha$ for all $i = 1,2,\dots,d$. Multiplying (2.9) by $r_1 \cdots r_d$ and integrating each r_i over $[0,\epsilon)$ shows

$$f(z) = (\pi\epsilon^2)^{-d} \int_{D_\epsilon} f(z+\xi)\lambda(d\xi),$$

where λ denotes Lebesgue measure on $\mathbb{C}^d$. In particular, for each $\alpha < 1$,

$$\sup_{z \in D_\alpha} |f(z)| \le (\pi(1-\alpha)^2)^{-d} \|f\|_{\mathrm{L}^2}$$

which can be pieced together to prove Eq. (2.7). Therefore, an L^2– convergent sequence of holomorphic functions is uniformly convergent on compact subsets of D_1 and so the limit is also holomorphic. Since the derivatives of locally uniformly convergent holomorphic functions are locally uniformly convergent (see your proof of Exercise 2.2), it follows that L^2 convergence also implies uniform convergence of the differentials on compact sets. ■

Part II

Gaussian Measure Structures

Gaussian Measure Preliminaries

Definition 3.1. *A measure μ on $(W, \mathcal{B}_W)$ is called a (mean zero)* ***Gaussian measure*** *provided that every element $\alpha \in W^*$ is Gaussian random variable. We will refer to $(W, \mathcal{B}_W, \mu)$ as a Gaussian space.*

A word of warning: I am going to drop the adjective, "mean zero" and simply refer to a Gaussian measure or Gaussian probability and intend for you to understand that it is a mean zero.

The condition that every $\alpha \in W^*$ is Gaussian relative to μ is equivalent to assuming that $W^* \subset L^2(\mu)$ and

$$\hat{\mu}(\alpha) := e^{-\frac{1}{2}q_\mu(\alpha)} \text{ for all } \alpha \in W^* \tag{3.1}$$

where $q_\mu(\alpha) := q_\mu(\alpha, \alpha)$ and

$$q_\mu(\alpha, \beta) = \int_W \alpha(x)\,\beta(x)\,d\mu(x) \text{ for all } \alpha, \beta \in W^*. \tag{3.2}$$

For any finite subset $\{\alpha_k\}_{k=1}^n$ and $a_k \in \mathbb{R}$ we have $\sum_{k=1}^n a_k \alpha_k \in W^*$ and is therefore Gaussian and in particular

$$\int_W e^{i \sum_{k=1}^n a_k \alpha_k} d\mu = e^{-\frac{1}{2} q_\mu\left(\sum_{k=1}^n a_k \alpha_k\right)} = \exp\left(-\frac{1}{2}\sum_{k,l=1}^n a_k a_l q_\mu(\alpha_k, \alpha_l)\right).$$

Hence we see that $\{\alpha_k\}_{k=1}^n$ are jointly Gaussian random variables.

Definition 3.2. *We say that a Gaussian measure μ on $(W, \mathcal{B}_W)$ is non-degenerate if q_μ is positive definite on W^*, i.e. (W^*, q_μ) is an inner product space. (This condition turns out to be equivalent to the support of μ being all of W.)*

Corollary 3.3. *The quadratic form $q : W^* \times W^* \to \mathbb{R}$ in Eq. (3.2) is continuous, i.e. there exists $C_2 \in (0, \infty)$ such that*

$$|q(\alpha, \beta)| \leq C_2 \|\alpha\|_W \cdot \|\beta\|_W \text{ for all } \alpha, \beta \in W^*. \tag{3.3}$$

Proof. Because of the Cauchy–Schwarz inequality

$$|q(\alpha, \beta)| \leq \sqrt{q(\alpha)}\sqrt{q(\beta)}$$

and so it suffices to show

$$C_2 := \sup_{\alpha \in W^*} \frac{|q(\alpha, \alpha)|}{\|\alpha\|_{W^*}^2} < \infty. \tag{3.4}$$

(By convention we will typically define $0/0 = 0$ in this type of ratio.) If $C_2 = \infty$ in Eq. (3.4) we could find $\{\alpha_n\}_{n=1}^\infty \subset W^*$ such that $\mathbb{E}\left[\alpha_n^2\right] = q(\alpha_n, \alpha_n) = 1$ while $\lim_{n\to\infty} \|\alpha_n\|_{W^*} = 0$. However this is not possible since

$$q(\alpha_n, \alpha_n) = -2\ln[\hat{\mu}(\alpha_n)] = -2\ln \int_W e^{i\alpha_n(x)} d\mu(x) \to -2\ln(1) = 0$$

by the DCT. while $q(\alpha_n, \alpha_n) \to \infty$ ■

Exercise 3.1. Suppose that μ is a Gaussian measure on $(W, \mathcal{B})$ and for $\theta \in \mathbb{R}$, let $R_\theta : W \times W \to W \times W$ be the "rotation" map[1] given by

$$R_\theta(x, y) = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta). \tag{3.5}$$

Then for all $f \in (\mathcal{B}_{(W\times W)})_b = (\mathcal{B}_W \otimes \mathcal{B}_W)_b$ and $\theta \in \mathbb{R}$,

$$\int_{W\times W} f(x, y)d\mu(x)d\mu(y) = \int_{W\times W} f(R_\theta(x, y))d\mu(x)d\mu(y), \tag{3.6}$$

i.e. $\mu \times \mu$ is invariant under the rotations R_θ for all $\theta \in \mathbb{R}$. (See Theorem 3.5 for a strong converse to this exercise.)

Hint: compute both sides of Eq. (3.6) when $f(x, y) = e^{i\psi(x,y)}$ and $\psi \in (W \times W)^*$. You will also want to use that

$$W^* \times W^* \ni (\alpha, \beta) \to \alpha \circ p_1 + \beta \circ p_2 \in (W \times W)^*$$

is a linear isomorphism of Banach spaces where $p_i : W \times W \to W$ is the projection map onto the i^{th} – factor. Thus

$$(\alpha \circ p_1 + \beta \circ p_2)(x, y) = \alpha(x) + \beta(y) \;\; \forall\, x, y \in W.$$

[1] If $p_i : W \times W \to W$ is projection onto the i^{th} – factor for $i = 1, 2$, then $p_i \circ R_\theta(x, y)$ is linear combination of x and y. As the vector space operations are measurable it follows that $p_i \circ R_\theta$ is measurable for $i = 1$ and 2 and therefore R_θ is $\mathcal{B}_W \otimes \mathcal{B}_W$ – measurable. Alternatively one observes that $R_\theta : W \times W \to W \times W$ is continuous and $\mathcal{B}_{(W\times W)} = \mathcal{B}_W \otimes \mathcal{B}_W$ which again shows that R_θ is measurable.

Theorem 3.4 (Ferniques Theorem). *Suppose that* μ *is any measure on* $(W, \mathcal{B}_W)$ *such that* $\mu \times \mu$ *is invariant under rotation by* 45^o, *i.e.* $(\mu \times \mu) \circ R^{-1} = \mu \times \mu$ *where*

$$R(x,y) = \frac{1}{\sqrt{2}}(x-y, y+x). \tag{3.7}$$

Then there exists $\varepsilon = \varepsilon(\mu) > 0$ *such that*

$$\int_W e^{\varepsilon \|x\|_W^2} d\mu(x) < \infty. \tag{3.8}$$

In particular μ *has moments to all orders. (For the proof of this theorem see [6, Theorem* **??**] *(also see [4, Theorem 2.8.5] or [24, Theorem 3.1]).)*

Proof. Since

$$e^z = 1 + \int_0^z e^y dy = 1 + \int_{\mathbb{R}} 1_{0 \le y \le z} e^y dy,$$

$$\begin{aligned}\int_W e^{\varepsilon\|x\|^2} d\mu(x) &= \int_W \left(1 + \int_{\mathbb{R}} 1_{0 \le y \le \varepsilon\|x\|^2} e^y dy\right) d\mu(x) \\ &= 1 + \int_0^\infty dy\ e^y \mu(\varepsilon\|x\|^2 \ge y). \end{aligned} \tag{3.9}$$

Because of this formula it suffices to show that there are constants, $C \in (0,\infty)$ and $\beta \in (1,\infty)$ such that

$$\mu(\varepsilon\|x\|^2 \ge y) \le Ce^{-\beta y} \text{ for all } y \ge 0. \tag{3.10}$$

For if we use Eq. (3.10) in Eq. (3.9), we will have

$$\int_W e^{\varepsilon\|x\|^2} d\mu(x) \le 1 + C\int_0^\infty dy\ e^y e^{-\beta y} = 1 + \frac{C}{\beta - 1} < \infty. \tag{3.11}$$

We will now prove Eq. (3.10). By replacing y with εt^2, Eq. (3.10) is equivalent to showing

$$\mu(\|x\| \ge t) \le Ce^{-\beta\varepsilon t^2} = Ce^{-\gamma t^2} \text{ for all } t \ge 0, \tag{3.12}$$

where $\gamma := \beta\varepsilon$. Because we are free to choose $\varepsilon > 0$ as small as we like, it suffices to prove that Eq. (3.12) for some $\gamma > 0$. Let $P = \mu \times \mu$ on $W \times W$ and let $t \ge s \ge 0$. Then by the $R_{\pi/4}$ invariance of P,

$$\begin{aligned}\mu(\|x\| \le s)\mu(\|x\| \ge t) &= P(\|x\| \le s \text{ and } \|y\| \ge t) \\ &= P\left(\left\|\frac{x+y}{\sqrt{2}}\right\| \le s \text{ and } \left\|\frac{x-y}{\sqrt{2}}\right\| \ge t\right) \\ &\le P(|\|x\| - \|y\|| \le \sqrt{2}s \text{ and } \|x\| + \|y\| \ge \sqrt{2}t).\end{aligned}$$

Let $a = \|x\|$ and $b = \|y\|$ and notice that if $|a - b| \le \sqrt{2}s$ and $a + b \ge \sqrt{2}t$ then

$$\begin{aligned}&\sqrt{2}t \le a + b \le b + \sqrt{2}s + b = 2b + \sqrt{2}s \text{ and} \\ &\sqrt{2}t \le a + b \le a + a + \sqrt{2}s = 2a + \sqrt{2}s\end{aligned}$$

from which it follows that

$$a \ge \frac{t-s}{\sqrt{2}} \text{ and } b \ge \frac{t-s}{\sqrt{2}},$$

as seen in Figure 3.1 below. Combining these expressions shows

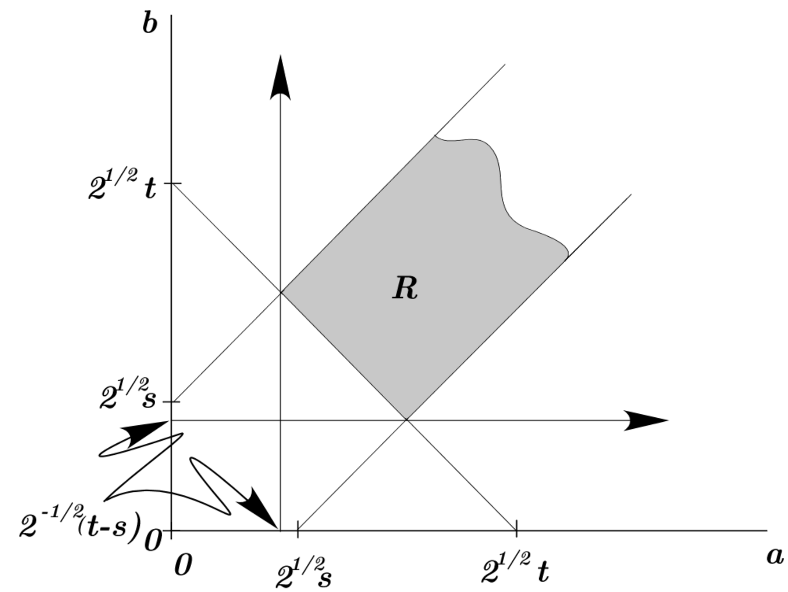

Fig. 3.1. The region R is contained in the region $a \ge \frac{t-s}{\sqrt{2}}$ and $b \ge \frac{t-s}{\sqrt{2}}$.

$$\begin{aligned}\mu(\|x\| \le s)\mu(\|x\| \ge t) &\le P\left(\|x\| \ge \frac{t-s}{\sqrt{2}} \text{ and } \|y\| \ge \frac{t-s}{\sqrt{2}}\right) \\ &= \left[\mu(\|x\| \ge \frac{t-s}{\sqrt{2}})\right]^2,\end{aligned}$$

which is to say for $t \ge s \ge 0$,

$$\mu(\|x\| \ge t) \le \frac{1}{\mu(\|x\| \le s)}\left[\mu(\|x\| \ge \frac{t-s}{\sqrt{2}})\right]^2. \tag{3.13}$$

We will now complete the proof by iterating Eq. (3.13). Define $t_0 = s$ and then define $\{t_n\}_{n=1}^{\infty}$ inductively so that

$$\frac{t_{n+1}-s}{\sqrt{2}} = t_n \text{ for all } n \tag{3.14}$$

i.e.

$$t_{n+1} = s + \sqrt{2}t_n \tag{3.15}$$

and (by a simple induction argument)

$$t_n = s\sum_{i=0}^{n} 2^{i/2} = s\frac{2^{\frac{n+1}{2}}-1}{2^{1/2}-1} \le s\frac{2^{\frac{n+1}{2}}}{2^{1/2}-1}. \tag{3.16}$$

Then by Eq. (3.13)

$$\mu(\|x\| \ge t_{n+1}) \le \frac{1}{\mu(\|x\| \le s)}\left[\mu(\|x\| \ge \frac{t_{n+1}-s}{\sqrt{2}})\right]^2 = \frac{[\mu(\|x\| \ge t_n)]^2}{\mu(\|x\| \le s)}$$

or equivalently

$$\alpha_{n+1}(s) := \frac{\mu(\|x\| \ge t_{n+1})}{\mu(\|x\| \le s)} \le \left(\frac{\mu(\|x\| \ge t_n)}{\mu(\|x\| \le s)}\right)^2 =: \alpha_n^2(s).$$

Iterating this inequality implies

$$\alpha_n(s) \le \alpha_0^{2^n}(s) \text{ with } \alpha_0(s) = \frac{\mu(\|x\| \ge s)}{\mu(\|x\| \le s)}, \tag{3.17}$$

i.e.

$$\mu(\|x\| \ge t_n) \le \mu(\|x\| \le s)\,(\alpha_0(s))^{2^n} \text{ for all } n. \tag{3.18}$$

We now **fix** an $s > 0$ sufficiently large so that $\alpha_0(s) < 1$ and suppose $t \ge 2s$ is given. Choose n so that $t_n \le t \le t_{n+1} = s + \sqrt{2}t_n$ in which case (as $0 \le t - s \le \sqrt{2}t_n$)

$$\left(\frac{t-s}{\sqrt{2}}\right)^2 \le t_n^2 \le \frac{2s^2}{\left(2^{1/2}-1\right)^2}2^n,$$

i.e.

$$2^n \ge \frac{\left(2^{1/2}-1\right)^2}{4s^2}(t-s)^2.$$

Combining this with Eq. (3.18), using $\alpha_0(s) < 1$, shows

$$\begin{aligned}\mu(\|x\| \ge t) &\le \mu(\|x\| \ge t_n)\\ &\le \mu(\|x\| \le s)\,(\alpha_0(s))^{2^n} \le \mu(\|x\| \le s)\rho^{(t-s)^2}\end{aligned}$$

where

$$\rho := (\alpha_0(s))^{\frac{\left(2^{1/2}-1\right)^2}{4s^2}} \in (0,1).$$

Since $t \ge 2s$ is equivalent to $(t-s) \ge t/2$, we have $(t-s)^2 \ge (t/2)^2$ and therefore

$$\mu(\|x\| \ge t) \le \mu(\|x\| \le s)e^{-\frac{|\ln \rho|}{4}t^2}$$

which is sufficient to prove Eq. (3.12) with $\gamma = \frac{|\ln \rho|}{4}$ and C chosen to be a sufficiently large constant. ■

As stated Theorem 3.4 seems to say something about measures more general than Gaussian measures but as is well know this not the case.

Theorem 3.5. *If μ is a probability measure such that $\mu \times \mu$ is invariant under $R_{\pi/4}$, then μ is a Gaussian measure on $(W, \mathcal{B}_W)$. As usual we will have $\hat{\mu} = e^{-\frac{1}{2}q}$ where $q(\alpha) := q(\alpha, \alpha)$ and*

$$q(\alpha, \beta) := \int_W \alpha(x)\beta(x)\,d\mu(x) \text{ for all } \alpha, \beta \in W^*.$$

(See Feller [11, Section III.4, pp. 77-80] for more general theorems along these lines.)

Proof. By Fernique's Theorem 3.4 bound in Eq. (3.8), μ has moments to all orders and $\hat{\mu}(\alpha)$ is infinitely differentiable in α. In fact for any $\alpha \in W^*$, the function,

$$\mathbb{C} \ni z \to \text{“}\hat{\mu}(z\alpha)\text{”} := \int_W e^{iz\alpha(x)}d\mu(x)$$

is holomorphic. The invariance of $\mu \times \mu$ under rotation by $\pi/4$ and the formula, $\widehat{\mu \times \mu}(\alpha \circ p_1 + \beta \circ p_1) = \hat{\mu}(\alpha)\hat{\mu}(\beta)$, implies[2]

$$\hat{\mu}(\alpha)\hat{\mu}(\beta) = \hat{\mu}\left(\frac{1}{\sqrt{2}}(\alpha+\beta)\right)\hat{\mu}\left(\frac{1}{\sqrt{2}}(-\alpha+\beta)\right) \;\forall\, \alpha, \beta \in W^*. \tag{3.19}$$

Taking $\alpha = 0$ and then $\beta = 0$ in this equation implies,

$$\hat{\mu}(\beta) = \hat{\mu}\left(\frac{1}{\sqrt{2}}\beta\right)\hat{\mu}\left(\frac{1}{\sqrt{2}}\beta\right) \;\forall\, \beta \in W^* \tag{3.20}$$

$$\hat{\mu}(\alpha) = \hat{\mu}\left(\frac{1}{\sqrt{2}}\alpha\right)\hat{\mu}\left(-\frac{1}{\sqrt{2}}\alpha\right) \;\forall\, \alpha \in W^* \tag{3.21}$$

[2] If $\psi = \alpha \circ p_1 + \beta \circ p_1$ then with $R = R_{\pi/4}$ as in Eq. (3.7) we find,

$$\begin{aligned}\psi(R(x,y)) &= \frac{1}{\sqrt{2}}\psi(x-y, y+x)\\ &= \frac{1}{\sqrt{2}}\left[(\alpha+\beta)(x) + (\beta-\alpha)(y)\right].\end{aligned}$$

and in particular we learn

$$\hat{\mu}\left(\frac{1}{\sqrt{2}}\alpha\right)\hat{\mu}\left(-\frac{1}{\sqrt{2}}\alpha\right)=\hat{\mu}\left(\frac{1}{\sqrt{2}}\alpha\right)\hat{\mu}\left(\frac{1}{\sqrt{2}}\alpha\right)\ \forall\ \alpha\in W^{*}.$$

For $t\in\mathbb{R}$ near zero we know that $\hat{\mu}\left(\frac{t}{\sqrt{2}}\alpha\right)\neq 0$ and hence we may conclude that

$$\hat{\mu}\left(-\frac{t}{\sqrt{2}}\alpha\right)=\hat{\mu}\left(\frac{t}{\sqrt{2}}\alpha\right)$$

for t near zero and hence by the principle of analytic continuation this equation in fact holds for all $t\in\mathbb{C}$ and in particular for $t=\sqrt{2}$ from which we learn that $\hat{\mu}(-\alpha)=\hat{\mu}(\alpha)$ for all $\alpha\in W^{*}$. Since $\overline{\hat{\mu}(\alpha)}=\hat{\mu}(-\alpha)=\hat{\mu}(\alpha)$ we may now conclude that $\hat{\mu}$ is real.

Iterating Eq. (3.20) implies,

$$\hat{\mu}(\beta)=\hat{\mu}\left(\left(\frac{1}{\sqrt{2}}\right)^{n}\beta\right)^{2^{n}}\quad\text{for all } n\in\mathbb{N}_{0}. \tag{3.22}$$

Let $f(t):=\hat{\mu}(t\beta)$ so that $f(0)=1$, $\dot{f}(0)=0$ (since $f(t)$ is odd), and

$$\ddot{f}(0)=\left(\frac{d}{dt}\right)^{2}_{t=0}\int_{W}e^{it\beta(x)}d\mu(x)=-\int_{W}\beta^{2}(x)\,d\mu(x):=-q(\beta).$$

So by Taylor's theorem;

$$f(t)=1-\frac{1}{2}q(\beta)t^{2}+O\left(t^{3}\right)$$

while by Eq. (3.22);

$$f(1)=\left[f\left(2^{-n/2}\right)\right]^{2^{n}}=\left[1-\frac{1}{2}q(\beta)2^{-n}+O\left(2^{-3n/2}\right)\right]^{2^{n}}.$$

A simple calculus exercise now shows

$$\ln f(1)=2^{n}\cdot\ln\left(1-\frac{1}{2}q(\beta)2^{-n}+O\left(2^{-3n/2}\right)\right)\to-\frac{1}{2}q(\beta)\ \text{ as } n\to\infty.$$

Thus we have shown

$$\hat{\mu}(\beta)=\hat{\mu}(\beta)=f(1)=e^{-\frac{1}{2}q(\beta)}=e^{-\frac{1}{2}\operatorname{Var}_{\mu}(\beta)},$$

i.e. $\hat{\mu}=e^{-q/2}$ and so μ is a (possibly degenerate) Gaussian measure. ∎

Corollary 3.6. *Suppose that $\mathcal{L}$ is a linear subspace of W^{*} such that $\sigma(\mathcal{L})=\mathcal{B}_{W}$ and μ is a probability measure on $(W,\mathcal{B}_{W})$ such that every element $\alpha\in\mathcal{L}$ is a mean - zero Gaussian random variable. Then μ is a Gaussian measure.*

Proof. Let

$$\begin{aligned}\mathcal{L}\oplus\mathcal{L}&=\left\{\psi\in(W\times W)^{*}:\alpha:=\psi(\cdot,0)\in\mathcal{L}\text{ and }\beta:=\psi(0,\cdot)\in\mathcal{L}\right\}\\&=\left\{\alpha\circ p_{1}+\beta\circ p_{2}\in(W\times W)^{*}:\alpha,\beta\in\mathcal{L}\right\}\end{aligned}$$

where $p_{i}:W\times W\to W$ is projection onto the i^{th} – factor. We then have that

$$\begin{aligned}\sigma(\mathcal{L}\oplus\mathcal{L})&=\sigma\left(\{\alpha\circ p_{1}:\alpha\in\mathcal{L}\}\cup\{\beta\circ p_{2}:\beta\in\mathcal{L}\}\right)\\&=\sigma(\mathcal{L})\otimes\sigma(\mathcal{L})=\mathcal{B}_{W}\otimes\mathcal{B}_{W}=\mathcal{B}_{(W\times W)}.\end{aligned}$$

So in order to show $(\mu\times\mu)\circ R_{\pi/4}^{-1}=\mu\times\mu$ it suffices to show see item 7 of Theorem 2.2) $\widehat{\left[(\mu\times\mu)\circ R_{\pi/4}^{-1}\right]}=\widehat{\mu\times\mu}$ on $\mathcal{L}\oplus\mathcal{L}$ which we now do.

Let $\psi(x,y)=\alpha(x)+\beta(y)$ with $\alpha,\beta\in\mathcal{L}$. Then

$$\psi\circ R_{\pi/4}(x,y)=\frac{1}{\sqrt{2}}\psi(x-y,x+y)=\frac{1}{\sqrt{2}}\left[\alpha(x)+\beta(x)+\beta(y)-\alpha(y)\right]$$

so that

$$\begin{aligned}&\widehat{\left[(\mu\times\mu)\circ R_{\pi/4}^{-1}\right]}(\psi)\\&=\widehat{\mu\times\mu}\left(\psi\circ R_{\pi/4}\right)\\&=\int_{W\times W}\exp\left(i\frac{1}{\sqrt{2}}\left[\alpha(x)+\beta(x)+\beta(y)-\alpha(y)\right]\right)d\mu(x)\,d\mu(y)\\&=e^{-\frac{1}{4}\left[q(\alpha+\beta)+q(\beta-\alpha)\right]}=e^{-\frac{1}{4}\left[2q(\alpha)+2q(\beta)\right]}=\widehat{\mu\times\mu}(\psi).\end{aligned}$$

∎

Example 3.7 (Gaussian processes and Gaussian measures). Suppose that $\{Y_{t}\}_{0\leq t\leq T}$ is a stochastic process with continuous sample paths, $W:=C([0,T],\mathbb{R})$, and $\alpha_{t}(x)=x(t)$ for all $x\in W$ and $0\leq t\leq T$. Recall from Example 2.3 we know that $\sigma\left(\{\alpha_{t}\}_{t\in\mathbb{Q}\cap[0,T]}\right)=\mathcal{B}_{W}$. Therefore we may view Y as a W – valued random variable and define $\mu:=\operatorname{Law}(Y_{\cdot})$ as a measure on $(W,\mathcal{B}_{W})$.

If we further assume that $\{Y_{t}\}_{0\leq t\leq T}$ is a mean zero Gaussian process, then $(W,\mathcal{B}_{W},\mu)$ is a Gaussian measure space. To prove this apply Corollary 3.6 with $\mathcal{L}:=\operatorname{span}\{\alpha_{t}:0\leq t\leq T\}$. Note that for $\alpha=\sum_{i=1}^{n}a_{i}\alpha_{t_{i}}$ in $\mathcal{L}$ we have that $\alpha\overset{d}{=}\sum_{i=1}^{n}a_{i}Y_{t_{i}}$, i.e. $\operatorname{Law}_{\mu}(\alpha)=\operatorname{Law}_{P}\left(\sum_{i=1}^{n}a_{i}Y_{t_{i}}\right)$. By assumption $\sum_{i=1}^{n}a_{i}Y_{t_{i}}$ is Gaussian and therefore so is α.

Example 3.8 (Brownian Motion). Suppose that $\{B_{t}\}_{t\geq 0}$ is a Brownian motion. Then B induces a Gaussian measure, μ on $W:=\{x\in C([0,T],\mathbb{R}):x(0)=0\}$ which is uniquely specified by its covariances;

$$\int_W x(s)\, x(t)\, d\mu(x) = s \wedge t \text{ for all } 0 \le s,t \le T.$$

Let us end this chapter now with a simple but useful lemma regarding the convergence properties of jointly Gaussian random variables.

Lemma 3.9. *If for $\{Y_n\}_{n=1}^\infty \cup \{Y\}$ are random variables such that $\{Y_n, Y\}$ is a mean zero Gaussian vector for each n and $Y_n \xrightarrow{P} Y$ as $n \to \infty$, then $Y_n \to Y$ in $L^p(P)$ for all $1 \le p < \infty$ and $e^{Y_n} \to e^Y$ in $L^2(P)$.*

Proof. By assumption, $aY_n + bY$ is Gaussian for all $a, b \in \mathbb{R}$ and therefore

$$\begin{aligned}\mathbb{E}e^{(aY_n+bY)} &= e^{\frac{1}{2}\mathbb{E}(aY_n+bY)^2} \text{ and}\\ \mathbb{E}e^{i(aY_n+bY)} &= e^{-\frac{1}{2}\mathbb{E}(aY_n+bY)^2}.\end{aligned}$$

In particular,

$$\mathbb{E}\left[e^{i(Y-Y_n)}\right] = \exp\left(-\frac{1}{2}\mathbb{E}(Y-Y_n)^2\right)$$

which along with the DCT shows

$$\delta_n := \mathbb{E}(Y-Y_n)^2 = -2\ln \mathbb{E}\left[e^{i(Y-Y_n)}\right] \to 0 \text{ as } n \to \infty.$$

If N is a standard normal random variable then $Y - Y_n \overset{d}{=} \sqrt{\delta_n} N$ and therefore,

$$\mathbb{E}(Y-Y_n)^p = \delta_n^{p/2}\mathbb{E}[N^p] \to 0 \text{ as } n \to \infty$$

which is the first assertion. The second assertion now follows by the following simple computation;

$$\begin{aligned}\mathbb{E}\left(\left|e^Y - e^{Y_n}\right|^2\right) =& \mathbb{E}\left(e^{2Y} + e^{2Y_n} - 2e^{Y_n+Y}\right)\\ =& e^{\frac{1}{2}4\mathbb{E}Y^2} + e^{\frac{1}{2}4\mathbb{E}Y_n{}^2} - 2e^{\frac{1}{2}\mathbb{E}(Y_n+Y)^2}\\ &\to e^{2\mathbb{E}Y^2} + e^{2\mathbb{E}Y^2} - 2e^{\frac{1}{2}\mathbb{E}(2Y)^2} = 0.\end{aligned}$$

■

Corollary 3.10. *If $\{Y_t\}_{t\ge 0}$ is a Gaussian process with continuous sample paths then $\mathbb{R}_+ \ni t \to Y_t \in L^p(P)$ is continuous for all $1 \le p < \infty$.*

Exercise 3.2. Suppose that $\{Y_n\}_{n=1}^\infty$ is a sequence of random variables such that (Y_n, Y_m) is a mean zero Gaussian random vector for all $m \ne n$ and Y_n converges to some Y in probability. Show that Y is Gaussian and that (Y, Y_n) is again a mean zero Gaussian random vector for each $n \in \mathbb{N}$.

Gaussian basics I

In this chapter suppose that W is a finite dimensional real Banach space and $q : W^* \times W^* \to \mathbb{R}$ is a non-negative symmetric quadratic form on W^*.

Notation 4.1 *Let*

$$\operatorname{Nul}(q) := \{\alpha \in W^* : q(\alpha) = 0\} \tag{4.1}$$

be the null space of q and

$$H = H_q = \operatorname{Nul}(q)^0 := \{\xi \in W : \langle \alpha, \xi \rangle = 0 \quad \forall\ \alpha \in \operatorname{Nul}(q)\} \tag{4.2}$$

be the backwards annihilator of $\operatorname{Nul}(q)$. *(If q is non-degenerate, i.e.* $\operatorname{Nul}(q) = \{0\}$, *then $H = W$.) We call H the **"horizontal space"** associated to q. We may also refer to H as the Cameron-Martin space associated to q.*

Lemma 4.2. *There is a unique inner product, $(\cdot,\cdot)_H$, on H such that for any orthonormal base $\{h_k\}_{k=1}^m$ $(m := \dim(H))$ of H we have*

$$q(\alpha, \beta) = \sum_{k=1}^m \langle \alpha, h_k \rangle \langle \beta, h_k \rangle \text{ for all } \alpha, \beta \in W^*. \tag{4.3}$$

In particular

$$q(\alpha) = (\alpha, \alpha)_q = \sum_{k=1}^m |\langle \alpha, h_k \rangle|^2. \tag{4.4}$$

Moreover, let

$$\|x\|_H := \sup_{\alpha \in W^*} \frac{|\alpha(x)|}{\sqrt{q(\alpha)}} \text{ (with } 0/0 := 0), \tag{4.5}$$

then

$$H = \{h \in W : \|h\|_H < \infty\} \tag{4.6}$$

and $\|h\|_H^2 = (h,h)_H$ for all $h \in H$.

Proof. The form q descends to a strictly positive definite quadratic form, $\bar{q}$, on $W^*/\operatorname{Nul}(q)$ and the map

$$W^*/\operatorname{Nul}(q) \ni (\alpha + \operatorname{Nul}(q)) \to \alpha|_H \in H^* \tag{4.7}$$

is a linear isomorphism of vectors spaces.[1] Using this isometry, $\bar{q}$ induces an inner product, $(\cdot,\cdot)_{H^*}$, on H^* and hence, by the Riesz theorem, an inner product, $(\cdot,\cdot)_H$, on H. Suppose that $\{h_k\}_{k=1}^m$ is any orthonormal basis of $(H, (\cdot,\cdot)_H)$ and $\alpha, \beta \in W^*$. Then

$$q(\alpha, \beta) = \bar{q}(\alpha + \operatorname{Nul}(q), \beta + \operatorname{Nul}(q)) = (\alpha|_H, \beta|_H)_{H^*} = \sum_{k=1}^m \langle \alpha, h_k \rangle \langle \beta, h_k \rangle.$$

If $x \notin H$ there exists $\alpha \in \operatorname{Nul}(q)$ such that $\alpha(x) \neq 0$ and therefore from Eq. (4.5),

$$\|x\|_H \geq \frac{|\alpha(x)|}{\sqrt{q(\alpha)}} = \frac{|\alpha(x)|}{0} = \infty.$$

For $h \in H$ we have and $\alpha \in W^*$ we have $h = \sum_{k=1}^m (h_k, h)_H h_k$ and so

$$|\alpha(h)|^2 = \left| \sum_{k=1}^m (h_k, h)_H \alpha(h_k) \right|^2 \leq \sum_{k=1}^m (h_k, h)_H^2 \cdot \sum_{k=1}^m \alpha(h_k)^2 = (h,h)_H q(\alpha)$$

with equality if we choose $\alpha \in W^*$ such that $\alpha(h_k) = (h_k, h)$ for all $k = 1, \ldots, m$. From these observations it follows that

$$\|h\|_H^2 := \sup_{\alpha \in W^*} \frac{|\alpha(h)|^2}{q(\alpha)} = (h,h)_H \text{ for all } h \in H.$$

■

[1] Here is the argument. Let $N := \dim W$ and $\{\varepsilon^i\}_{i=1}^N$ be a basis for W^* such that $\{\varepsilon^i : m < i \leq N\}$ is a basis for K and let $\{e_i\}_{i=1}^N$ be the corresponding dual basis. Since for any $x \in W$ we have $x = \sum_{i=1}^N \langle \varepsilon^i, x \rangle e_i$ and

$$H = \left\{ x \in W : \left\langle \varepsilon^i, x \right\rangle = 0 \text{ for } m < i \leq N \right\},$$

it follows that $H = \operatorname{span}\{e_i\}_{i=1}^m$. So letting $R : W^* \to H^*$ be the restriction map, $Ra = a|_H$, it follows that $Ra = 0$ iff $\langle a, e_i \rangle = 0$ for $1 \leq i \leq n$ iff $a \in \operatorname{span}\{\varepsilon^i : m < i \leq N\}$ iff $a \in K$. Thus it follows that Eq. (4.7) indeed defines an isomorphism of vector spaces.

Theorem 4.3. *As above assume that* $\dim W < \infty$ *and* $q : W^* \times W^* \to \mathbb{R}$ *is a non-negative symmetric quadratic form and let* $\{h_k\}_{k=1}^m$ *be an orthonormal basis for* $H \subset W$ *as in Lemma 4.2. If* $\{N_k\}_{k=1}^m$ *is any i.i.d. sequence of standard normal random variables, then* $\mu := \mathrm{Law}\,(\sum_{k=1}^m N_k h_k)$ *is a Gaussian measure with* $\hat{\mu} = e^{-q/2}$, *i.e. Eq. (3.1) holds. Moreover, for every bounded measurable function,* $f : W \to \mathbb{R}$ *we have*

$$\int_W f d\mu = \int_H f d\mu = \frac{1}{Z}\int_H f(h)\exp\left(-\frac{1}{2}\|h\|_H^2\right)dh \tag{4.8}$$

where dh *denotes Lebesgue measure on* H *and* $Z = (2\pi)^{m/2}$ *is the normalization constant so that*

$$\frac{1}{Z}\int_H \exp\left(-\frac{1}{2}\|h\|_H^2\right)dh = 1.$$

Proof. Let $S := \sum_{k=1}^m N_k h_k$ and $\alpha \in W^*$. Then

$$\begin{aligned}\hat{\mu}(\alpha) &= \int_W e^{i\alpha(x)}d\mu(x) = \mathbb{E}\left[e^{i\alpha(S)}\right] = \mathbb{E}\left[e^{i\sum_{k=1}^m N_k\alpha(h_k)}\right]\\ &= \exp\left(-\frac{1}{2}\mathrm{Var}\left(\sum_{k=1}^m N_k\alpha(h_k)\right)\right) = \exp\left(-\frac{1}{2}\sum_{k=1}^m \alpha^2(h_k)\right) = e^{-\frac{1}{2}q(\alpha)}.\end{aligned}$$

Similarly,

$$\int_W f d\mu = \mathbb{E}[f(S)] = \int_{\mathbb{R}^m}\frac{1}{(2\pi)^{m/2}} f\left(\sum_{k=1}^m x_k h_k\right)e^{-\frac{1}{2}\sum_{k=1}^m x_k^2}dx$$

from which Eq. (4.8) easily follows by making the orthogonal change of variables, $h = \sum_{k=1}^m x_k h_k$. ∎

Remark 4.4. Notice from the above theorem we learn that $\mu(H_\mu) = 1$ when $\dim H_\mu < \infty$. It turns out that the same ideas of this proof work for infinite dimensional Gaussian measure space $(W, \mathcal{B}_W, \mu)$ as well. In this case we will have

$$S = \sum_{k=1}^\infty N_k h_k \text{ convergent in } W$$

and $\mathrm{Law}\,(S) = \mu$, see Theorem 6.4 and Theorem 8.6 below. However, let us note that S is convergent in H iff $\sum_{k=1}^\infty N_k^2 < \infty$ while by the strong law of large numbers we know that

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^m N_k^2 = 1$$

and therefore $\sum_{k=1}^\infty N_k^2 = \infty$ a.s. This is a strong indication that $\mu(H_\mu) = 0$ which is indeed always true when $\dim H_\mu = \infty$, see Lemma 6.4 below for more details.

Theorem 4.5 (Baby Fernique's Theorem). *Suppose that* (W,μ) *is a Gaussian measure space with* $\dim W < \infty$, *then there exists* $\varepsilon > 0$ *such that*

$$\int_W e^{\varepsilon\|x\|_W^2}d\mu(x) < \infty.$$

(The general version of this theorem removes the restriction that $\dim W < \infty$.*)*

Proof. Recall that $q(\alpha) \le C^2\|\alpha\|_W^2$ for some $C < \infty$ and therefore,

$$\|h\|_H = \sup_{\alpha\in W^*}\frac{|\alpha(h)|}{\sqrt{q(\alpha)}} \ge \sup_{\alpha\in W^*}\frac{|\alpha(h)|}{C\|\alpha\|_W} = \frac{1}{C}\|h\|_W.$$

Thus it follows that

$$\begin{aligned}\int_W e^{\varepsilon\|x\|_W^2}d\mu(x) &= \int_H e^{\varepsilon\|x\|_W^2}d\mu(x)\\ &\le \int_H e^{C\varepsilon\|h\|_H^2}d\mu(h) = \left(\frac{1}{2\pi}\right)^{m/2}\int_H e^{C\varepsilon\|h\|_H^2}\exp\left(-\frac{1}{2}\|h\|_H^2\right)dh\\ &= \left(\frac{1}{2\pi}\right)^{m/2}\int_H e^{C\varepsilon\|h\|_H^2}\exp\left(-\frac{1}{2}(1-2C\varepsilon)\|h\|_H^2\right)dh\\ &= \left(\frac{1}{2\pi}\right)^{m/2}\left(\frac{2\pi}{1-2C\varepsilon}\right)^{m/2} = \left(\frac{1}{1-2C\varepsilon}\right)^{m/2} < \infty\end{aligned}$$

which is valid provided that $2C\varepsilon < 1$. ∎

Theorem 4.6 (Characterization of H). *Suppose that* (W,μ) *is a Gaussian measure space with* $\dim W < \infty$ *and define* $J = J_\mu : L^2(\mu) \to W$ *by*

$$Jf := \int_W f(x)\,x\,d\mu(x).$$

Further let K *be the subspace of* $L^2(\mu)$ *defined by*

$$K := \left\{[\alpha] \in L^2(\mu) : \alpha \in W^*\right\}.$$

Then $J(L^2(\mu)) = H$ *and* $J_K := J|_K : K \to H$ *is a unitary map such that*

$$(h, J\alpha)_H = \alpha(h) \quad \forall\, \alpha \in W^* \text{ and } h \in H. \tag{4.9}$$

In particular,

$$H_\mu = \left\{ \int_W f(x)\, x \; d\mu(x) : f \in K \right\} = \left\{ \int_W f(x)\, x \; d\mu(x) : f \in L^2(\mu) \right\}.$$

The adjoint map for $J^* : H \to L^2(\mu)$ is given by $J^* = J_K^{-1}$ an satisfies

$$(J^* h)(x) = 1_H(x)\,(h, x)_H \text{ for } \mu - a.e.\ x. \tag{4.10}$$

(The value of $J^* h$ on $W \setminus H$ is not uniquely determined as this is a set of μ – measure zero. Notice that $JJ^* = id_H$.)

Proof. Since $\mu(W \setminus H) = 0$ we also have

$$Jf = \int_H f(x)\, x \; d\mu(x) \in H$$

which shows that $J\left(L^2(\mu)\right) \subset H$. Moreover if $\alpha \in W^*$, then using the notation in the proof of Theorem 4.3 we find

$$J\alpha = \mathbb{E}\left[\alpha(S)\, S\right] = \mathbb{E}\left[\sum_{k,l=1}^m N_k N_l \alpha(h_k)\, h_l\right] = \sum_{k1}^m \alpha(h_k)\, h_k.$$

As we may choose $\alpha \in W^*$ such that $(\alpha(h_1), \dots, \alpha(h_m))$ is any m – tuple we please, we may now conclude that $JK = H$. Moreover J is unitary since

$$\|J\alpha\|_H^2 = \sum_{k=1}^m \alpha(h_k)^2 = q(\alpha) = \|\alpha\|^2_{L^2(\mu)}.$$

If $\alpha \in W^*$ and $h = Jf \in H$, then

$$\begin{aligned}(h, J\alpha)_H &= (Jf, J\alpha)_H = (f, \alpha)_{L^2(\mu)} \\ &= \int_W f(x)\, \alpha(x) \; d\mu(x) \\ &= \alpha\left(\int_W f(x)\, x \; d\mu(x)\right) = \alpha(Jf) = \alpha(h).\end{aligned}$$

which is Eq. (4.9).

Now suppose that $h = J\alpha$ with $\alpha \in W^*$ and $f \in L^2(\mu)$, then

$$(J^* h, f)_{L^2(\mu)} = (h, Jf)_H = (J\alpha, Jf)_H = (\alpha, f)_{L^2(\mu)} = \left(J_K^{-1} h, f\right)$$

which shows $J^* h = J_K^{-1} h$. If $k \in H$, then

$$(J^* h)(k) \overset{\text{a.s.}}{=} \alpha(k) = (J\alpha, k)_H = (h, k)_H$$

from which gives Eq. (4.10). ■

Notation 4.7 *We sometimes abuse notation and simply write $(h, x)_H$ for $(J^* h)(x)$ even when H is a proper subspace of W.*

Theorem 4.8 (Baby Cameron Martin Theorem). *For $h \in W$ let $\mu_h(A) := \mu(A - h)$. The $\mu_h \ll \mu$ iff $h \in H$ and if $h \in H$ then*

$$\frac{d\mu_h}{d\mu} = \exp\left(J^* h - \frac{1}{2}\|h\|_H^2\right). \tag{4.11}$$

Proof. This is a simple matter of using the change of variables formula. Let $h \in H$ and $f : W \to \mathbb{R}$ be a bounded measurable function. Then

$$\begin{aligned}\int_W f(x)\, d\mu_h(x) &= \int_W f(x+h)\, d\mu(x) = \int_H f(x+h)\, d\mu(x) \\ &= \frac{1}{Z}\int_H f(x+h) \exp\left(-\frac{1}{2}\|x\|_H^2\right) dx \\ &= \frac{1}{Z}\int_H f(x) \exp\left(-\frac{1}{2}\|x-h\|_H^2\right) dx \\ &= \int_H f(x) \exp\left((h, x)_H - \frac{1}{2}\|h\|_H^2\right) d\mu(x)\end{aligned}$$

from which Eq. (4.11) follows. If $h \notin H$, then $\mu_h(H + h) = \mu(H + h - h) = \mu(H) = 1$ while $\mu(H) = 1$ and $H \cap (H + h) = \emptyset$. ■

5

The Heat Equation Interpretation

For this section, suppose that $(W, \mathcal{B}_W, \mu)$ is a Gaussian measure space with $\hat{\mu} = e^{-q/2}$ and $H = H_\mu$.

Lemma 5.1. *Suppose $\alpha_i \in W^*$ for $1 \le i \le n$, $F \in C^\infty(\mathbb{R}^n)$, and*

$$f = F(\alpha_1, \ldots, \alpha_n) \in \mathcal{F}C^\infty(\alpha_1, \ldots, \alpha_n) \subset \mathcal{F}C^\infty(W^*). \tag{5.1}$$

Then for any choice of orthonormal basis $S \subset H$ we have

$$\sum_{h \in S} \partial_h^2 f = \sum_{i,j=1}^n q(\alpha_i, \alpha_j)(\partial_j \partial_i F)(\alpha_1, \ldots, \alpha_n). \tag{5.2}$$

where $\partial_h f(\omega) := \frac{d}{dt}|_0 f(\omega + th)$.

Proof. The proof consists of the chain rule along with the fact that $q(\alpha, \beta) = \sum_{h \in S} \alpha(h)\beta(h)$. Indeed,

$$\partial_h f = \sum_{i=1}^n (\partial_i F)(\alpha_1, \ldots, \alpha_n) \cdot \alpha_i(h)$$

$$\partial_h^2 f = \sum_{i,j=1}^n (\partial_j \partial_i F)(\alpha_1, \ldots, \alpha_n) \cdot \alpha_i(h)\alpha_j(h)$$

and therefore and

$$\sum_{h \in S} \partial_h^2 f = \sum_{i,j=1}^n (\partial_j \partial_i F)(\alpha_1, \ldots, \alpha_n) \cdot \sum_{h \in S} \alpha_i(h)\alpha_j(h)$$

$$= \sum_{i,j=1}^n q(\alpha_i, \alpha_j)(\partial_j \partial_i F)(\alpha_1, \ldots, \alpha_n).$$

■

Remark 5.2. Equation (5.2) demonstrates that its left member is independent of the choice of orthonormal basis, S, for H while its right member is independent of how f is represented in the form of Eq. (5.1). For this reason the following definition makes sense. (Also see Exercise 5.1.)

Definition 5.3. *Associated to μ is the second order differential operator $L = L_\mu$ acting on $\mathcal{F}C^\infty(W^*)$ defined by either;*

$$L_\mu f = \sum_{h \in S} \partial_h^2 f \text{ for all } f \in \mathcal{F}C^\infty(W^*)$$

or by

$$L_\mu [F(\alpha_1, \ldots, \alpha_n)] = \sum_{i,j=1}^n q(\alpha_i, \alpha_j)(\partial_j \partial_i F)(\alpha_1, \ldots, \alpha_n)$$

where S is any orthonormal basis for H and $\{\alpha_i\}_{i=1}^n \subset W^*$ and $F \in C^\infty(\mathbb{R}^n)$ are arbitrary.

Exercise 5.1. Show that

$$L_\mu f(x) = \int_W (\partial_y^2 f)(x)\, d\mu(y) \text{ for all } x \in W.$$

This gives another proof that L_μ is well defined.

Remark 5.4. We may recover $q = q_\mu$ from $L := L_\mu$ since for $\alpha_1, \alpha_2 \in W^*$ we have with $F(x_1, x_2) = x_1 x_2$ that

$$L(\alpha_1 \cdot \alpha_2) = \sum_{i,j=1}^2 q(\alpha_i, \alpha_j)(\partial_j \partial_i F)(\alpha_1, \ldots, \alpha_n) = 2q(\alpha_1, \alpha_2).$$

In this way we see that the map $q \to L^q$ is injective.

Lemma 5.5. *Suppose that $N = \dim W < \infty$ and L is a constant coefficient purely second order semi-elliptic differential operators, i.e. $L = \sum g^{ij} \partial_{e_i} \partial_{e_j}$ for some basis $\{e_i\}_{i=1}^N$ of W and $(g^{ij}) \ge 0$. Then $L = L_\mu$ where μ is the unique Gaussian measure on W such that $\hat{\mu} = e^{-q/2}$ where*

$$q(\alpha, \beta) := \frac{1}{2} L(\alpha\beta) = \sum g^{ij} \langle \alpha, e_i \rangle \langle \beta, e_j \rangle. \tag{5.3}$$

This allows us to conclude that the map $q \to L^q$ is a one to one correspondence between the non-negative quadratic forms on W^* and the the constant coefficient purely second order semi-elliptic differential operators acting on $C^2(W)$.

Proof. It is clear that q defined by Eq. (5.3) is a non-negative quadratic form on W^* and therefore

$$q(\alpha,\beta)=\sum_{k=1}^{m}\langle\alpha,h_k\rangle\langle\beta,h_k\rangle \tag{5.4}$$

for some linearly independent subset of W. Letting $\{\varepsilon^i\}_{i=1}^N$ be the dual basis to $\{e_i\}_{i=1}^N$ it follows by comparing Eqs. (5.3) and (5.4) with $\alpha=\varepsilon^i$ and $\beta=\varepsilon^j$ that

$$g^{ij}=\sum_{k=1}^{m}\langle\varepsilon^i,h_k\rangle\langle\varepsilon^j,h_k\rangle.$$

Since

$$\sum_i\langle\varepsilon^i,h_k\rangle\partial_{e_i}=\partial_{\sum_i\langle\varepsilon^i,h_k\rangle e_i}=\partial_{h_k}$$

we may now conclude,

$$L=\sum_{i,j}g^{ij}\partial_{e_i}\partial_{e_j}=\sum_{k=1}^{m}\sum_{i,j}\langle\varepsilon^i,h_k\rangle\langle\varepsilon^j,h_k\rangle\partial_{e_i}\partial_{e_j}=\sum_{k=1}^{m}\partial_{h_k}^2=L_q.$$

■

Definition 5.6. *Given $\mathcal{L}\subset W^*$ let $\mathcal{P}(\mathcal{L})=\mathbb{R}[\mathcal{L}]$ denote the space polynomial functions on W based on $\mathcal{L}$. Thus $f\in\mathcal{P}(\mathcal{L})$ iff there exists $n\in\mathbb{N}$, $\alpha_i\in\mathcal{L}$ for $1\le i\le n$, and a polynomial function $p:\mathbb{R}^n\to\mathbb{R}$ such that $f=p(\alpha_1,\dots,\alpha_n)$.*

Theorem 5.7 (Heat Interpretation). *Let $(W,\mathcal{B}_W,\mu)$ be a Gaussian space and $L=L_\mu$. Then*

$$\int_W f\left(x+\sqrt{t}y\right)d\mu(y)=\left(e^{tL/2}f\right)(x)\ \text{ for all } f\in\mathcal{P}(W^*) \tag{5.5}$$

where

$$e^{tL/2}f:=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{tL}{2}\right)^n f.\qquad (\text{finite sum}) \tag{5.6}$$

Proof. For $\alpha\in W^*$ and $h\in W$ we have, $\partial_h e^{i\alpha}=i\alpha(h)e^{i\alpha}$ and therefore it follows that

$$Le^{i\alpha}=\sum_{h\in S}(i\alpha(h))^2\cdot e^{i\alpha}=-q(\alpha)e^{i\alpha}$$

and therefore

$$e^{tL/2}e^{i\alpha}=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left(\frac{L}{2}\right)^n e^{i\alpha}=e^{-tq(\alpha)/2}e^{i\alpha}$$

while

$$\int_W e^{i\alpha\left(x+\sqrt{t}y\right)}d\mu(y)=e^{i\alpha(x)}\int_W e^{i\sqrt{t}\alpha(y)}d\mu(y)$$
$$=e^{i\alpha(x)}e^{-\frac{1}{2}q\left(\sqrt{t}\alpha\right)}=e^{-tq(\alpha)/2}e^{i\alpha(x)}.$$

So we have shown

$$\left(e^{tL/2}e^{i\alpha}\right)(x)=\int_W e^{i\alpha\left(x+\sqrt{t}y\right)}d\mu(y).$$

Differentiating this equation in α relative to $\alpha_1,\dots,\alpha_n\in W^*$ we find,

$$e^{tL/2}\left[(i\alpha_1\dots i\alpha_n)e^{i\alpha}\right]=\partial_{\alpha_1}\dots\partial_{\alpha_n}e^{tL/2}e^{i\alpha}$$
$$=\int_W i\alpha_1\left(\cdot+\sqrt{t}y\right)\dots i\alpha_n\left(\cdot+\sqrt{t}y\right)e^{i\alpha\left(\cdot+\sqrt{t}y\right)}d\mu(y).$$

Evaluating this equation at $\alpha=0\in W^*$ then shows

$$e^{L/2}(\alpha_1\dots\alpha_n)=\int_W\alpha_1\left(\cdot+\sqrt{t}y\right)\dots\alpha_n\left(\cdot+\sqrt{t}y\right)\,d\mu(y).$$

You are asked to justify these computations in Exercise 5.3 and 5.6 below. ■

Exercise 5.2. Show

$$e^{tL/2}\left[(i\alpha_1\dots i\alpha_n)e^{i\alpha}\right]=\partial_{\alpha_1}\dots\partial_{\alpha_n}e^{tL/2}e^{i\alpha}. \tag{5.7}$$

Hint: you might use the Cauchy estimates to simplify your life.

Exercise 5.3. Use the following outline in order to prove Eq. (5.5) with $t=1$. Let $\alpha,\alpha_1,\dots,\alpha_k\in W^*$ for some $k\in\mathbb{N}$.

1. Show for all $z\in\mathbb{R}$ that

$$\int_W\alpha^k e^{iz\alpha}d\mu=e^{\frac{L}{2}}\left((i\alpha)^k e^{iz\alpha}\right)(0) \tag{5.8}$$

by differentiating the identity

$$\int_W e^{iz\alpha}d\mu=e^{-z^2q(\alpha)/2}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{q(\alpha)}{2}\right)^n z^{2n} \tag{5.9}$$

$k-$ times in z making use of the identity

$$\left(-\frac{q(\alpha)}{2}\right)^n=\left[\left(\frac{L}{2}\right)^n e^{iz\alpha}\right](0) \tag{5.10}$$

2. Taking $z=0$ in Eq. (5.8) shows

$$\int_W \alpha^k d\mu = e^{\frac{L}{2}}\left(\alpha^k\right)(0).$$

Polarize this identity by computing $\partial_{\alpha_1}\dots\partial_{\alpha_k}$ of both sides in order to conclude

$$\int_W \alpha_1\dots\alpha_k \; d\mu = \left[e^{\frac{L}{2}}\left(\alpha_1\dots\alpha_k\right)\right](0). \tag{5.11}$$

3. Use Eq. (5.11) along with linearity and the translation invariance of L (i.e. $L\left[f\left(x+\cdot\right)\right]=\left(Lf\right)\left(x+\cdot\right)$) to prove Eq. (5.5).

Corollary 5.8. *For all $g\in\mathcal{P}\left(W^*\right)$, $x\in W$, and $t\geq 0$,*

$$\int_W g\left(x+\sqrt{t}y\right)d\mu\left(y\right)=\left(e^{tL/2}g\right)(x). \tag{5.12}$$

Proof. When $t=0$ both sides of Eq. (5.12) are equal to $g\left(x\right)$ and so we may assume that $t>0$. Applying Eq. (5.5) with $f\left(x\right)=g\left(\sqrt{t}x\right)$ and using $\left(Lf\right)(x)=t\left(Lg\right)\left(\sqrt{t}x\right)$ shows,

$$\int_W g\left(\sqrt{t}x+\sqrt{t}y\right)d\mu\left(y\right)=\left(e^{L/2}f\right)(x)=\left(e^{tL/2}g\right)\left(\sqrt{t}x\right).$$

The proof is then complete by replacing x by $\frac{1}{\sqrt{t}}x$ in this equation. ■

Exercise 5.4 (Integration by Parts). Suppose that $(x,y)\in\mathbb{R}\times\mathbb{R}^{d-1}\to f(x,y)\in\mathbb{C}$ and $(x,y)\in\mathbb{R}\times\mathbb{R}^{d-1}\to g(x,y)\in\mathbb{C}$ are measurable functions such that for each fixed $y\in\mathbb{R}^d$, $x\to f(x,y)$ and $x\to g(x,y)$ are continuously differentiable. Also assume $f\cdot g$, $\partial_x f\cdot g$ and $f\cdot\partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R}\times\mathbb{R}^{d-1}$, where $\partial_x f(x,y):=\frac{d}{dt}f(x+t,y)|_{t=0}$. Show

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}}\partial_x f(x,y)\cdot g(x,y)dxdy=-\int_{\mathbb{R}\times\mathbb{R}^{d-1}}f(x,y)\cdot\partial_x g(x,y)dxdy. \tag{5.13}$$

Hints: Let $\psi\in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of $0\in\mathbb{R}$ and set $\psi_\varepsilon(x)=\psi(\varepsilon x)$. First verify Eq. (5.13) with $f(x,y)$ replaced by $\psi_\varepsilon(x)f(x,y)$ by doing the x – integral first. Then use the dominated convergence theorem to prove Eq. (5.13) by passing to the limit, $\varepsilon\downarrow 0$.

Solution to Exercise (5.4). By assumption, $\partial_x\left[\psi_\varepsilon(x)f(x,y)\right]\cdot g(x,y)$ and $\psi_\varepsilon(x)f(x,y)\partial_x g(x,y)$ are in $L^1(\mathbb{R}^n)$, so we may use Fubini's theorem and follow the hint to learn

$$\begin{aligned}\int_{\mathbb{R}^{n-1}}dy\int_{\mathbb{R}}\partial_x\left[\psi_\varepsilon(x)f(x,y)\right]\cdot g(x,y)dx\\ =-\int_{\mathbb{R}^{n-1}}dy\int_{\mathbb{R}}\left[\psi_\varepsilon(x)f(x,y)\right]\cdot\partial_x g(x,y)dx,\end{aligned} \tag{5.14}$$

wherein we have done and integration by parts. (There are no boundary terms because ψ_ε is compactly supported.) Now

$$\begin{aligned}\partial_x\left[\psi_\varepsilon(x)f(x,y)\right]&=\partial_x\psi_\varepsilon(x)\cdot f(x,y)+\psi_\varepsilon(x)\partial_x f(x,y)\\ &=\varepsilon\psi'(\varepsilon x)f(x,y)+\psi_\varepsilon(x)\partial_x f(x,y)\end{aligned}$$

and by the dominated convergence theorem and the given assumptions we have, as $\varepsilon\downarrow 0$, that

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\varepsilon\psi'(\varepsilon x)f(x,y)\cdot g(x,y)dxdy\right|\leq C\varepsilon\int_{\mathbb{R}^n}\left|f(x,y)\cdot g(x,y)\right|dxdy\to 0\\ &\int_{\mathbb{R}^n}\psi_\varepsilon(x)\partial_x f(x,y)\cdot g(x,y)dxdy\to\int_{\mathbb{R}^n}\partial_x f(x,y)\cdot g(x,y)dxdy \text{ and}\\ &\int_{\mathbb{R}^n}\psi_\varepsilon(x)f(x,y)\cdot\partial_x g(x,y)dxdy\to\int_{\mathbb{R}^n}f(x,y)\cdot\partial_x g(x,y)dxdy\end{aligned}$$

where $C=\sup_{x\in\mathbb{R}}\left|\psi'(x)\right|$. Combining the last three equations with Eq. (5.14) shows

$$\int_{\mathbb{R}\times\mathbb{R}^{n-1}}\partial_x f(x,y)\cdot g(x,y)dxdy=-\int_{\mathbb{R}\times\mathbb{R}^{n-1}}f(x,y)\cdot\partial_x g(x,y)dxdy$$

as desired.

Exercise 5.5 (Gaussian Integration by parts). If $h\in H$ and $f\in\mathcal{P}\left(W^*\right)$, then

$$\int_W\partial_h f\left(x\right)d\mu\left(x\right)=\int_W\left(x,h\right)_H f\left(x\right)d\mu\left(x\right).$$

This formula actually holds for any $f\in C^1\left(W\right)$ such that f, $\partial_h f$, and $\left(\cdot,h\right)_H f$ are μ – integrable.

Exercise 5.6. Let $f\in C^2\left(W\right)$ such that f and its first and second derivatives grow (for example) at most exponentially at infinity. Show that

$$F\left(t,x\right):=\int_W f\left(x+\sqrt{t}y\right)d\mu\left(y\right)\ \forall\ t>0\text{ and }x\in W. \tag{5.15}$$

satisfies the heat equation,

$$\frac{\partial}{\partial t}F\left(t,x\right)=\frac{1}{2}\left(LF\right)\left(t,x\right). \tag{5.16}$$

Hint: use the fact that $\mu\left(W\setminus H\right)=0$ and the results of Exercise 5.5.

Definition 5.9 (Convolutions). *The convolution $\mu*\nu$ of two probability measures, μ and ν on $(W,\mathcal{B}_W)$ is the probability measure defined by*

$$\mu*\nu(A):=\int_{W\times W}1_A(x+y)\,d\mu(x)\,d\nu(y)$$

for all $A\in\mathcal{B}_W$. The convolution may also be written as;

$$\mu*\nu(A)=\int_W\nu(A-x)\,d\mu(x)=\int_W\mu(A-y)\,d\nu(y).$$

If $f:W\to\mathbb{C}$ is a bounded (or non-negative) measurable function we define

$$\mu*f(x)=\int_W f(x-y)\,d\mu(y).$$

It is a simple matter to check that $\mu*\nu$ is the unique probability measure on $(W,\mathcal{B}_W)$ such that

$$\int_W fd(\mu*\nu)=\int_{W\times W}f(x+y)\,d\mu(x)\,d\nu(y)$$

for all bounded measurable functions $f:W\to\mathbb{C}$. We also use below that

$$\int_W(f*\mu)\,d\nu=\int_{W\times W}f(x-y)\,d\mu(y)\,d\nu(x)=\int_W fd(\mu^r*\nu)$$

where $\mu^r(A):=\mu(-A)$ for all $A\in\mathcal{B}_W$.

Exercise 5.7. Suppose that μ and ν are two Gaussian measures on W. Show $\mu*\nu$ is again Gaussian with $q_{\mu*\nu}=q_\mu+q_\nu$ and $L_{\mu*\nu}=L_\mu+L_\nu$. (Hint: you might use Exercise 5.1 for the last assertion.)

Definition 5.10 (Dilating μ). *If $(W,\mathcal{B}_W,\mu)$ is a Gaussian probability space and $t>0$ let*

$$\mu_t(A):=\int_W 1_A\left(\sqrt{t}x\right)d\mu(x)=\mu\left(t^{-1/2}A\right)\ \text{for all } A\in\mathcal{B}_W.$$

When μ_t*f is defined we write

$$e^{tL/2}f=\mu_t*f.$$

It is easily verified that

$$\int_W fd\mu_t=\int_H f\left(\sqrt{t}y\right)d\mu(y)$$

for all bounded and measurable functions $f:W\to\mathbb{C}$. This result then implies $\hat{\mu}_t=e^{-tq/2}$ so that μ_t is again a Gaussian measure with $q_{\mu_t}=tq_\mu$. When $\dim W<\infty$, the change of variables formula shows that

$$d\mu_t(y)=1_H(y)\frac{1}{(2\pi t)^m}e^{-\frac{1}{2t}\|y\|_H^2}dy$$

where this last equation is short hand for

$$\mu_t(A)=\int_{A\cap H}\frac{1}{(2\pi t)^m}e^{-\frac{1}{2t}\|y\|_H^2}dy\ \text{for all } A\in\mathcal{B}_W.$$

Exercise 5.8. Let $(W,\mathcal{B}_W,\mu)$ be a Gaussian probability space and $\{\mu_t\}_{t>0}$ be as in Definition 5.10. Show $\mu_t*\mu_s=\mu_{t+s}$ for all $s,t>0$. In particular conclude that $\mu*\mu=\mu_2$.

Remark 5.11. All of the above results reflect the fact that

$$p_t(x):=\left(\frac{1}{2\pi t}\right)^{m/2}e^{-\frac{1}{2t}\|x\|_H^2}$$

is the fundamental solution to the heat equation (5.16) on $H\subset W$.

Example 5.12. Let $W=\mathbb{R}^2$ and

$$d\mu(x,y)=\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx\,\delta_0(dy)$$

and $f\in L^{1+}(\mu_t)$. Then $H=\mathbb{R}\times\{0\}$,

$$d\mu_t(x,y)=\frac{e^{-\frac{1}{2t}x^2}}{\sqrt{2\pi t}}dx\,\delta_0(dy)$$

and

$$\begin{aligned}(\mu_t*f)(a,b)&=\int_{\mathbb{R}^2}f((a,b)+(x,y))\frac{e^{-\frac{1}{2t}x^2}}{\sqrt{2\pi t}}dx\,\delta_0(dy)\\&=\int_{\mathbb{R}}f((a,b)+(x,0))\frac{e^{-\frac{1}{2t}x^2}}{\sqrt{2\pi t}}dx\\&=\int_{\mathbb{R}}f(x+a,b)\frac{e^{-\frac{1}{2t}x^2}}{\sqrt{2\pi t}}dx\\&=\int_{\mathbb{R}}f(x,b)\frac{e^{-\frac{1}{2t}(x-a)^2}}{\sqrt{2\pi t}}dx.\end{aligned}$$

By assumption we have

$$\infty > \int_{\mathbb{R}^2} f(x,y)^p \frac{e^{-\frac{1}{2t}x^2}}{\sqrt{2\pi t}} dx\, \delta_0(dy) = \int_{\mathbb{R}^2} f(x,0)^p \frac{e^{-\frac{1}{2t}x^2}}{\sqrt{2\pi t}} dx$$

and so there is no control of $f(x,y)$ for y not equal to zero. (Notice that $f = g$ $\gamma \otimes \delta_0$ – a.e. iff $f(\cdot,0) = g(\cdot,0)$ γ – a.e.) Hence it is clear that we can only expect $\mu_t * f(a,b)$ to make sense for $b = 0$. It his case it follows that for $a \in \mathbb{C}$ we have

$$(\mu_t * f)(a,0) = \int_{\mathbb{R}} f(x,0) \frac{e^{-\frac{1}{2t}(x-a)^2}}{\sqrt{2\pi t}} dx$$

which is analytic in a.

6

The Gaussian Basics II

Most of the finite dimensional statements above hold in infinite dimensions with a few exceptions. The most notable exception is that the Cameron-Martin space, H_μ in Theorem 6.1, has measure zero whenever $\dim H_\mu = \infty$. In what follows we frequently make use of the fact that

$$C_p := \int_W \|x\|_W^p \, d\mu(x) < \infty \text{ for all } 1 \leqslant p < \infty \tag{6.1}$$

which certainly holds in light of Fernique's Theorem 3.4. (Also see Skorohod's inequality,

$$\int_W e^{\lambda \|x\|_W} d\mu(x) < \infty \text{ for all } \lambda < \infty; \tag{6.2}$$

see for example [24, Theorem 3.2].)

6.1 Gaussian Structures

The next theorem forms the natural extension of Lemma 4.2 and Theorem 4.6 to the infinite dimensional setting. This material is well known and may be (mostly) found in the books [24] and [4]. In particular, the following theorem is based in part on [4, Lemma 2.4.1 on p. 59] and [4, Theorem 3.9.6 on p. 138].

Theorem 6.1. *Let W be a real separable Banach space and $(W, \mathcal{B}_W, \mu)$ be a Gaussian measure space as in Definition 3.1 For $x \in W$ let*

$$\|x\|_{H_\mu} := \sup_{u \in W^*} \frac{|u(x)|}{\sqrt{q(u,u)}} \text{ (with } 0/0 := 0) \tag{6.3}$$

and define the ***Cameron-Martin subspace,*** *$H = H_\mu \subset W$, by*

$$H = \{h \in W : \|h\|_H < \infty\}. \tag{6.4}$$

Then;

1. *$(H, \|\cdot\|_H)$ is a normed space such that*

$$\|h\|_W \leqslant \sqrt{C_2}\, \|h\|_H \quad \text{for all } h \in H, \tag{6.5}$$

where C_2 is as in (6.1.

2. *For $f \in \operatorname{Re} L^2(\mu)$, let*

$$Jf = J_\mu f := h_f := \int_W x \; f(x) \; d\mu(x) \in W, \tag{6.6}$$

where the integral is to be interpreted as a Bochner integral.[1]*. Then $Jf \in H$ and $J : \operatorname{Re} L^2(\mu) \to H$ is a contraction.*

3. *Now suppose that K is the closure of W^* in $\operatorname{Re} L^2(\mu)$. Then $J_K := J|_K : K \to H$ is an isometry.*
4. *Moreover, $J(K) = H$ and therefore $J_K : K \to H$ is an isometric isomorphism of real Banach spaces. Since K is a real Hilbert space it follows that $\|\cdot\|_H$ is a Hilbertian norm on H.*
5. *H is a separable Hilbert space and*

$$(Ju, h)_H = u(h) \text{ for all } u \in W^* \text{ and } h \in H. \tag{6.7}$$

6. *The quadratic form q may be computed as*

$$q(u,v) = \sum_{k=1}^{\infty} u(h_k) v(h_k) \tag{6.8}$$

where $\{h_k\}_{k=1}^\infty$ is any orthonormal basis for H.

7. *If q is non-degenerate, the Cameron-Martin space, H, is dense in W. Notice that by Item 1. $H \overset{i}{\hookrightarrow} W$ is continuous and hence so is $W^* \overset{i^{tr}}{\hookrightarrow} H^* \cong H = (\cdot,\cdot)_{H^*}$. Eq. (6.8 asserts that*

$$q = (\cdot,\cdot)_{H^*}\big|_{W^* \times W^*}.$$

8. *If $\{h_k\}_{k=1}^\infty$ is any orthonormal basis for H and $\{N_k\}_{k=1}^\infty$ are a sequence of i.i.d. standard normal random variables, then;*
   a) *$S := \sum_{k=1}^\infty N_k h_k$ converges in W a.s. and in $L^p(\mu; W)$ for all $1 \le p < \infty$. (Also see Theorem 8.6 below.)*

[1] Notice that

$$\int_X \|x f(x)\| \, d\mu(x) \le \sqrt{C_2}\, \|f\|_{L^2(\mu)} < \infty,$$

so the integrand is indeed Bochner integrable.

b) Law $(S) = \mu$.
c) $J\alpha = \sum_{k=1}^{\infty} \alpha(h_k) h_k$ *for all* $\alpha \in W^*$.
d) *If* $f \in L^2(\mu)$ *and* $h \in H$, *then*

$$(Jf, h)_H = \left(f, J_K^{-1} h\right)_{L^2(\mu)}. \tag{6.9}$$

Alternatively stated $J^* = J_K^{-1}$ *where* $J^* : H \to L^2(\mu)$ *is the adjoint of* J. *(Incidentally, as we will see later,* $J^* h$ *is the Wiener integral of* h *in the Brownian motion setting.)*
e) *If* $f \in L^2(\mu)$ *then* $Jf = \sum_{k=1}^{\infty} \left(f, J_K^{-1} h_k\right) h_k$.

9. *Let* $W_0 := \bar{H}^W$ *be the closure of* H *inside of* W. *Then* W_0 *is again a separable Banach space and* $\mu(W_0) = 1$. *If we let* $\mu_0 := \mu|_{\mathcal{B}_{W_0}}$ *then* $(W_0, \mathcal{B}_{W_0}, \mu_0)$ *is a non-degenerate Gaussian measure space. Moreover,* $q(u, v) = q_0(u|_{W_0}, v|_{W_0})$ *for all* $u, v \in W^*$ *where* $q_0 = q_{\mu_0}$, *i.e.*

$$q_0(u, v) := \int_{W_0} u(x) v(x) \, d\mu_0(x).$$

Proof. See Theorem 6.1 in [6]. We will prove each item in turn.

1. Using Eq. (3.3) we find

$$\|h\|_W = \sup_{u \in W^* \setminus \{0\}} \frac{|u(h)|}{\|u\|_{W^*}} \leqslant \sup_{u \in W^* \setminus \{0\}} \frac{|u(h)|}{\sqrt{q(u,u)/C_2}} \leqslant \sqrt{C_2} \|h\|_H,$$

and hence if $\|h\|_H = 0$ then $\|h\|_W = 0$ and so $h = 0$. If $h, k \in H$, then for all $u \in W^*$, $|u(h)| \leqslant \|h\|_H \sqrt{q(u)}$ and $|u(k)| \leqslant \|k\|_H \sqrt{q(u)}$ so that

$$|u(h+k)| \leqslant |u(h)| + |u(k)| \leqslant (\|h\|_H + \|k\|_H) \sqrt{q(u)}.$$

This shows $h + k \in H$ and $\|h+k\|_H \leqslant \|h\|_H + \|k\|_H$. Similarly, if $\lambda \in \mathbb{R}$ and $h \in H$, then $\lambda h \in H$ and $\|\lambda h\|_H = |\lambda| \|h\|_H$. Therefore H is a subspace of W and $(H, \|\cdot\|_H)$ is a normed space.

2. For $f \in \operatorname{Re} L^2(\mu)$ and $u \in W^*$

$$u(Jf) = u\left(\int_W x f(x) d\mu(x)\right) = \int_W u(x) f(x) d\mu(x) = (u, f)_{L^2(\mu)} \tag{6.10}$$

and hence

$$|u(Jf)| \leqslant \|u\|_{L^2(\mu)} \|f\|_{L^2(\mu)} = \sqrt{q(u)} \|f\|_{L^2(\mu)}$$

which shows that $Jf \in H$ and $\|Jf\|_H \leqslant \|f\|_{L^2(\mu)}$.

3. Let $f \in K$ and choose $u_n \in W^*$ such that $L^2(\mu) - \lim_{n\to\infty} u_n = f$. Then

$$\lim_{n\to\infty} \frac{|u_n(Jf)|}{\sqrt{q(u_n)}} = \lim_{n\to\infty} \frac{\left|\int_W u_n(x) f(x) d\mu(x)\right|}{\|u_n\|_{L^2(\mu)}} = \frac{\|f\|^2_{L^2(\mu)}}{\|f\|_{L^2(\mu)}} = \|f\|_{L^2(\mu)}$$

from which it follows $\|Jf\|_H = \|f\|_K$. So we have shown that $J : K \to H$ is an isometry.

4. We now wish to show that $J_K := J|_K : K \to H$ is surjective, i.e. given $h \in H$ we are looking for an $f \in K$ such that

$$h = Jf = \int_W x f(x) \, d\mu(x).$$

This will be the case iff

$$\hat{h}(u) := u(h) = u(Jf) = \int_W u(x) f(x) \, d\mu(x) = (u, f)_K \text{ for all } u \in W^*.$$

In order to see that this equation has a solution f, notice that

$$\left|\hat{h}(u)\right| = |u(h)| \leqslant \sqrt{q(u)} \|h\|_H = \|u\|_{L^2(\mu)} \|h\|_H = \|u\|_K \|h\|_H$$

for all $u \in W^*$ which is dense in K. Therefore $\hat{h}$ extends continuously to K and so by the Riesz representation theorem for Hilbert spaces, there exists an $f \in K$ such that $\hat{h}(u) = (u, f)_K$ for all $u \in W^* \subset K$.

5. H is a separable since it is unitarily equivalent to $K \subset L^2(W, \mathcal{B}, \mu)$ and $L^2(W, \mathcal{B}, \mu)$ is separable. Suppose that $u \in W^*$, $f \in K$ and $h = Jf \in H$. Then

$$\begin{aligned}(Ju, h)_H &= (Ju, Jf)_H = (u, f)_K \\ &= \int_W u(x) f(x) d\mu(x) = u\left(\int_W x f(x) d\mu(x)\right) \\ &= u(Jf) = u(h).\end{aligned}$$

6. Let $\{h_i\}_{i=1}^{\infty}$ be an orthonormal basis for H, then for $u, v \in W^*$,

$$\begin{aligned}q(u, v) = (u, v)_K = (Ju, Jv)_H &= \sum_{i=1}^{\infty} (Ju, h_i)_H (h_i, Jv)_H \\ &= \sum_{i=1}^{\infty} u(h_i) v(h_i)\end{aligned}$$

wherein the last equality we have again used Eq. (6.7).

7. If $u \in W^*$ such that $u|_H = 0$, then by Eq. (6.8) it follows that $q(u,u) = 0$ and since q is an inner product we must have $u = 0$. Alternatively this last assertion follows from Eq. (6.7);

$$q(u,u) = (Ju, Ju)_H = u(Ju) = 0.$$

It now follows as a consequence of the Hahn–Banach theorem that H must be a dense subspace of W.

a) I will omit the proof of 8a. which relies on basic Martingale theory for Gaussian measures on Banach spaces which may be found in [6, Part VIII] and [29].

b) As simple computation shows that

$$\mathbb{E}\left[e^{i\alpha(S)}\right] = \exp\left(-\frac{1}{2}\sum_{k=1}^{\infty}\alpha(h_k)^2\right) = \exp\left(-\frac{1}{2}q(\alpha,\alpha)\right)$$

which is enough to show $\text{Law}(S) = \mu$.

c) Making use of 8b. we have

$$\begin{aligned} J\alpha = \mathbb{E}[\alpha(S)S] &= \lim_{N\to\infty}\mathbb{E}\left[\alpha\left(\sum_{k=1}^{N}N_k h_k\right)\sum_{l=1}^{N}N_l h_l\right] \\ &= \lim_{N\to\infty}\sum_{k,l=1}^{N}\mathbb{E}[N_k N_l]\,\alpha(h_k)\,h_l = \sum_{k=1}^{\infty}\alpha(h_k)\,h_k. \end{aligned}$$

d) For $\alpha \in W^*$ and $f \in L^2(\mu)$ we have

$$(Jf, J\alpha)_H = \alpha(Jf) = (f,\alpha)_{L^2(\mu)}.$$

By continuity it then follows that $(Jf, Jg)_{L^2(\mu)} = (f,g)_{L^2(\mu)}$ for all $g \in K$. Taking $g = J_K^{-1}h$ then implies Eq. (6.9).

e) For $f \in L^2(\mu)$ we have

$$Jf = \sum_{k=1}^{\infty}(Jf, h_k)_H\,h_k = \sum_{k=1}^{\infty}(f, J^*h_k)_{L^2(\mu)}\,h_k = \sum_{k=1}^{\infty}\left(f, J_K^{-1}h_k\right)_{L^2(\mu)}h_k$$

wherein we have used Eq. (6.9) for the last equality.

8. As S in part 8a takes values in W_0 a.s., it follows that $\mu = \text{Law}(S)$ is concentrated on W_0. The remaining assertions are all easy and will be left to the reader.

■

Exercise 6.1. Suppose that $(W, \mathcal{B}_W, \mu)$ and $(V, \mathcal{B}_V, \nu)$ are two Gaussian measure spaces. Show;

1. $(W \times V, \mathcal{B}_{W\times V}, \mu\times\nu)$ is a Gaussian measure space.
2. $q_{\mu\times\nu}(\psi) = q_\mu(\psi(\cdot,0)) + q_\nu(\psi(0,\cdot))$ for all $\psi \in (W\times V)^*$.
3. $H_{\mu\times\nu} = H_\mu \times H_\nu$ as Hilbert spaces.

Exercise 6.2. Let $t > 0$ and $D_t : W \to W$ be the dilation operator given by $D_t(x) = \sqrt{t}x$ and let $\mu_t := \mu\circ D_t^{-1}$. In more detail we have $\mu_t(A) := \mu\left(t^{-1/2}A\right)$ for all $A \in \mathcal{B}_W$ and

$$\int_W f\,d\mu_t = \int_W f\left(\sqrt{t}x\right)d\mu(x) = \int_W f\circ D_t\,d\mu(x) \tag{6.11}$$

for all bounded measurable functions on W. Let $T_t f := f\circ D_t$ and $J_t := J_{\mu_t}$ with $J = J_1$. Show;

1. $\|x\|_H^2 = t\,\|x\|_{H_t}^2$ for all $x \in W$ and hence $H_{\mu_t} = H_\mu$ as sets with $(h,k)_{H_\mu} = t\,(h,k)_{H_{\mu_t}}$ for all $h,k \in H_\mu$.
2. $T_t : L^p(\mu_t) \to L^p(\mu)$ is isometric isomorphism of Banach spaces for all $1 \le p < \infty$.
3. $J_t = D_t|_H J T_t$ on $L^2(\mu_t)$.
4. $J^* = T_t J_t^* D_t$, i.e. for $h \in H$ and $x \in W$ we have
$$(J^*h)(x) = \frac{1}{\sqrt{t}}(J_t^*h)\left(\sqrt{t}x\right) \qquad (\mu - \text{a.e. } x).$$
5. If $\alpha \in W^*$ and $h = J_t\alpha \in H$ show that J_t^*h and J^*h have a unique continuous version and for these versions we have $J_t^*h = \frac{1}{t}J^*h$.

Hint: if you get stuck this exercise is mostly a special case of Proposition 6.3 below.

Remark 6.2. By Lemma 3.9 and Exercise 3.2 it follows that each $f \in K$ is a mean zero Gaussian random variable. As K is a subspace it follow that $K \subset L^2(\mu)$ consists of jointly Gaussian random variables.

Proposition 6.3 (Pushing forward (may omit)). *Suppose that $(W, \mathcal{B}_W, \mu)$ is a Gaussian probability space and let $T : W \to W_0$ be a bounded linear transformation to a separable Banach space W_0. Let $\mu_0 := T_*\mu$ and for $f_0 \in K_0 := \overline{W_0^*}^{L^2(\mu_0)}$ let*

$$J_0 f_0 := \int_{W_0} f_0(x_0)\,x_0\,d\mu_0(x_0)\; f_0 \in K_0 := \overline{W_0^*}^{L^2(\mu_0)}.$$

Then;

1. *μ_0 is a Gaussian measure on $(W_0, \mathcal{B}_{W_0})$ with $q_0(\alpha_0) = q(\alpha_0 \circ T)$,*
2. *$H_0 = T(H)$,*
3. *$TJT^{\rm tr} = J_0$ and $T|_H^* J = J_0 T^{\rm tr}$,*
4. *$TT|_H^* = Id_{H_0}$, and*
5. *$T|_{\mathrm{Nul}(T|_H)^\perp} : \mathrm{Nul}(T|_H)^\perp \to H_0$ is unitary.*

Proof. Let $\alpha_0 \in W_0^*$, then

$$\hat{\mu}_0(\alpha_0) = \int_{W_0} e^{i\alpha(x_0)} d(T_*\mu)(x_0) = \int_W e^{i\alpha\circ T} d\mu = e^{-q(\alpha\circ T)/2}$$

which shows that μ_0 is a Gaussian measure with $q_0 = q \circ T^{\rm tr}$. Given $x \in W$ we have

$$\|Tx\|_{H_0}^2 = \sup_{\alpha_0 \in W_0^*} \frac{|\alpha_0(Tx)|^2}{q_0(\alpha_0)} = \sup_{\alpha_0 \in W_0^*} \frac{|\alpha_0 \circ T(x)|^2}{q(\alpha_0 \circ T)} \le \|x\|_H^2$$

which shows that $T(H) \subset H_0$ and $T|_H : H \to H_0$ is a contraction.

Now suppose that $\alpha_0 \in W_0^*$, then

$$\begin{aligned} TJ[\alpha_0 \circ T] &= T\int_W \alpha_0 \circ T(x)\, x d\mu(x) = \int_W \alpha_0 \circ T(x)\, Tx \, d\mu(x) \\ &= \int_{W_0} \alpha_0(y)\, y \, d\mu_0(y) = J_0 \alpha_0. \end{aligned}$$

By a simple limiting argument it now follows that $TJT^{\rm tr} = J_0$ and from this identity we learn that

$$H_0 = J_0 K_0 = TJT^{\rm tr} K_0 \subset TJK = TH$$

from which we may now conclude that $H_0 = T(H)$.

Let us simply write T^* for $T|_H^*$. Then for $\alpha \in W_0^*$ and $\beta \in W^*$ we have

$$(T^* J_0 \alpha, J\beta)_H = (J_0\alpha, TJ\beta)_{H_0} = (\alpha \circ T)(J\beta) = (J(\alpha\circ T), J\beta)_H$$

from which it follows that $T^* J_0 = JT^{\rm tr}$. Using this identity and item 2. we learn that $TT^* J_0 = TJT^{\rm tr} = J_0$ which implies $TT^* = Id_{H_0}$.

The last identity implies that T^* is an isometry since

$$(T^* h_0, T^* k_0)_H = (h_0, TT^* k_0)_{H_0} = (h_0, Id_{H_0} k_0)_{H_0} = (h_0, k_0)_{H_0}.$$

Since T^* is an isometry $\mathrm{Ran}(T^*)$ is closed and hence $\mathrm{Ran}(T^*) = \mathrm{Nul}(T)^\perp$. It now follows that

$$T|_{\mathrm{Nul}(T)^\perp} = (T^*)^{-1} : \mathrm{Nul}(T)^\perp = \mathrm{Ran}(T^*) \to H_0$$

is also an isometry and therefore $T|_{\mathrm{Nul}(T|_H)^\perp} : \mathrm{Nul}(T|_H)^\perp \to H_0$ is unitary.

Better way: Let $\{h_n\}_{n=1}^\infty \subset \mathrm{Nul}(T|_H)^\perp \subset H$ be an orthonormal basis for $\mathrm{Nul}(T|_H)^\perp$ and $\{k_n\}_{n=1}^\infty$ be an orthonormal basis for $\mathrm{Nul}(T|_H)$. It then follows that $\mathrm{Law}\left(\sum_{n=1}^\infty N_n h_n + \sum_{n=1}^\infty N_n' k_n\right) = \mu$ where $\{N_n, N_n'\}_{n=1}^\infty$ are i.i.d. standard normal random variables and the sums are convergent in W. Therefore

$$\begin{aligned} \mu_0 = T_*\mu &= \mathrm{Law}\left(T\left(\sum_{n=1}^\infty N_n h_n + \sum_{n=1}^\infty N_n' k_n\right)\right) \\ &= \mathrm{Law}\left(\sum_{n=1}^\infty N_n T h_n\right) \end{aligned}$$

where the latter sum is convergent in W_0. Thus if $\alpha_0 \in W_0^*$ then $\alpha_0 \overset{d}{=} \sum_{n=1}^\infty N_n \alpha_0(Th_n)$ (α_0 distributed by $\mu_0 = T_*\mu$) and hence is Gaussian. Moreover it follows that

$$q_0(\alpha_0) = \mathbb{E}\left[\left[\sum_{n=1}^\infty N_n \alpha_0(Th_n)\right]^2\right] = \sum_{n=1}^\infty [\alpha_0(Th_n)]^2.$$

As the $\{Th_n\}_{n=1}^\infty$ are linearly independent, it follows that H_0 is the Hilbert space with $\{Th_n\}_{n=1}^\infty$ being an orthonormal basis for H_0. Notice that if $\{a_n\} \subset \ell^2$, then $\sum_{n=1}^\infty a_n h_n$ converges in H and hence in W and therefore

$$T\left(\sum_{n=1}^\infty a_n h_n\right) = \sum_{n=1}^\infty a_n T h_n$$

showing the latter sum is convergent in W_0. Thus we have show that $H_0 = T\left(\mathrm{Nul}(T|_H)^\perp\right) = TH$ and that $T|_{\mathrm{Nul}(T|_H)^\perp} : \mathrm{Nul}(T|_H)^\perp \to H_0$ is a unitary map. ■

6.2 Cameron-Martin Theorem

Lemma 6.4. *Let $(W, \mathcal{B}, \mu)$ be a non-degenerate (for simplicity) Gaussian measure space . Then there exists $\{u_k\}_{k=1}^\infty \subset W^* \subset K$ which is an orthonormal basis for K and satisfies*

$$\|x\|_H^2 = \sum_{k=1}^\infty |u_k(x)|^2 \text{ for all } x \in W. \tag{6.12}$$

In particular,

$$H := H_\mu = \left\{x \in W : \sum_{k=1}^\infty |u_k(x)|^2 < \infty\right\} \in \mathcal{B}_W. \tag{6.13}$$

Moreover, if $\dim W = \infty$ then $\mu(H) = 0$.

Proof. If $\{u_k\}_{k=1}^{\infty} \subset W^*$ is any orthonormal basis for K, then for any $x \in H \subset W$ we will have

$$\|x\|_H^2 = \sum_{k=1}^{\infty} (x, Ju_k)_H^2 = \sum_{k=1}^{\infty} |u_k(x)|^2. \tag{6.14}$$

To get this identity to hold for all $x \in W$ we will choose the $\{u_k\}_{k=1}^{\infty}$ more carefully. To this first choose $\{\alpha_n\}_{n=1}^{\infty} \subset W^*$ such that $\|x\|_W = \sup_n |\alpha_n(x)|$ for all $x \in W$ as mentioned in the proof of Theorem 2.2. Now from the $\{u_k\}_{k=1}^{\infty}$ by applying the Graham-Schmidt process to the $\{\alpha_n\} \subset W^* \subset K \subset L^2(\mu)$. I claim the resulting sequence $\{u_k\}_{k=1}^{\infty}$ is complete and hence an orthonormal basis for K. To see this suppose that $f \in K$ is perpendicular to $\{u_k\}_{k=1}^{\infty}$, then

$$0 = (f, u_k)_K = (Jf, Ju_k)_H = u_k(Jf) \text{ for all } k. \tag{6.15}$$

Now for each $n \in \mathbb{N}$ we know that $(\alpha_n, u_k) = 0$ for all $k > n$ so that $\alpha_n = \sum_{k=1}^n (\alpha_n, u_k) u_k$ as elements of K which implies

$$q\left(\alpha_n - \sum_{k=1}^n (\alpha_n, u_k) u_k\right) = \left\|\alpha_n - \sum_{k=1}^n (\alpha_n, u_k) u_k\right\|_{L^2(\mu)}^2 = 0.$$

Because q is non-degenerate we may now conclude that $\alpha_n = \sum_{k=1}^n (\alpha_n, u_k) u_k$ as element of W^*. So we may now conclude from this remark and Eq. (6.15) that $\alpha_n(Jf) = 0$ for all n and therefore $\|Jf\|_W = \sup_n |\alpha_n(Jf)| = 0$. Having shown $Jf = 0$ also shows $f = 0$ ($J : K \to H$ is isometric) which proves the assertion that $\{u_k\}_{k=1}^{\infty}$ is complete. We now fix this choice for $\{u_k\}$ for the rest of the proof.

If $x \in W$ satisfies $\sum_{k=1}^{\infty} |u_k(x)|^2 < \infty$ we may define $h := \sum_{k=1}^{\infty} u_k(x) Ju_k \in H$. The sum converges in H and hence also in W and therefore for all $m \in \mathbb{N}$,

$$u_m(h) = \sum_{k=1}^{\infty} u_k(x) u_m(Ju_k) = \sum_{k=1}^{\infty} u_k(x) (Ju_k, Ju_m)_H = u_m(x).$$

Since $u_k(x-h) = 0$ for all k it follows from the argument above that $x = h \in H$. Thus we have shown $\sum_{k=1}^{\infty} |u_k(x)|^2 < \infty$ implies $x \in H$ which combined with Eq. (6.14) shows that $\sum_{k=1}^{\infty} |u_k(x)|^2 < \infty$ iff $x \in H$. As Eq. (6.12) holds on H and both $\|x\|_H^2 = \infty$ and $\sum_{k=1}^{\infty} |u_k(x)|^2 = \infty$ for $x \notin H$ we can conclude that Eq. (6.12) holds for all $x \in W$.

Since $\{u_k\}_{k=1}^{\infty} \subset W^*$ is an orthonormal basis for K, the $\{u_k\}_{k=1}^{\infty}$ are i.i.d. standard normal random variables. Therefore by an application of the strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n |u_k(x)|^2 = \mathbb{E}|u_1|^2 = 1 \quad (\mu - \text{a.e. } x). \tag{6.16}$$

On the other hand if $x \in H$, then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n |u_k(x)|^2 = 0$ and so $\mu(H) = 0$.

■

Theorem 6.5 (Cameron-Martin). *Let (H, W, μ) be a Gaussian measure space as above and for $h \in W$ let $\mu_h(A) = \mu(A - h)$ for all $A \in \mathcal{B}_W$. Then $\mu_h \ll \mu$ iff $h \in H$ and if $h \in H$ then*

$$\frac{d\mu_h}{d\mu} = \exp\left(J^* h - \frac{1}{2}\|h\|_H^2\right).$$

Moreover if $h \in W \setminus H$ (i.e. $\|h\|_H = \infty$), then $\mu_h \perp \mu$. (Since $J^* = J_K^{-1}$, if $h = J\alpha$, then $J^* h = \alpha$ where $\alpha(k) = (J\alpha, k)_H = (h, k)$ for all $k \in H$. For this reason it is often customary to abuse notation and write $J^* h = (h, \cdot)_H$.)

Proof. I will only prove here that $\mu_h \ll \mu$ when $h \in H$. See, for example, [6, Proposition **??**] for a proof of the orthogonality assertions.

We must show

$$\int_W f(x+h) d\mu(x) = \int_W e^{J^* h - \frac{1}{2}\|h\|^2} f(x) d\mu(x)$$

for all $f \in (\mathcal{B}_W)_b$. It suffices to show that

$$\int_W e^{i\varphi(x+h)} d\mu(x) = \int e^{i\varphi(x)} e^{J^* h - \frac{1}{2}\|h\|_H^2} d\mu(x) \tag{6.17}$$

for all $\varphi \in W^*$.

We will start by verifying Eq. (6.17) when $h = J\psi \in JW^*$ for some $\psi \in W^*$. For h of this form $J^* h = \psi$ a.s. and

$$\varphi(h) = \varphi(J\psi) = (J\varphi, J\psi)_H = q(\varphi, \psi).$$

Therefore the left side of Eq. (6.17) is given by

$$e^{i\varphi(h)} e^{-\frac{1}{2}q(\varphi,\varphi)} = e^{iq(\varphi,\psi) - \frac{1}{2}q(\varphi,\varphi)}$$

while the right side by;

$$\begin{aligned}
\int_W e^{i\varphi(x)} e^{J^* h - \frac{1}{2}\|h\|_H^2} d\mu(x) &= \int_W e^{i\varphi(x)} e^{\psi(x) - \frac{1}{2}q(\psi,\psi)} d\mu(x) \\
&= e^{-\frac{1}{2}q(\psi,\psi)} \exp\left(\frac{1}{2}\int_W (\psi + i\varphi)^2 d\mu\right) \\
&= e^{-\frac{1}{2}q(\varphi,\varphi) + iq(\varphi,\psi) + \frac{1}{2}q(\psi,\psi) - \frac{1}{2}q(\psi,\psi)} \\
&= e^{iq(\varphi,\psi) - \frac{1}{2}q(\varphi,\varphi)}.
\end{aligned}$$

For general $h \in H$ we may choose $\psi_n \in W^*$ so that $h_n := J\psi_n \to h$ in H or equivalently so that $\psi_n \to J^*h$ in $K \subset L^2(\mu)$. It then follows from Lemma 3.9 that $e^{\psi_n} \to e^{J^*h}$ in $L^2(\mu)$. Using this remark, it is easy to pass to the limit in Eq. (6.17) with h replaced by h_n in order to show Eq. (6.17) holds for all $h \in H$. ■

Remark 6.6. Despite the fact that $\mu(H) = 0$ and infinite dimensional Lebesgue measure does not exists and that Z below should be $(2\pi)^{\dim H/2} = \infty$, one should still morally think that μ is the measure given by

$$\text{“}d\mu(x) = 1_H(x)\frac{1}{Z}\exp\left(-\frac{1}{2}\|x\|_H^2\right)\mathcal{D}x.\text{”}$$

The Cameron-Martin theorem is easy to understand with this heuristic; namely by making the formal change of variables, $x \to x - h$ we “find,”

$$\begin{aligned} d\mu_h(x) &= \frac{1}{Z}\exp\left(-\frac{1}{2}\|x-h\|_H^2\right)\mathcal{D}(x-h) \\ &= \frac{1}{Z}\exp\left(-\frac{1}{2}\|x\|_H^2 - \frac{1}{2}\|h\|_H^2 + (x,h)_H\right)\mathcal{D}x \\ &= \exp\left((h,x)_H - \frac{1}{2}\|h\|_H^2\right)d\mu(x). \end{aligned}$$

We have use the formal translation invariance of $\mathcal{D}x$ in the second line.

See L. Gross [15] for a historical perspective and a nice analogy between Gaussian measures and Lebesgue measure on $\mathbb{R}$ where $\mathbb{Q}$ plays the role of H.

Exercise 6.3. Let $(W, \mathcal{B}_W, \mu)$ be a non-degenerate Gaussian probability space. Show that $\mu(B(x,\varepsilon)) > 0$ for all $x \in W$ and $\varepsilon > 0$ where $B(x,\varepsilon)$ is the open ball in W of radius ε centered at x.

Theorem 6.7 (Integration by Parts). *Let $h \in H$ and $f \in \mathcal{F}C^\infty(W^*)$ such that f and $\partial_h f$ does not grow too fast at infinity. Then*

$$\int_W \partial_h f d\mu = \int_W J^*h \cdot f d\mu.$$

First Proof.. Assuming enough regularity on f to justify the interchange of the derivatives involved with the integral we have, using the Cameron-Martin theorem, that

$$\begin{aligned} \int_W \partial_h f d\mu &= \int_W \frac{d}{dt}|_0 f(x+th)\,d\mu(x) = \frac{d}{dt}|_0 \int_W f(x+th)\,d\mu(x) \\ &= \frac{d}{dt}|_0 \int_W f\exp\left(tJ^*h - \frac{t^2}{2}\|h\|_H^2\right)d\mu = \int_W f\cdot J^*h\ d\mu(x). \end{aligned}$$

It is not really necessary in this proof that f be a cylinder function. ■

Second Proof.. By replacing W by $W_0 = \bar{H}^W$ if necessary we may assume that $(W, \mathcal{B}_W, \mu)$ is a non-degenerate Gaussian probability space. Given a cylinder function $f = F(\alpha_1, \ldots, \alpha_n)$ we may assume (if not apply Gram–Schmidt to the $\{\alpha_k\}_{k=1}^n$) that the $\{\alpha_k\}_{k=1}^n$ form an orthonormal subset of (W^*, q). Extend this set to an orthonormal basis, $\{\alpha_k\}_{k=1}^\infty$, for K. Then for any $N \geq n$, by finite dimensional integration by parts,

$$\begin{aligned} \int_W \partial_h f d\mu &= \int_W \sum_{k=1}^n (D_k F)(\alpha_1,\ldots,\alpha_n)\,\alpha_k(h)\,d\mu \\ &= \sum_{k=1}^n \alpha_k(h)\int_{\mathbb{R}^n} D_k F(a_1,\ldots,a_n)\left(\frac{1}{2\pi}\right)^{n/2} e^{-\frac{1}{2}|a|^2_{\mathbb{R}^n}}da \\ &= \sum_{k=1}^n \alpha_k(h)\int_{\mathbb{R}^n} a_k F(a_1,\ldots,a_n)\left(\frac{1}{2\pi}\right)^{n/2} e^{-\frac{1}{2}|a|^2_{\mathbb{R}^n}}da \\ &= \sum_{k=1}^n \alpha_k(h)\int_W \alpha_k\cdot F(\alpha_1,\ldots,\alpha_n)\,d\mu \\ &= \sum_{k=1}^N \alpha_k(h)\int_W \alpha_k\cdot F(\alpha_1,\ldots,\alpha_n)\,d\mu \end{aligned} \tag{6.18}$$

wherein we have used

$$\int_W \alpha_k\cdot F(\alpha_1,\ldots,\alpha_n)\,d\mu = \int_W \alpha_k d\mu\cdot\int_W F(\alpha_1,\ldots,\alpha_n)\,d\mu = 0 \text{ for all } k > n$$

because α_k is independent of $\{\alpha_1, \ldots, \alpha_n\}$. Since

$$\lim_{N\to\infty}\sum_{k=1}^N \alpha_k(h)\,J\alpha_k = \sum_{k=1}^\infty (J\alpha_k, h)_H\,J\alpha_k = h$$

it follows from Theorem 6.1 that

$$\sum_{k=1}^N \alpha_k(h)\,\alpha_k \to J_K^{-1}h = J^*h \text{ in } L^2(\mu).$$

Thus letting $N \to \infty$ in Eq. (6.18) completes the second proof. ■

6.3 The Heat Interpretation

The heat interpretation of a Gaussian measure remains essentially unchanged when going to the infinite dimensional setting. In fact, when acting on cylinder functions the results really come back to the finite dimensional – see the second proof of integration by parts in Theorem 6.7.

Exercise 6.4 (Compare with Exercise 5.6). Let $(W, \mathcal{B}_W, \mu)$ be a Gaussian probability space ($\dim W = \infty$ permissible). Suppose $f \in \mathcal{F}C^2(W^*)$ such that f and its first and second derivatives grow (for example) at most exponentially at infinity. Show that

$$F(t,x) := \int_W f\left(x + \sqrt{t}y\right) d\mu(y) \ \forall\, t > 0 \text{ and } x \in W. \tag{6.19}$$

satisfies the heat equation,

$$\frac{\partial}{\partial t} F(t,x) = \frac{1}{2}(LF)(t,x). \tag{6.20}$$

Gaussian Process as Gaussian Measures

In this chapter let $\{Y_t\}_{0\le t\le T}$ be a mean zero Gaussian random process having continuous sample paths. As in Example 2.3, let $W := C([0,T],\mathbb{R})$ and $\mu := \mathrm{Law}(Y_\cdot)$ be the associated Gaussian measure on $(W,\mathcal{B}_W)$. Recall that we have also defined $\alpha_t \in W^*$ to be the evaluation map $\alpha_t(x) = x(t)$ for all $x \in W$ and $0 \le t \le T$. Our immediate goal in this chapter is to better understand $(W,\mathcal{B}_W,\mu)$. In particular we want to describe the associate Cameron-Martin spaces for such process. Of course the case where $\{Y_t\}$ is a Brownian motion holds special interest for us.

7.1 Reproducing Kernel Hilbert Spaces

In what follows (as in Lemma 7.1) let

$$\Lambda = \{0 = t_0 < t_1 < \cdots < t_n = T\}$$

and $\pi_\Lambda : W \to W$ be the projection map defined by $\pi_\Lambda x = y$ where $y = x$ on Λ and $y''(t) = 0$ for $t \notin \Lambda$. (So $\pi_\Lambda(x)$ is a piecewise linear approximation to x.) Further let

$$H_\Lambda := \pi_\Lambda(W) = \{h \in H = H_\mu : h''(t) = 0 \text{ if } t \notin \Lambda\}.$$

Lemma 7.1. *To each $n \in \mathbb{N}$ let $\Lambda_n := \left\{\frac{k}{n}T : 0 \le k \le n\right\}$ and let $\pi_n = \pi_{\Lambda_n} : W \to W$ be as above. Then;*

1. *$\pi_n(x) \to x$ in W as $n \to \infty$ for each $x \in W$.*
2. *If $\alpha \in W^*$, then $\alpha_n := \alpha \circ \pi_n$ converges to α as $n \to \infty$ both pointwise on W and in $L^2(\mu)$.*
3. *If $K := \overline{W^*}^{L^2(\mu)}$, then $\mathrm{span}\{\alpha_t : 0 \le t \le T\}$ is a dense subspace of K.*

Proof. We prove each item in turn.

1. The convergence is a simple consequence of the fact that every $x \in W$ is uniformly continuous.
2. The pointwise convergence follows directly from item 1. The $L^2(\mu)$ – convergence may be deduced from Lemma 3.9 or using the DCT along with the uniform estimate;

$$|\alpha_n(x)| \le \|\alpha\|_{W^*} \|\pi_n(x)\|_W \le \|\alpha\|_{W^*} \|x\|_W.$$

Notice that $\|\cdot\|_W \in L^2(\mu)$ by Fernique's theorem.

3. Because $\pi_n(x)$ is completely determined by the values of x on Λ_n it follows that $\alpha_n = \alpha \circ \pi_n$ is a linear combination of $\{\alpha_t\}_{t\in\Lambda_n}$. Thus it follows that every $\alpha \in W^*$ is in the $L^2(\mu)$ – closure of $\mathrm{span}\{\alpha_t\}_{0\le t\le T}$ which suffices to prove item 3.

■

Theorem 7.2 (The Cameron-Martin Space for Y). *Let $\tilde{K} = \overline{\left\{\mathrm{span}\{Y_t\}_{0\le t\le T}\right\}}^{L^2(P)}$. Then the Cameron-Martin space H_μ associate to μ is given by $\left\{h_{\tilde{Z}} : \tilde{Z} \in \tilde{K}\right\}$ where for $\tilde{Z} \in \tilde{K}$;*

$$h_{\tilde{Z}}(t) = \mathbb{E}\left[Y_t \tilde{Z}\right] \text{ for } 0 \le t \le T.$$

Moreover if $\tilde{Z}_1, \tilde{Z}_2 \in \tilde{K}$, then

$$\left(h_{\tilde{Z}_1}, h_{\tilde{Z}_2}\right)_{H_\mu} = \mathbb{E}\left[\tilde{Z}_1 \cdot \tilde{Z}_2\right].$$

Proof. Letting $K := \overline{W^*}^{L^2(\mu)}$ we know from Theorem 6.1 that $H_\mu = J_\mu(K) = J_\mu\left(L^2(\mu)\right)$. For $\tilde{Z} \in \tilde{K}$ we can find a $Z \in K$ such that $\tilde{Z} = Z \circ Y$. Therefore

$$h_{\tilde{Z}}(t) = \mathbb{E}[Y_t Z \circ Y] = \int_W \alpha_t(x) Z(x)\, d\mu(x) = \alpha_t(J_\mu Z)$$

and hence $h_{\tilde{Z}} = J_\mu Z \in H_\mu$. Similarly if $\tilde{Z}_1, \tilde{Z}_2 \in \tilde{K}$ there exists $Z_1, Z_2 \in K$ such that $\tilde{Z}_i = Z_i \circ Y$ and we have

$$\left(h_{\tilde{Z}_1}, h_{\tilde{Z}_2}\right)_{H_\mu} = (J_\mu Z_1, J_\mu Z_2)_{H_\mu} = \int_W [Z_1 Z_2]\, d\mu = \mathbb{E}\left[\tilde{Z}_1 \cdot \tilde{Z}_2\right].$$

■

By Lemma 3.9 we know that $[0,T] \ni t \to Y_t \in L^2(P)$ is continuous. Since $L^2(P) \times L^2(P) \ni (f,g) \to (f,g)_{L^2(P)} \in \mathbb{R}$ is a continuous function it follows directly that $h_{\tilde{Z}}(t) = \mathbb{E}\left[Y_t \tilde{Z}\right]$ is a continuous function of t. Of course this is a consequence of the general theory as well.

Definition 7.3 (Reproducing Kernel). *Let $G = G_Y : [0,T]^2 \to \mathbb{R}$ be the* ***reproducing kernel associate to*** *Y define by*

$$G(s,t) := \mathbb{E}[Y_s Y_t] = \int_W x(s)\, x(t)\, d\mu(x).$$

Proposition 7.4. *The reproducing kernel satisfies;*

1. *(Continuity.) $G : [0,T]^2 \to \mathbb{R}$ is continuous and $G(t,\cdot) \in H_\mu$ for all $0 \le t \le T$.*
2. *(Reproducing property.) $(G(t,\cdot), h)_{H_\mu} = h(t)$ for all $h \in H_\mu$ and $t \in [0,T]$.*
3. *(Pointwise bounds.) If $h \in H_\mu$ and $0 \le t \le T$, then $|h(t)| \le \sqrt{G(t,t)} \cdot \|h\|_{H_\mu}$. In particular,*
$$\|h\|_W \le \sqrt{\max_{0 \le t \le T} G(t,t)} \cdot \|h\|_{H_\mu}.$$
Moreover these bounds are sharp.
4. *(Totality.) $\{G(t,\cdot) : 0 \le t \le T\}$ is a total subset of H_μ, i.e. $\operatorname{span}\{G(t,\cdot) : 0 \le t \le T\}$ is dense in H_μ.*

Proof. The continuity of G follows from the comments before Definition 7.3 and moreover $G(t,\cdot) = h_{Y_t} \in H_\mu$. If $h = h_{\tilde{Z}} \in H_\mu$ we will have,

$$(G(t,\cdot), h)_{H_\mu} = (h_{Y_t}, h_{\tilde{Z}})_{H_\mu} = \mathbb{E}\left[Y_t \tilde{Z}\right] = h_{\tilde{Z}}(t)$$

which proves the reproducing property. By the Cauchy–Schwarz inequality and the reproducing property,

$$\begin{aligned} |h(t)|^2 &= \left|(G(t,\cdot), h)_{H_\mu}\right|^2 \\ &\le \|G(t,\cdot)\|^2_{H_\mu} \|h\|^2_{H_\mu} \\ &= (G(t,\cdot), G(t,\cdot))_{H_\mu} \|h\|^2_{H_\mu} = G(t,t)\, \|h\|^2_{H_\mu}. \end{aligned}$$

If we choose a $t_0 \in [0,T]$ such that $G(t_0,t_0) = \max_{0 \le t \le T} G(t,t)$ and let $h = G(t_0,\cdot)$, then $\|h\|_{H_\mu} = \sqrt{G(t_0,t_0)}$ and

$$h(t_0) = G(t_0,t_0) = \left(\max_{0 \le t \le T} \sqrt{G(t,t)}\right) \cdot \|h\|_{H_\mu}$$

which show that the given bounds are sharp.

The fact that $\{G(t,\cdot) : 0 \le t \le T\}$ is a total in H_μ follows from the fact that $\{Y_t : 0 \le t \le T\}$ is total in $\tilde{K}$. Alternatively, if $h \in H_\mu$ is perpendicular to $\{G(t,\cdot) : 0 \le t \le T\}$ then $0 = (h, G(t,\cdot))_{H_\mu} = h(t)$ for all t which shows that h must be zero. ■

Much of what we have just proved for G holds more generally as you are asked to show in the next exercise.

Exercise 7.1 (Reproducing Kernel Hilbert Spaces). Let H be a subspace of $W := C([0,T], \mathbb{R})$ (can replace $[0,T]$ by a more general topological space if you wish) which is equipped with a Hilbertian norm, $\|\cdot\|_H$, such that $\|h\|_W \le C\|h\|_H$ for all $h \in H$. Then;

1. for each $t \in [0,T]$ there exists $G(t,\cdot) \in H$ such that $h(t) = (G(t,\cdot), h)_H$ for all $h \in H$. Moreover $G(t,s) = (G(t,\cdot), G(s,\cdot))$ showing that G is a symmetric function of (s,t).
2. The map $[0,T] \ni t \to G(t,\cdot) \in H$ is continuous.
3. $(s,t) \to G(s,t)$ is continuous.
4. $\{G(t,\cdot) : 0 \le t \le T\}$ is total in H.
5. H is necessarily a separable Hilbert space.
6. If $\{h_n\}_{n=1}^\infty$ is any orthonormal basis for H and $0 \le s,t \le T$, then
$$\sum_{n=1}^\infty h_n(s)\, h_n(t) = G(s,t)$$
where the sum is absolutely convergent.
7. Each $h \in H$ satisfies the continuity estimate,
$$|h(t) - h(s)| \le \|h\|_H \cdot \sqrt{G(t,t) + G(s,s) - 2G(s,t)}.$$

As a check on computation so far recall that $C([0,T] \to \mathbb{R})^*$ is isomorphic to the space of signed measures on $[0,T]$ under the identification;

$$\alpha(x) = \int_0^T x(t)\, d\alpha(t)$$

when α is a signed measure on $[0,T]$. So if α and β are two singed measures, then

$$\begin{aligned} q_\mu(\alpha,\beta) &= \mathbb{E}\left[\int_0^T Y_s d\alpha(s) \int_0^T Y_t d\beta(t)\right] \\ &= \int_0^T \int_0^T \mathbb{E}[Y_s Y_t]\, d\alpha(s)\, d\beta(t) = \int_0^T \int_0^T G(s,t)\, d\alpha(s)\, d\beta(t) \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_0^T \int_0^T G(s,t)\, d\alpha(s)\, d\beta(t) &= \int_0^T \int_0^T \sum_{n=1}^\infty h_n(s)\, h_n(t)\, d\alpha(s)\, d\beta(t) \\ &= \sum_{n=1}^\infty \int_0^T h_n(s)\, d\alpha(s) \int_0^T h_n(t)\, d\beta(t) \\ &= \sum_{n=1}^\infty \alpha(h_n)\, \beta(h_n) \end{aligned}$$

where I leave it to you to justify the interchanges of sum and integrals used above. The last two displayed equations are consistent with Eq. (6.8), i.e. that $q_\mu(\alpha,\beta) = \sum_{n=1}^{\infty} \alpha(h_n)\beta(h_n)$.

Example 7.5 (The Classical Cameron – Martin Space). Let

$$W = \{x \in C([0,T] \to \mathbb{R}) : x(0) = 0\}$$

and let H denote the set of functions $h \in W$ which are absolutely continuous and satisfy $(h,h) = \int_0^T |h'(s)|^2 ds < \infty$. The space H is called the Cameron-Martin space and is a Hilbert space when equipped with the inner product

$$(h,k) = \int_0^T h'(s)k'(s)ds \text{ for all } h,k \in H.$$

By the fundamental theorem of calculus we have for $h \in H$ that

$$h(t) = \int_0^t h'(\sigma)\,d\sigma = \int_0^T 1_{\sigma \le t} h'(\sigma)\,d\sigma = (G(t,\cdot),h)_H$$

provided we define

$$G(t,s) = \int_0^s 1_{\sigma \le t} d\sigma = \min(s,t). \tag{7.1}$$

The function in Eq. (7.1) is the reproducing kernel for H. Consequently we may conclude that

$$\min(s,t) = \sum_{n=1}^{\infty} h_n(s)h_n(t)$$

for any orthonormal basis $\{h_n\}_{n=1}^{\infty}$ of H and we have the "Sobolev" inequality for $h \in H$;

$$|h(t) - h(s)| = \|h\|_H \sqrt{s + t - 2s \wedge t} = \|h\|_H \sqrt{|t-s|}$$

for all $0 \le s,t \le T$. Of course we could also prove this inequality directly;

$$\begin{aligned}|h(t) - h(s)| &= \left|\int_s^t h'(\tau)\,d\tau\right| \\ &\le \sqrt{\int_s^t |h'(\tau)|^2\,d\tau} \cdot \sqrt{\int_s^t 1^2 d\tau} \le \|h\|_H \sqrt{t-s}\end{aligned}$$

for all $0 \le s \le t \le T$.

7.2 The Example of Brownian Motion

Theorem 7.6 (Brownian Motion). *Let $W = \{x \in C([0,T] \to \mathbb{R}) : x(0) = 0\}$, $\{B_t\}_{0 \le t \le T}$ be a Brownian motion, and let $\mu := \mathrm{Law}(B_\cdot)$ as a measure on $(W, \mathcal{B}_W)$. Then H_μ is the classical Cameron-Martin space described in Example 7.5.*

Proof. The reproducing kernel for H_μ is

$$G(s,t) = \mathbb{E}[B_s B_t] = s \wedge t.$$

which is also that reproducing kernel of H described in Example 7.5. As $G(t,\cdot)$ is a total subset of both H_μ and H and the inner products of $G(t,\cdot)$ and $G(s,\cdot)$ is $G(s,t)$ when computed in either Hilbert space it follows that $H = H_\mu$ as Hilbert spaces. ■

Remark 7.7. If we did not know about H ahead of time how might of we determined H_μ explicitly? Here is one method. By the general theory we know that $\{G(t,\cdot) : 0 \le t \le T\}$ is a total subset of H_μ. If $h = \sum_{i=1}^n \lambda_i G(t_i,\cdot)$ then

$$\|h\|_{H_\mu}^2 = \sum_{i,j=1}^n \lambda_i \lambda_j (G(t_i,\cdot), G(t_j,\cdot))_{H_\mu} = \sum_{i,j=1}^n \lambda_i \lambda_j G(t_i,t_j).$$

But

$$G(s,t) = s \wedge t = \int_0^T 1_{\sigma \le s} \cdot 1_{\sigma \le t} d\sigma$$

and so it follows that

$$\begin{aligned}\|h\|_{H_\mu}^2 &= \sum_{i,j=1}^n \lambda_i \lambda_j \int_0^T 1_{\sigma \le t_i} \cdot 1_{\sigma \le t_j} d\sigma \\ &= \int_0^T \sum_{i,j=1}^n \lambda_i \lambda_j 1_{\sigma \le t_i} \cdot 1_{\sigma \le t_j} d\sigma \\ &= \int_0^T \left|\sum_{j=1}^n \lambda_j 1_{\sigma \le t_j}\right|^2 d\sigma = \int_0^T |h'(\sigma)|^2\,d\sigma.\end{aligned}$$

I now want to convince you that it is reasonable to heuristically view Wiener measure $\mu := \mathrm{Law}(B)$ as

$$d\mu(x) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_0^T x'(s)^2\,ds\right)\mathcal{D}x.$$

where $\mathcal{D}x$ is the non-existent Lebesgue measure on $H = H_\mu$ and Z is an ill-defined a normalizing constant.

If B is a Brownian motion and $f(x) = F(x|_\Lambda)$, then with $y_0 = 0$ and $\Delta_j := t_j - t_{j-1}$ we have

$$\begin{aligned}\mathbb{E}[f(B)] &= \int_{\mathbb{R}^n} F(y_1,\dots,y_n) \prod_{j=1}^n p_{t_j - t_{j-1}}(y_{j-1}, y_j)\, dy_j \\ &= \frac{1}{Z_\Lambda} \int_{\mathbb{R}^n} F(y_1,\dots,y_n) \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{1}{\Delta_j}(y_j - y_{j-1})^2\right) dy_1 \dots dy_n.\end{aligned}$$

Letting $T = T_\Lambda : \mathbb{R}^n \overset{\cong}{\to} H_\Lambda$ be defined by $T_\Lambda(y) = x$ iff $x \in H_\Lambda$ and $x(t_j) = y_j$ for all j we see that

$$\sum_{j=1}^n \frac{1}{\Delta_j}(y_j - y_{j-1})^2 = \sum_{j=1}^n \left(\frac{x(t_j) - x(t_{j-1})}{\Delta_j}\right)^2 \Delta_j = \int_0^T |x'(s)|^2 \, ds.$$

Therefore we may conclude that

$$\mathbb{E}[f(B)] = \frac{1}{Z_\Lambda}\int_{H_\Lambda} f(x) \exp\left(-\frac{1}{2}\int_0^T |x'(s)|^2\, ds\right) dm_\Lambda(x)$$

where $m_\Lambda(dy) = T_*(dy_1 \dots dy_n)$ is a Lebesgue measure on H_Λ.

Corollary 7.8. *For all bounded measurable functions, $f : W \to \mathbb{R}$, we have*

$$\mathbb{E}[f(\pi_\Lambda(B))] = \frac{1}{Z_\Lambda}\int_{H_\Lambda} f(x)\exp\left(-\frac{1}{2}\|x\|_H^2\right) dm_\Lambda(x),$$

where

$$\|x\|_H^2 := \int_0^T |x'(s)|^2\, ds.$$

Corollary 7.9. *If $f : W \to \mathbb{R}$ is a bounded continuous function on W then*

$$\lim_{|\Lambda|\to 0} \frac{1}{Z_\Lambda}\int_{H_\Lambda} f(x)\exp\left(-\frac{1}{2}\|x\|_H^2\right) dm_\Lambda(x) = \mathbb{E}[f(B)].$$

Proof. This follows directly from Corollary 7.8 and the DCT as $\pi_\Lambda(B) \to B$ as $|\Lambda| \to 0$. ∎

Exercise 7.2 (Projection Lemma). Show $\pi_\Lambda|_H$ is an orthogonal projection onto H_Λ and that $H_\Lambda^\perp = \{k \in H : k|_\Lambda = 0\}$.

Theorem 7.10. *If $\alpha \in W^*$ then*

$$\mathbb{E}\left[e^{i\alpha(B)}\right] = e^{-\frac{1}{2}\|\alpha\|_{H^*}^2}.$$

Proof. By Corollary 7.9 along with the projection lemma of Exercise 7.2 we have

$$\begin{aligned}\mathbb{E}\left[e^{i\alpha(B)}\right] &= \lim_{|\Lambda|\to 0}\frac{1}{Z_\Lambda}\int_{H_\Lambda} e^{i\alpha(x)}\exp\left(-\frac{1}{2}\|x\|_H^2\right) dm_\Lambda(x) \\ &= \lim_{|\Lambda|\to 0}\exp\left(-\frac{1}{2}\|\alpha\circ\pi_\Lambda\|_{H^*}^2\right) = \exp\left(-\frac{1}{2}\|\alpha\|_{H^*}^2\right).\end{aligned}$$

∎

Path Integral Quantization (A word from a sponsor)

Let $V : \mathbb{R}^N \to \mathbb{R}$ is a smooth potential such that $V(x) \to \infty$ as $|x| \to \infty$ with associated classical equations of motion,

$$\ddot{x}(t) = -\operatorname{grad} V(x(t)) \text{ with } x(t) \in \mathbb{R}^N.$$

The corresponding quantum mechanical Hamiltonian is

$$H_0 := -\frac{1}{2}\Delta + V \text{ acting on } L^2\left(\mathbb{R}^N, m\right).$$

Let

$$\lambda_0 = \inf \sigma(H_0), \text{ and } \quad H_0\Omega_0 = \lambda_0\Omega_0$$

to be the lowest eigenvalue and corresponding ground state (Ω_0) for H_0.

Lemma 8.1. *If $\psi \in L^2\left(\mathbb{R}^d\right)$, then $e^{-t(H_0-\lambda_0)}\psi \to (\psi, \Omega_0)\Omega_0$ as $t \uparrow \infty$.*

Proof. Use the spectral theorem. ■

Notation 8.2 *Now let μ be the "measure" on $C\left(\mathbb{R}, \mathbb{R}^N\right)$ given informally by*

$$d\mu(x) = \frac{1}{Z}\exp\left(-\int_{\mathbb{R}} E(\dot{x}(t), x(t))\,dt\right)\mathcal{D}x \tag{8.1}$$

where

$$E(\dot{x}(t), x(t)) = \frac{1}{2}|\dot{x}(t)|^2 + V(x(t))$$

is the classical energy os the classical system. Here Z is a normalization constant so that μ is a probability measure.

Theorem 8.3 (Heuristic). *For nice f and g on $\mathbb{R}^N$,*

$$\int_{C(\mathbb{R},\mathbb{R}^N)} f(x(0))\,g(x(t))\,d\mu(x) = (f, Q_t g)_{L^2(\lambda)}$$

where

$$d\lambda := \Omega_0^2 dm \text{ and } Q_t := M_{\Omega_0^{-1}}e^{-t(H_0-\lambda_0)}M_{\Omega_0}.$$

Note that Q_t acting on $L^2\left(\mathbb{R}^d, \lambda\right)$ is unitarily equivalent to $e^{-t(H_0-\lambda_0 I)}$ acting on $L^2\left(\mathbb{R}^d, dm\right)$.

Proof. Let $\varphi, \psi > 0$ be nice functions on $\mathbb{R}^d$ - say $\varphi, \psi \in L^2 \cap L^1$ and let $\{B_t\}_{-T\le t\le T}$ be a Brownian motion on with initial distribution φ. Then using the Feynman-Kac formula for Brownian motion,

$$\begin{aligned}
&\frac{1}{Z}\int_{C\left([-T,T]\to\mathbb{R}^N\right)} e^{-\int_{-T}^{T} E(\dot{x}(t),x(t))dt} f(x(0))\,g(x(t))\,\varphi(x(-T))\,\psi(x(T))\,\mathcal{D}x \\
&\text{“}=\text{”}\ \frac{1}{Z'}\mathbb{E}_\varphi\left[e^{-\int_{-T}^{T} V(B(t))dt} f(B_0)\,g(B_t)\,\varphi(B_{-T})\,\psi(B_T)\right] \\
&= \frac{(\varphi, P_T M_f P_t M_g P_{T-t}\psi)_m}{(\varphi, P_{2T}\psi)_m} = \frac{\left(P_T^0\varphi, M_f P_t^0 M_g P_{T-t}^0\psi\right)_m}{(\varphi, P_{2T}^0\psi)_m} \\
&\to \left(\Omega_0, f P_t^0(g\Omega_0)\right)_m = (f, Q_t g)_\lambda.
\end{aligned}$$

Note that the boundary conditions wash out – there are no phase transitions in one dimension. ■

The **moral** of this theorem is that from knowledge of classical quantities only we can recover the ground state by

$$\int_{C(\mathbb{R},\mathbb{R}^N)} f(x(0))\,d\mu(x) = \int_{\mathbb{R}^N} f(x)\,\Omega_0^2(x)\,dx$$

and the renormalized Hamiltonian, $H - \lambda_0$, via

$$\frac{d}{dt}|_0 \int_{C(\mathbb{R},\mathbb{R}^N)} f(x(0))\,g(x(t))\,d\mu(x) = (f\Omega_0, (H-\lambda_0 I)(g\Omega_0))_m.$$

8.1 Hints at Quantum Field Theoretic Complications

Example 8.4 (Klein-Gordon equation). The classical non-linear Klein-Gordon equation is;

$$\varphi_{tt} + (-\Delta_d + m^2)\varphi + v'(\varphi) = 0 \tag{8.2}$$

where $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. We now view (8.2) as an ordinary differential equation for $\varphi(t) \in \operatorname{Re}L^2(\mathbb{R}^d, dx)$ satisfying

$$\ddot{\varphi}(t) + (-\Delta + m^2)\varphi(t) + v'(\varphi) = 0 \tag{8.3}$$

or equivalently

$$\ddot{\varphi}(t) = -\operatorname{grad} V(\varphi(t))$$

where

$$V(\varphi) = \int_{\mathbb{R}^d} \left[\frac{1}{2}|\nabla\varphi(x)|^2 + \frac{m^2}{2}\varphi^2(x) + v(\varphi(x))\right] dx.$$

Here one typically thinks of v as being an even polynomial like $v(\varphi) = \lambda\varphi^4$ with $\lambda > 0$.

The path integral quantization method leads us to consider,

$$\begin{aligned} d\mu(\varphi) &= \frac{1}{Z}\exp\left(-\int_{-\infty}^{\infty} dt \left[\frac{1}{2}\|\dot{\varphi}(t)\|^2_{L^2(\mathbb{R}^d)} + V(\varphi(t))\right]\right)\mathcal{D}\varphi \\ &= \frac{1}{Z}\exp\left(-\int_{\mathbb{R}^{d+1}}\left[\frac{1}{2}|\nabla\varphi(y)|^2 + \frac{m^2}{2}\varphi(y)^2 + v(\varphi(y))\right]dy\right)\mathcal{D}\varphi \end{aligned}$$

where $y = (t,x) \in \mathbb{R}^{d+1}$. We want to consider what happens when we try to makes sense out of this expression as measure. One way to do this is to let

$$d\mu_0(\varphi) = \frac{1}{Z}\exp\left(-\int_{\mathbb{R}^{d+1}}\left[\frac{1}{2}|\nabla\varphi(y)|^2 + \frac{m^2}{2}\varphi(y)^2\right]dy\right)\mathcal{D}\varphi \tag{8.4}$$

and then try to write

$$d\mu(\varphi) = \frac{1}{Z_0}\exp\left(-\int_{\mathbb{R}^{d+1}} v(\varphi(y))\,dy\right)d\mu_0(\varphi). \tag{8.5}$$

The problem with this strategy is that unless $d = 0$ (the case of quantum mechanics) the measure μ_0 has to be realized on some space of distributions for which $\varphi(y)$ is not defined and certainly not $v(\varphi(y))$. See Glimm and Jaffe for a whole book on this subject. Here are a few more details to help spark your interest.

We are now going to explore what happens when we try to make sense out of Gaussian measures which are given informally as;

$$d\mu(\varphi) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}\left[|\nabla\varphi(x)|^2 + m^2\varphi(x)^2 + v(\varphi(x))\right]dx\right)\mathcal{D}\varphi.$$

(For notational simplicity I have now replaced $d+1$ by d so please remember that d is the dimension of space $\times$ time.) This is a prototypical expression which occurs in the physics literature on quantum field theory.

Theorem 8.5. *Suppose $(W, (\cdot,\cdot)_W)$ is a Hilbert space and μ is a Gaussian measure on $(W, \mathcal{B}_W)$. Let $H = H_\mu \subset W$ be the Cameron-Martin space associated to μ so that*

$$\hat{\mu}(\varphi) = e^{-\frac{1}{2}(\varphi|_H, \varphi|_H)_{H^*}} \quad \forall \quad \varphi \in W^*. \tag{8.6}$$

Then the inclusion map, $i : H \hookrightarrow W$, is Hilbert Schmidt (H.S.). In fact,

$$\|i\|^2_{HS} := \sum_k \|h_k\|^2_W = \int_W \|x\|^2_W d\mu(x) < \infty$$

where $\{h_k\}_{k=1}^\infty$ is any orthonormal basis for H.

Proof. Let $\{\xi_j\}_{j=1}^\infty \subset W$ be an orthonormal basis for W and $\{h_k\}_{k=1}^\infty$ be an orthonormal basis for H. Then integrating the identity,

$$\|x\|^2_W = \sum_{j=1}^\infty (x,\xi_j)^2_W \text{ for all } x \in W,$$

with respect to μ implies,

$$\begin{aligned} \int_W \|x\|^2_W d\mu &= \sum_{j=1}^\infty \int_W (x,\xi_j)^2_W d\mu = \sum_{j=1}^\infty ((\cdot,\xi_j)_W, (\cdot,\xi_j)_W)_{H^*} \\ &= \sum_{j=1}^\infty\sum_{k=1}^\infty (h_k,\xi_j)^2_W = \sum_{k=1}^\infty\sum_{j=1}^\infty (h_k,\xi_j)^2_W \\ &= \sum_{k=1}^\infty \|h_k\|^2_W = \|i\|^2_{HS}. \end{aligned}$$

This completes the proof since $\int_W \|x\|^2_W d\mu < \infty$ by Fernique's Theorem 3.4. ■

Theorem 8.6. *If H and W be Hilbert spaces such that the inclusion map, $H \overset{i}{\hookrightarrow} W$, is a Hilbert-Schmidt, then there exists a Gaussian measure μ on $(W, \mathcal{B}_W)$ such $\hat{\mu} = \exp\left(-\frac{1}{2}q(\alpha)\right)$ where*

$$q(\alpha,\alpha) = \|\alpha|_H\|^2_{H^*} = \sum_{k=1}^\infty |u(h_k)|^2$$

where $\{h_k\}_{k=1}^\infty$ is any O.N. basis for H.

Proof. Let $\{h_k\}_{k=1}^\infty$ be an orthonormal basis for H. The assumption that i is Hilbert-Schmidt means that

$$\sum_{k=1}^\infty \|h_k\|^2_W = \sum_{k=1}^\infty \|ih_k\|^2_W = \|i\|^2_{HS} < \infty.$$

Let $\{N_k\}_{k=1}^\infty$ be i.i.d. standard normal random variables on some probability space, $(\Omega, \mathcal{B}, P)$ and for $n \in \mathbb{N}$, let

$$S_m := \sum_{k=1}^{m} N_k h_k \in H \subset W.$$

We then have for $n \geq m$ that

$$\mathbb{E} \|S_n - S_m\|_W^2 = \mathbb{E} \left\| \sum_{k=m+1}^{n} N_k h_k \right\|_W^2$$
$$= \sum_{k,l=m+1}^{n} \mathbb{E}\left[N_k N_l\right] (h_k, h_l)_W = \sum_{k=m+1}^{n} \|h_k\|_W^2 \to 0 \text{ as } m, n \to \infty.$$

Therefore,

$$L^2(P) - \lim_{m,n\to\infty} \|S_n - S_m\|_W^2 = 0$$

from which one shows that $S = \lim_{n\to\infty} S_n$ exists in $L^2(P)$ and as in the proof of Theorem 4.3 we may conclude that

$$\mathbb{E}\left[e^{i\alpha(S)}\right] = e^{-\frac{1}{2}q(\alpha)}$$

so that $\mu = \text{Law}(S)$ is the desired Gaussian measure on W. ■

Remark 8.7. Suppose that $(W, (\cdot,\cdot)_W, \mu)$ is a Gaussian space with W being a Hilbert space and $H = H_\mu \subset W$. If $\{h_j\}_{j=1}^{\infty}$ is any orthonormal basis for H and $x_1, x_2 \in W$, then

$$\int_W (x_1, x)_W (x_2, x)_W \, d\mu(x) = ((x_1, \cdot)_W (x_2, \cdot)_W)_{H^*} = \sum_{j=1}^{\infty} (x_1, h_j)_W (x_2, h_j)_W$$
$$= \sum_{j=1}^{\infty} (x_1, ih_j)_W (x_2, ih_j)_W = \sum_{j=1}^{\infty} (i^* x_1, h_j)_H (i^* x_2, h_j)_H$$
$$= (i^* x_1, i^* x_2)_H = (x_1, ii^* x_2)_W = (x_1, i^* x_2)_W.$$

Remark 8.8. Let us recall that the trace of a positive operator, $B : H \to H$ is defined in the usual way as;

$$\text{tr}(B) := \sum_{k=1}^{\infty} (Be_k, e_k)_H.$$

The sum is independent of basis since, by the general spectral theorem we may write $B = A^2$ with $A \geq 0$ and then

$$\sum_{k=1}^{\infty} (Be_k, e_k)_H = \sum_{k=1}^{\infty} \left(A^2 e_k, e_k\right)_H = \sum_{k=1}^{\infty} (Ae_k, Ae_k)_H = \|A\|_{HS}^2$$

so that $\text{tr}(B) = \|A\|_{HS}^2$ which we know is basis independent. We say that $B \geq 0$ is trace class if $\text{tr}(B) < \infty$. Recall that we have seen that Hilbert-Schmidt operators are compact and therefore if $B \geq 0$ and is trace class then $B = A^2$ where A is Hilbert-Schmidt and hence compact and therefore B is also compact. As an aside here are some equivalent conditions for a linear operator, $A : H \to W$ to be Hilbert-Schmidt.

Proposition 8.9 (Trace formulas). *Suppose that H and W are Hilbert spaces and $A : H \to W$ is a bounded linear operator. Then;*

$$\text{tr}(A^* A) = \|A\|_{HS}^2 = \|A^*\|_{HS}^2 = \text{tr}(AA^*).$$

Proof. If $\{e_k\}$ is an O.N. basis for H, then

$$\|A\|_{HS}^2 = \sum_k \|Ae_k\|_W^2 = \sum_k (Ae_k, Ae_k)_W = \sum_k (A^* Ae_k, e_k)_H = \text{tr}(A^* A).$$

Replacing A by A^* above then shows $\|A^*\|_{HS}^2 = \text{tr}(AA^*)$ and we have already seen that $\|A^*\|_{HS}^2 = \|A\|_{HS}^2$. ■

Lemma 8.10. *If H is a Hilbert space and $A : H \to H$ is a positive trace class operator. We may define $(x, y)_A := (Ax, y)$ for all $x, y \in H$. Then let W denote the completion of H in the norm, $\|\cdot\|_A := \sqrt{(\cdot,\cdot)_A}$. Then the inclusion map, $i : H \to W$ will be Hilbert Schmidt and hence W will support a Gaussian measure with variance determined by H.*

Example 8.11. Now consider the "measure" $d\mu_0(\varphi)$ of Eq. (8.4). In this setting one may show that μ can not be constructed on $L^2(\mathbb{R}^d)$ no matter the dimension d.[1] In this case H is the Sobolev space

$$H = \left\{\varphi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \left(|\nabla\varphi|^2 + m^2\varphi^2\right) dm < \infty\right\}.$$

Letting $i : H \to L^2(\mathbb{R}^d)$ be the inclusion map we want to compute

$$\|i\|_{HS}^2 = \text{tr}(i\, i^*) = \text{tr}(i^*).$$

I now claim that $i^* = \left(-\Delta + m^2\right)^{-1}$. Indeed, if $u := i^* g$, then

$$(f, g)_{L^2} = (if, g)_{L^2} = (f, u)_H = \int_{\mathbb{R}^d} \left(\nabla f \cdot \nabla u + m^2 f \cdot u\right) dx \ \forall\ f \in H,$$

[1] When $d = 1$ it comes close to working but there are problems due to the fact that $\mathbb{R}$ is not compact.

which, by Elliptic regularity or by the Fourier transform, implies the distribution, Δu, is an L^2 – function and

$$(f,g)_{L^2} = \left(f, \left(-\Delta + m^2\right) u\right).$$

Hence we have $\left(-\Delta + m^2\right) u = g$ or $i^* g = u = \left(-\Delta + m^2\right)^{-1} g$ as claimed.

Informally, now,

$$\mathrm{tr}\left(i\, i^*\right) = \int_{\mathbb{R}^d} \left(-\Delta + m^2\right)^{-1}(x,x)\, dx = \infty.$$

because when $d \geq 2$, $\left(-\Delta + m^2\right)^{-1}(x,x) = \infty$ and or $d = 1$ $\left(-\Delta + m^2\right)^{-1}(x,x) = c > 0$ for all $x \in \mathbb{R}$.

To give a rigorous proof, notice that $i\, i^*$ is unitarily equivalent to the the multiplication operator, $\left(k^2 + m^2\right)^{-1}$ which has continuous spectrum, it follows that $\left(-\Delta + m^2\right)^{-1}$ is not compact let alone trace class. In general on non-atomic spaces with no infinite atoms any non-zero multiplication operator is not trace class. Indeed, suppose M_f is the multiplication operator on $L^2(W,m)$ and observe that we may assume $f \geq 0$ since $|M_f| = M_{|f|}$. Then for some $\varepsilon > 0$ we will have $\mu(f \geq \varepsilon) > 0$. Since m is non-atomic, we may write $\{f \geq \varepsilon\}$ as a disjoint union of $\{A_j\}$ where $\infty > m(A_j) > 0$. Then the functions, $\left\{\frac{1}{\sqrt{m(A_j)}} 1_{A_j}\right\}_{j=1}^{\infty}$ forms an orthonormal subset of $L^2(m)$ and therefore,

$$\begin{aligned}\mathrm{tr}\left(M_f\right) &\geq \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt{m(A_j)}} 1_{A_j}, f \frac{1}{\sqrt{m(A_j)}} 1_{A_j}\right) = \sum_{j=1}^{\infty} \frac{1}{m(A_j)} \int_{A_j} f dm \\ &\geq \sum_{j=1}^{\infty} \frac{1}{m(A_j)} \int_{A_j} \varepsilon dm = \varepsilon \cdot \infty = \infty.\end{aligned}$$

Notation 8.12 *Let $\mathbb{T}^d$ – the d – dimensional torus which we identify with $[0,2\pi]^d / \backsim$ where $\backsim$ is the usual identification of the endpoints. We will denote points in $[0,2\pi]^d$ by θ and let $d\theta$ denote normalized Haar measure on $\mathbb{T}^d$. For any $s \in \mathbb{R}$ let and $f \in C^\infty\left(\mathbb{T}^d\right)$ let*

$$\|f\|_s^2 := \sum_{n \in \mathbb{Z}^d} \left(\|n\|^2 + m^2\right)^s \left|\hat{f}(n)\right|^2$$

where

$$\hat{f}(n) := \int_{\mathbb{T}^d} f(\theta)\, e^{-in\cdot\theta} d\theta.$$

Then let $L_s^2\left(\mathbb{T}^d\right)$ be the closure (completion) of $C^\infty\left(\mathbb{T}^d\right)$ in the $\|\cdot\|_s$ so that $L_s^2\left(\mathbb{T}^d\right)$ is the Sobolev space of "functions" with s – derivatives in $L^2\left(\mathbb{T}^d\right)$.

We will use below that $\|f\|_0 = \|f\|_{L^2(\mathbb{T}^d)}$ and

$$\|f\|_1^2 = \int_{\mathbb{T}^d} \left[\|\nabla f(\theta)\|_{\mathbb{R}^d}^2 + m^2 \left|f(\theta)\right|^2\right] d\theta.$$

Example 8.13. Part of the problem above was the non-compactness of $\mathbb{R}^d$. To avoid this issue, let us replace $\mathbb{R}^d$ by $\mathbb{T}^d$. In this case $\left\{\frac{e^{in\cdot\theta}}{\sqrt{\|n\|^2+m^2}}\right\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis for $H = L_1^2\left(\mathbb{T}^d\right)$ equipped with the inner product;

$$(f,g)_{L_1^2(\mathbb{T}^d)} := \int_{\mathbb{T}^d} \left[\nabla f(\theta)\cdot\nabla g(\theta) + m^2 f(\theta)\, g(\theta)\right] d\theta.$$

Hence if $i : L_1^2\left(\mathbb{T}^d\right) \to L_0^2\left(\mathbb{T}^d\right) = L^2\left(\mathbb{T}^d\right)$ is the inclusion map, then

$$\|i\|_{H.S.}^2 = \sum_{n\in\mathbb{Z}^d} \left\|\frac{e^{in\cdot\theta}}{\sqrt{\|n\|^2 + m^2}}\right\|_{L^2(\mathbb{T}^d)}^2 = \sum_{n\in\mathbb{Z}^d} \frac{1}{\|n\|^2 + m^2}$$

which is finite iff $d = 1$. When $d = 2$ the sum is logarithmically divergent and it is worse when $d = 3$. This can also be understood by noting that $\left(-\Delta + m^2\right)^{-1}(\theta - \alpha)$ has the same singularity structure as Example 8.11 above.

In light of this example we see that we need to take W even bigger than $L^2\left(\mathbb{T}^d\right)$ and in fact we have to take W to be a space of distributions. This is at the heart of the infinities arising in Quantum field theory. For example the expression,

$$\int_{\mathbb{R}^{d+1}} v(\varphi(y))\, dy$$

appearing in Eq. (8.5) does not make sense when φ is only a distribution. (For example of $v(\varphi) = \lambda\varphi^4$ then we might have to deal with $v(\delta(y)) = \delta(y)^4$ where δ is the delta distribution. As I hope you know $\delta(y)^4$ is very singular.)

However when $d = 2$ we just miss and need only take $W = L_s^2\left(\mathbb{T}^d\right)$ for any $s < 0$ as – however only just barely when $d = 2$. For example for any $s \in \mathbb{R}$ and $f \in C^\infty\left(\mathbb{T}^2\right)$,

$$\|f\|_s^2 := \sum_{n\in\mathbb{Z}^2} \left(\|n\|^2 + m^2\right)^s \left|\hat{f}(n)\right|^2$$

Where $\hat{f}(n) := \int_{\mathbb{T}^2} f(\theta)\, e^{-in\cdot\theta} d\theta$. So $\|f\|_0^2 = \|f\|_{L^2(\mathbb{T}^2)}^2$, $\|f\|_1^2 = \|f\|_H$ and for any $s < 0$, $\|f\|_s^2 \leq \|f\|_0^2$. Let W_s denote the completion of $C^\infty\left(\mathbb{T}^2\right)$ in the s – norm. This is the Sobolev space of s – derivatives in L^2. For any $s < 0$, the inclusion map, $i : H = W_1 \to W_s$ is Hilbert Schmidt. Indeed, we now have

$$\|i\|_{H.S.}^2 = \sum_{n\in\mathbb{Z}^2} \left\| \frac{e^{in\cdot\theta}}{\sqrt{\|n\|^2+m^2}} \right\|_{W_s}^2 = \sum_{n\in\mathbb{Z}^2} \frac{1}{\|n\|^2+m^2} \left(\|n\|^2+m^2\right)^s$$

$$= \sum_{n\in\mathbb{Z}^2} \frac{1}{\left(\|n\|^2+m^2\right)^{1+|s|}} < \infty.$$

Part III

Segal - Bargnann Theory

9

Heat smoothing and pointwise bounds

Notation 9.1 (Derivative operators) *If $f : W \to \mathbb{C}$ is a smooth function near some point $x \in W$ and $n \in \mathbb{N}$, let $D_x^n f : W^n \to \mathbb{C}$ be the n^{th} – order derivatives of f at x defined by*

$$D_x^n f(h_1, \dots, h_n) := (\partial_{h_1} \dots \partial_{h_n} f)(x),$$

where for any $h \in W$ we let

$$(\partial_h f)(x) = \frac{d}{dt}|_0 f(x + th)$$

be the directional derivative[1] *of f at x in the "direction" h.*

Theorem 9.2 (Heat Kernel Smoothing). *Let $(W, \mathcal{B}_W, \mu)$ be a Gaussian probability space and $p \in (1, \infty)$. Then;*

1. *For all $f \in L^p(\mu)$ and $h \in H := H_\mu$, $f(h + \cdot) \in L^{p-}(\mu)$ and*
$$\mu * f(h) = \int_W f(y + h)\, d\mu(y) = \int_H f \exp\left(J_\mu^* h - \frac{1}{2}\|h\|_H^2\right) d\mu$$
*is well defined and satisfies the following **pointwise bounds;***
$$|\mu * f(h)| \le \|f\|_{L^p(\mu)} e^{\frac{1}{2(p-1)}\|h\|_H^2} \text{ for all } h \in H. \tag{9.1}$$
2. *The resulting function, $\mu * f : H \to \mathbb{C}$ admits an analytic continuation to $H_\mathbb{C} := H + iH$ – the complexification of H. In particular $\mu * f$ is real analytic on H.*
3. *The derivatives may be computed as;*
$$D_0^n(\mu * f)(h_1, \dots, h_n) = \int_H f \partial_{h_1} \dots \partial_{h_n}\left(h \to e^{\left[J^* h - \frac{1}{2}\|h\|_H^2\right]}\right) d\mu.$$

(See [7, Corollary 1.9 on p. 507] for a Lie group version of Eq. (9.1).)

[1] The term directional derivative is a bit of a misnomer here since the derivative depends on both the direction and the length of h and not just the direction of h.

Proof. 1. The first technical point here is that an element of $L^2(\mu)$ is an equivalence class of functions rather than a fixed function so we need to know that the integral is independent of the choice of function in this equivalence class. However by the Cameron-Martin theorem we know that μ and $\mu(\cdot - h)$ have the same null sets so this technical point is OK. Moreover, by the Cameron Martin theorem we learn that

$$(\mu * f)(h) = \int_W f(h + x)\, d\mu(x) = \int_W f \exp\left(J_\mu^* h - \frac{1}{2}\|h\|_H^2\right) d\mu.$$

As $\exp\left(J_\mu^* h - \frac{1}{2}\|h\|_H^2\right) \in L^{\infty-}(\mu)$ it follows by Hölder's inequality that the latter integral is well defined and hence so is the first. In more detail let $q = p/(p-1)$ be the conjugate exponent to p. Then using $J^* h \overset{d}{=} N\left(0, \|h\|_H^2\right)$ and Hölder's inequality,

$$\begin{aligned}
|\mu * f(h)| &\le \int_W |f| \exp\left(J_\mu^* h - \frac{1}{2}\|h\|_H^2\right) d\mu \\
&\le \|f\|_{L^p(\mu)} \left\| e^{J^* h - \frac{1}{2}\|h\|_H^2} \right\|_{L^q(\mu)} \\
&= e^{-\frac{1}{2}\|h\|_H^2} \|f\|_{L^p(\mu)} \left\| e^{J^* h} \right\|_{L^q(\mu)} \\
&= e^{-\frac{1}{2}\|h\|_H^2} \|f\|_{L^p(\mu)} \sqrt[q]{\mathbb{E}_\mu\left[e^{qJ^* h}\right]} \\
&= e^{-\frac{1}{2}\|h\|_H^2} \|f\|_{L^p(\mu)} \sqrt[q]{e^{\frac{1}{2}q^2 \cdot \|h\|_H^2}} \\
&= e^{-\frac{1}{2}\|h\|_H^2} \|f\|_{L^p(\mu)} e^{\frac{1}{2}q \cdot \|h\|_H^2} \\
&= \|f\|_{L^p(\mu)} e^{\frac{1}{2}\|(q-1)h\|_H^2} = \|f\|_{L^p(\mu)} e^{\frac{1}{2(p-1)}\|h\|_H^2}
\end{aligned}$$

proving Eq. (9.1).

We may easily extend the latter expression to $H_\mathbb{C}$ by **defining**

$$(\mu * f)(z) = \int_W f \exp\left(J_\mu^* z - \frac{1}{2}B(z, z)\right) d\mu$$

where if $z = h + ik \in H_\mathbb{C}$, $J_\mu^* z := J_\mu^* h + iJ_\mu^* k$, and $B : H_\mathbb{C} \times H_\mathbb{C} \to \mathbb{C}$ is the complex bilinear (**not** sequilinear) extension of the inner product on H. If $w \in H_\mathbb{C}$ as well and $\lambda \in \mathbb{C}$ we have

$$(\mu * f)(z + \lambda w) = \int_W f \cdot e^{J_\mu^* z + \lambda J_\mu^* w - \frac{1}{2}\left[B(z,z) + 2\lambda B(z,w) + \lambda^2 B(w,w)\right]} \, d\mu$$

which is a holomorphic function of λ by Morera's and Fubini's theorem. (See the proof of Lemma 12.11 below for more details on this type of argument.) Hence we have shown $\mu * f$ has an analytic continuation to $H_{\mathbb{C}}$.

Moreover, from this expression we learn that we may differentiate $\mu * f(h)$ relative to h as many times as we like to find

$$\begin{aligned}
&(\partial_{k_1} \dots \partial_{k_n} \mu * f)(h) \\
&\quad = \int_W f \cdot \partial_{k_1} \dots \partial_{k_n} \left[h \to \exp\left(J_\mu^* h - \frac{1}{2} \|h\|_H^2 \right) \right] \, d\mu \\
&\quad = \int_W f \cdot p_n \left(J_{\mu^*} k_1, \dots, J_\mu^* k_n \right) \cdot \left[\exp\left(J_\mu^* h - \frac{1}{2} \|h\|_H^2 \right) \right] \, d\mu.
\end{aligned}$$

A key point here is that $\partial_k \left[h \to J_\mu^* h \right] = J_\mu^* k$ which is clear because J_μ^* is a linear map. Moreover by Lemma 3.9 we may conclude that $\partial_k \left(h \to \exp\left(J_\mu^* h \right) \right) = J_\mu^* k \cdot \exp\left(J_\mu^* h \right)$ where the derivative holds in all L^p for all $1 \le p < \infty$. ■

Exercise 9.1. The pointwise bounds in Eq. (9.1) are tight in the sense that it is not possible to decrease the coefficient of $\frac{1}{2(p-1)}$ appearing in the exponent in Eq. (9.1). In your proof you may as well assume that $W = \mathbb{R}^d$ and $d\mu(x) = (2\pi)^{-d/2} e^{-\frac{1}{2} x \cdot x} dx$. **Suggestion:** compute $\mu * f_z$ where $f_z(x) = \exp\left(\frac{z}{2} x \cdot x \right)$ and observe that $f_z \in L^p(\mu)$ for all $z < \frac{1}{p}$.

Fock Spaces

Suppose that H and K are two Hilbert spaces and $\alpha : H \times K \to \mathbb{C}$ is a multilinear form which is continuous in each variable separately. Then for each $k \in K$ we have $\sup_{\|h\|=1} |\alpha(h,k)| = \|\alpha(\cdot,k)\|_{H^*} < \infty$ by assumption. Therefore $\{\alpha(h,\cdot)\}_{\|h\|=1}$ are point wise bounded collection of continuous linear functionals on K and so by the uniform boundedness principle they are uniformly bounded, i.e.

$$\sup_{\|h\|=1} \sup_{\|k\|=1} |\alpha(h,k)| = \sup_{\|h\|=1} \|\alpha(h,\cdot)\|_{K^*} =: C < \infty.$$

Thus $\alpha : H \times K \to \mathbb{C}$ is continuous.

Lemma 10.1. *Suppose that $\alpha : H \times K \to \mathbb{R}$ is a continuous bilinear form, then*

$$\|\alpha\|^2 := \sum_{h \in S, k \in \Lambda} |\alpha(h,k)|^2$$

is independent of choice of orthonormal bases S for H and Λ for K.

Proof. Let $\tilde{\alpha} : H \to K$ be the unique linear operator such that

$$\alpha(h,k) = (\tilde{\alpha}h, k)_K \text{ for all } h \in H \text{ and } k \in K.$$

That is it $\beta : K^* \to K$ is the inverse or the map $K \ni k \to (k,\cdot) \in K^*$, then $\tilde{\alpha}h = \beta \circ \alpha(h,\cdot)$. Notice that

$$\|\tilde{\alpha}h\|_K = \sup_{\|k\|=1} |(\tilde{\alpha}h, k)_K| = \sup_{\|k\|=1} |\alpha(h,k)| \le C \|h\|_H$$

so that $\tilde{\alpha}$ is a bounded operator. Moreover we have

$$\sum_{h \in S, k \in \Lambda} |\alpha(h,k)|^2 = \sum_{h \in S, k \in \Lambda} |(\tilde{\alpha}h, k)_K|^2 = \sum_{h \in S} |\tilde{\alpha}h|^2 = \|\tilde{\alpha}\|_{HS}^2$$

and the latter quantity is known to be basis independent. ■

Proposition 10.2. *Suppose that $\alpha : H_1 \times \cdots \times H_n \to \mathbb{R}$ is a continuous*[1] *multi-linear form. Then*

$$\|\alpha\|^2 := \sum_{h_1 \in S_1, \dots, h_n \in S_n} |\alpha(h_1, \dots, h_n)|^2$$

is independent of the choices of orthonormal bases, S_i, for H_i.

Proof. The best way do this is as follows. Let Λ_i be other bases for H_i, then for any $u_l \in H_l$ and $1 \le i \le n$ we have $h \to \alpha(u_1, \dots, u_{i-1}, h, u_{i+1}, \dots, u_n)$ is a continuous linear functional on H_i and therefore

$$\sum_{h \in S_i} |\alpha(u_1, \dots, u_{i-1}, h, u_{i+1}, \dots, u_n)|^2 = \|\alpha(u_1, \dots, u_{i-1}, \cdot, u_{i+1}, \dots, u_n)\|_{H_i^*}^2$$

which is independent of the choice of basis. Thus we find that

$$\begin{aligned}
&\sum_{\substack{(k_1,\dots,k_{i-1}) \in \Lambda_1 \times \cdots \times \Lambda_{i-1} \\ (h_i,\dots,h_n) \in S_i \times \cdots \times S_n}} |\alpha(k_1, \dots, k_{i-1}, h_i, \dots, h_n)|^2 \\
&= \sum_{\substack{(k_1,\dots,k_{i-1}) \in \Lambda_1 \times \cdots \times \Lambda_{i-1} \\ (h_{i+1},\dots,h_n) \in S_{i+1} \times \cdots \times S_n}} \sum_{h_i \in S_i} |\alpha(k_1, \dots, k_{i-1}, h_i, \dots, h_n)|^2 \\
&= \sum_{\substack{(k_1,\dots,k_{i-1}) \in \Lambda_1 \times \cdots \times \Lambda_{i-1} \\ (h_{i+1},\dots,h_n) \in S_{i+1} \times \cdots \times S_n}} \sum_{k_i \in \Lambda_i} |\alpha(k_1, \dots, k_i, h_{i+1}, \dots, h_n)|^2 \\
&= \sum_{\substack{(k_1,\dots,k_i) \in \Lambda_1 \times \cdots \times \Lambda_i \\ (h_{i+1},\dots,h_n) \in S_{i+1} \times \cdots \times S_n}} |\alpha(k_1, \dots, k_i, h_{i+1}, \dots, h_n)|^2
\end{aligned}$$

and so it follows by induction that

$$\sum_{(k_1,\dots,k_n) \in \Lambda_1 \times \cdots \times \Lambda_n} |\alpha(k_1, \dots, k_n)|^2 = \sum_{h_1 \in S_1, \dots, h_n \in S_n} |\alpha(h_1, \dots, h_n)|^2.$$

Even better suppose that $\alpha : H_1 \times \cdots \times H_n \to H$ is another Hilbert space is multi-linear form which is continuous in each of its variables. Then

$$\begin{aligned}
&\sum_{h \in S_i} \|\alpha(u_1, \dots, u_{i-1}, h, u_{i+1}, \dots, u_n)\|_H^2 \\
&\quad = \|\alpha(u_1, \dots, u_{i-1}, \cdot, u_{i+1}, \dots, u_n)\|_{HS(H_i, H)}
\end{aligned}$$

[1] The continuity assertion is equivalent to the existstence of a $C < \infty$ such that

$$|\alpha(h_1, \dots, h_n)| \le C \|h_1\|_{H_1} \dots \|h_n\|_{H_n}$$

for all $h_i \in H_i$.

which is basis independent. The same argument as above now allows us to change the basis one slot at a time to see that the whole thing is basis independent. ■

Definition 10.3. *If $\rho : H^n \to \mathbb{C}$ be a multilinear form which is continuous in each of its variables and we let*

$$\|\rho\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} := \sum_{(h_1,\dots h_n)\in S_1\times\cdots\times S_n} |\rho(h_1,\dots,h_n)|^2 \tag{10.1}$$

where $S \subset H$ is any orthonormal basis of H. Further let $\mathrm{Mult}_n(H,\mathbb{C})$ denote those ρ such that $\|\rho\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} < \infty$ where by convention, $\mathrm{Mult}_0(H,\mathbb{C}) = \mathbb{C}$ with inner product, $(z,w)_{\mathrm{Mult}_0(H,\mathbb{C})} := z\bar{w}$. As we saw above these definitions are well defined. We further let

$$\mathrm{Sym}_n(H,\mathbb{C}) = \{\alpha \in \mathrm{Mult}_n(H,\mathbb{C}) : \alpha \text{ is symmetric}\}.$$

Below we will usually just write $\|\rho\|$ for $\|\rho\|_{\mathrm{Mult}_n(H,\mathbb{C})}$ as it should be clear from context which norm we mean.

Lemma 10.4. $\mathrm{Mult}_n(H,\mathbb{C})$ *is a complex Hilbert space in when equipped with the inner product,*

$$(\rho_1,\rho_2)_{\mathrm{Mult}_n(H,\mathbb{C})} = \sum_{h_1,\dots h_n\in S} \rho_1(h_1,\dots,h_n)\cdot\overline{\rho_2(h_1,\dots,h_n)}. \tag{10.2}$$

Proof. The sum defining the inner product in Eq. (10.2) converges by the Cauchy - Schwarz inequality and clearly defines an inner product on $\mathrm{Mult}_n(H,\mathbb{C})$ whose associated norm is given by Eq. (10.1). Since the inner product may be recovered from the norm by polarization it must be basis independent. So it only remains to show $\mathrm{Mult}_n(H,\mathbb{C})$ is a complete space.

To simplify notation let $\|\rho\| := \|\rho\|_{\mathrm{Mult}_n(H,\mathbb{C})}$. So let $\{\rho_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathrm{Mult}_n(H,\mathbb{C})$. As any unit vectors $h \in H$ is part of an orthonormal basis for H it easily follows that $\{\rho_k(h_1,\dots,h_n)\}_{k=1}^\infty$ is a Cauchy sequence for all $(h_1,\dots,h_n) \in H^n$. Thus we know that $\rho(h_1,\dots,h_n) := \lim_{k\to\infty}\rho_k(h_1,\dots,h_n)$ exists and the resulting function, $\rho : H^n \to \mathbb{C}$ is still multi-linear. By the uniform boundedness principle it is continuous in each of its variables as well. We now use Fatou's lemma to learn,

$$\begin{aligned}
\|\rho-\rho_k\|^2 &= \sum_{(h_1,\dots h_n)\in S_1\times\cdots\times S_n} |\rho(h_1,\dots,h_n)-\rho_k(h_1,\dots,h_n)|^2\\
&= \sum_{(h_1,\dots h_n)\in S_1\times\cdots\times S_n} \liminf_{l\to\infty}|\rho_l(h_1,\dots,h_n)-\rho_k(h_1,\dots,h_n)|^2\\
&\le \liminf_{l\to\infty}\sum_{(h_1,\dots h_n)\in S_1\times\cdots\times S_n} |\rho_l(h_1,\dots,h_n)-\rho_k(h_1,\dots,h_n)|^2\\
&= \liminf_{l\to\infty}\|\rho_l-\rho_k\|^2 \to 0 \text{ as } k\to\infty.
\end{aligned}$$

■

Example 10.5. Suppose that $\alpha_1,\dots,\alpha_n \in H^*$, then we may define an element $\alpha_1\otimes\cdots\otimes\alpha_n \in \mathrm{Mult}_n(H,\mathbb{C})$ by

$$\alpha_1\otimes\cdots\otimes\alpha_n(h_1,\dots,h_n) = \prod_{i=1}^n \alpha_i(h_i).$$

Similarly $\alpha_1\vee\cdots\vee\alpha_n := \sum_\sigma \alpha_{\sigma1}\otimes\cdots\otimes\alpha_{\sigma n} \in \mathrm{Sym}_n(H,\mathbb{C})$ where the sum is over all permutations of $\{1,2,\dots,n\}$. Observe that

$$\|\alpha_1\otimes\cdots\otimes\alpha_n\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} = \prod_{i=1}^n \|\alpha_i\|^2_{H^*}$$

and that

$$\begin{aligned}
\|\alpha_1\vee\cdots\vee\alpha_n\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} &= \left(\sum_\sigma \alpha_{\sigma1}\otimes\cdots\otimes\alpha_{\sigma n}, \sum_\tau \alpha_{\tau1}\otimes\cdots\otimes\alpha_{\tau n}\right)_{\mathrm{Mult}_n(H,\mathbb{C})}\\
&= \sum_{\sigma,\tau}\prod_{i=1}^n(\alpha_{\sigma i},\alpha_{\tau i})_{H^*} = \sum_{\sigma,\tau}\prod_{i=1}^n(\alpha_{\sigma\tau i},\alpha_{\tau i})_{H^*}\\
&= \sum_{\sigma,\tau}\prod_{i=1}^n(\alpha_{\sigma i},\alpha_i)_{H^*} = n!\sum_\sigma\prod_{i=1}^n(\alpha_{\sigma i},\alpha_i)_{H^*}.
\end{aligned}$$

Example 10.6. Suppose that $f : H \to \mathbb{C}$ is a smooth function near $x \in H$, then $D^n_x f : H^n \to \mathbb{C}$ (see Notation 9.1) defines a multi-linear symmetric function on H since mixed partial derivatives commute. If we further assume that $f \in \mathcal{FC}^\infty(H^*)$ so that $f = F((k_1,\cdot),\dots,(k_m,\cdot))$ for some $k_i \in H$, then $D^n_x f \in \mathrm{Sym}_n(H,\mathbb{C})$ and

$$D^n_x f = \sum_{i_1,\dots,i_n=1}^m (\partial_{i_1}\dots\partial_{i_n}F)((k_1,\cdot),\dots,(k_m,\cdot))(k_{i_1},\cdot)\otimes\cdots\otimes(k_{i_n},\cdot).$$

Remark 10.7. Let $\mathcal{P}_n(H^*)$ be the space of homogeneous polynomials of degree n on H. When $\dim H < \infty$. the map

$$\mathrm{Sym}_n(H, \mathbb{C}) \ni \alpha \to (x \to \alpha(x, x, \ldots, x)) \in \mathcal{P}_n(H^*) \tag{10.3}$$

is a linear isomorphism with inverse map given by

$$\mathcal{P}_n(H^*) \ni p \to \frac{1}{n!} D_0^n p \in \mathrm{Sym}_n(H, \mathbb{C}). \tag{10.4}$$

When $\dim H = \infty$ it is no longer true that $\mathcal{P}_n(H^*)$ and $\mathrm{Sym}_n(H, \mathbb{C})$ are isomorphic. To describe better what is going on in this case let $\mathrm{Mult}_n^{\mathrm{alg}}(H, \mathbb{C})$ denote those $\alpha \in \mathrm{Mult}_n(H, \mathbb{C})$ such that $\alpha = P^*\alpha$ for some finite rank orthonormal projection on H where $P^*\alpha(h_1, \ldots, h_n) := \alpha(Ph_1, \ldots, Ph_n)$. We also let $\mathrm{Sym}_n^{\mathrm{alg}}(H, \mathbb{C}) = \mathrm{Mult}_n^{\mathrm{alg}}(H, \mathbb{C}) \cap \mathrm{Sym}_n(H, \mathbb{C})$.

Proposition 10.8. *The map in Eq. (10.4) is a linear isomorphism from* $\mathcal{P}_n(H^*)$ *onto* $\mathrm{Sym}_n^{\mathrm{alg}}(H, \mathbb{C})$ *and* $\mathrm{Sym}_n^{\mathrm{alg}}(H, \mathbb{C})$ *is a dense subspace in* $\mathrm{Sym}_n(H, \mathbb{C})$.

Proof. Let S be an orthonormal basis for H and $\{S_l\}_{l=1}^\infty \subset S$ such that $S_l \uparrow S$ with $\#(S_l) < \infty$ for all l. Further let $P_l x := \sum_{h \in S_l} (x, h) h$ be orthogonal projection onto $H_l := \operatorname{span} S_l$. Then give $\alpha \in \mathrm{Sym}_n(H, \mathbb{C})$ let $\alpha_l := P_l^* \alpha \in \mathrm{Sym}_n^{\mathrm{alg}}(H, \mathbb{C})$ and $p_l(x) := \alpha_l(x, \ldots, x)$ in $\mathcal{P}_n(H)$ with $\frac{1}{n!} D_0^n p_l = \alpha_l$. So to finish the proof of the assertion it suffices to show $\alpha_l \to \alpha$ in $\mathrm{Sym}_n(H, \mathbb{C})$. This however follows from the DCT for sums. Indeed, for $h_1, \ldots, h_n \in S$, we have

$$|(\alpha - \alpha_l)(h_1, \ldots, h_n)|^2 \le 2 |\alpha(h_1, \ldots, h_n)|^2$$

while

$$\lim_{l \to \infty} |(\alpha - \alpha_l)(h_1, \ldots, h_n)|^2 = 0.$$

■

Definition 10.9 (Fock spaces). *Given a real Hilbert space,* H *and* $t > 0$, *let*

$$\mathcal{T}(H; t) := \left\{ \alpha = (\alpha_n)_{n=0}^\infty : \alpha_n \in \mathrm{Mult}_n(H, \mathbb{C}) \text{ and } \|\alpha\|_t^2 < \infty \right\} \text{ and}$$
$$\mathcal{F}(H; t) := \{ \alpha \in \mathcal{T}(H; t) : \alpha_n \in \mathrm{Sym}_n(H, \mathbb{C}) \}$$

where

$$\|\alpha\|_t^2 := \sum \frac{t^n}{n!} \|\alpha_n\|_{\mathrm{Mult}_n(H, \mathbb{C})}^2 .$$

We call $\mathcal{T}(H; t)$ *the full* ***Fock space*** *over* H *and* $\mathcal{F}(H; t)$ *the* ***Bosonic Fock space*** *over* H.

The full Fock space $\mathcal{T}(H; t)$ is a Hilbert space when given the inner product,

$$(\alpha, \beta)_t := \sum \frac{t^n}{n!} (\alpha_n, \beta_n)_{\mathrm{Mult}_n(H, \mathbb{C})}$$

for all $\alpha, \beta \in \mathcal{T}(H; t)$ and $\mathcal{F}(H; t)$ is a Hilbert subspace of $\mathcal{T}(H; t)$.

11

Segal Bargmann Transforms

11.1 Three key identities

Let $(W, \mathcal{B}_W, v)$ be a real Gaussian probability space with Cameron-Martin space $H := H_v$ and $B := L_v := \sum_{h \in S_v} \partial_h^2$ where S_ν is an orthonormal basis for H. (We assume $\dim W < \infty$ for the moment but it is not really needed). Given $f, g \in \mathcal{P}(W^*)$, $n \in \mathbb{N}$, and $h_1, \dots, h_n \in H$, let

$$D_x^n f(h_1, \dots, h_n) := (\partial_{h_1} \dots \partial_{h_n} f)(x).$$

As mixed partial derivatives commute it follows that $D_x^n f \in \mathrm{Sym}_n(H, \mathbb{C})$. Further let $\alpha_x^f := (D_x^n f)_{n=0}^\infty$ which is in$\mathcal{F}(H; t)$ for all $t > 0$ when $f \in \mathcal{P}(W^*)$.

Theorem 11.1 (Key identity 1). *Let $f, g \in \mathcal{P}(W^*)$, $x \in W$, and $t > 0$, then*

$$e^{tB/2}\left[e^{-tB/2} f \cdot e^{-tB/2} \bar{g}\right](x) = \left(\alpha_x^f, \alpha_x^g\right)_t = \sum_{n=0}^\infty \frac{t^n}{n!} (D_x^n f, D_x^n \bar{g})_{\mathrm{Mult}_n(H,\mathbb{C})} \tag{11.1}$$

and in particular

$$\left(e^{-B/2} f \cdot e^{-B/2} g\right)_{L^2(\nu)} = \left(\alpha_0^f, \alpha_0^g\right)_1 = \sum_{n=0}^\infty \frac{1}{n!} (D_0^n f, D_0^n \bar{g})_{\mathrm{Mult}_n(H,\mathbb{C})}. \tag{11.2}$$

Proof. Below we will suppress x from the notation with the understanding that all formulas are to evaluated at x. To further simplify the notion let

$$F_t := e^{-tB/2} f \text{ and } G_t := e^{-tB/2} \bar{g}.$$

By simple calculus we have,

$$\begin{aligned} \frac{d}{dt}\left(e^{tB/2}\left[F_t \cdot \bar{G}_t\right]\right) &= \frac{1}{2} e^{tB/2}\left(B\left(F_t \bar{G}_t\right) - BF_t \cdot \bar{G}_t - F_t \cdot B\bar{G}_t\right) \\ &= \sum_{h \in S} e^{tB/2}\left(\partial_h F_t \cdot \partial_h \bar{G}_t\right) \\ &= \sum_{h \in S} e^{tB/2}\left[e^{-tB/2} \partial_h f \cdot e^{-tB/2} \partial_h \bar{g}\right]. \end{aligned}$$

It now follows by induction that

$$\frac{d^n}{dt^n}\left(e^{tB/2}\left[F_t \cdot \bar{G}_t\right]\right) = \sum_{h_1, \dots, h_n \in S} e^{tB/2}\left[e^{-tB/2} \partial_{h_1} \dots \partial_{h_n} f \cdot e^{-tB/2} \partial_{h_1} \dots \partial_{h_n} \bar{g}\right].$$

Because $e^{tB/2}\left[e^{-tB/2} f \cdot e^{-tB/2} \bar{g}\right]$ is a polynomial in t, Taylor's theorem implies

$$\begin{aligned} e^{tB/2}\left[e^{-tB/2} f \cdot e^{-tB/2} \bar{g}\right] &= \sum_{n=0}^\infty \frac{t^n}{n!} \frac{d^n}{dt^n}\left(e^{tB/2}\left[F_t \cdot \bar{G}_t\right]\right)|_{t=0} \\ &= \sum_{n=0}^\infty \frac{t^n}{n!} (D^n f, D^n \bar{g})_{\mathrm{Mult}_n(H,\mathbb{C})} \end{aligned}$$

as claimed. Equation (11.2) follows immediately from Eq. (11.1) with $t = 1$ and $x = 0$. ■

Recall from Theorem 11.1 that for all $f, g \in \mathcal{P}(W^*)$;

$$e^{B/2}\left[e^{-B/2} f \cdot e^{-B/2} \bar{g}\right] = \sum_{n=0}^\infty \frac{1}{n!} (D^n f, D^n g)_{\mathrm{Mult}_n(H_\nu,\mathbb{C})}. \tag{11.3}$$

Notation 11.2 *Let $W_\mathbb{C} = W + iW$ denote the complexification of W, $W_\mathbb{C}^* = \mathrm{Hom}_\mathbb{C}(W, \mathbb{C})$ be the continuous complex linear functionals on W, and $W_\mathbb{C}^\dagger := \mathrm{Hom}_\mathbb{R}(W, \mathbb{C})$ be the continuous real linear functionals on $W_\mathbb{C}$. Further let $\tilde{B}$ be the operator acting on $\mathcal{P}\left(W_\mathbb{C}^\dagger\right)$ given by*

$$\tilde{B} := \sum_{h \in S_\nu} \partial_{ih}^2.$$

We will also abuse notation and view $B = \sum_{h \in S_\nu} \partial_{ih}^2$ as on operator on both $\mathcal{P}(W)$ and on $\mathcal{P}\left(W_\mathbb{C}^\dagger\right)$.

Theorem 11.3 (Key identity 2). *If $f, g \in \mathcal{P}(W_\mathbb{C}^*)$ (the holomorphic polynomial functions on $W_\mathbb{C}$), then*[1]

$$e^{\tilde{B}/2}\left[e^{B/2} f \cdot e^{B/2} \bar{g}\right] = \sum_{n=0}^\infty \frac{1}{n!} (D^n f, D^n g)_{\mathrm{Mult}_n(H_\nu,\mathbb{C})}. \tag{11.4}$$

[1] The identity in this theorem is not as key as the other two key identities. We will in fact only use it in the proof of the third key identity.

Proof. Let
$$u(t) := e^{t\tilde{B}/2}\left[e^{tB/2}f \cdot e^{tB/2}\bar{g}\right]$$
which is a polynomial in t and therefore as usual,
$$e^{\tilde{B}/2}\left[e^{B/2}f \cdot e^{B/2}\bar{g}\right] = u(1) = \sum_{n=0}^{\infty}\frac{1}{n!}u^{(n)}(0).$$
Moreover, we have for $f, g \in \mathcal{P}(W_{\mathbb{C}}^{*})$ and $h \in H_\nu$;
$$\partial_{ih}f = i\partial_h f,\ \partial_{ih}\bar{g} = \overline{\partial_{ih}g} = \overline{i\partial_h g} = -i\partial_h\bar{g},$$
$$\tilde{B}f = -Bf,\ \tilde{B}\bar{g} = -B\bar{g},\ \text{and}$$
$$\begin{aligned}\tilde{B}(f\bar{g}) &= \tilde{B}f\cdot\bar{g} + f\tilde{B}\bar{g} + 2\sum_{h\in S_\nu}\partial_{ih}f\cdot\partial_{ih}\bar{g}\\ &= -Bf\cdot\bar{g} - fB\bar{g} + 2\sum_{h\in S_\nu}\partial_h f\cdot\partial_h\bar{g}. \end{aligned} \tag{11.5}$$
If $f, g \in \mathcal{P}(\alpha_1, \ldots, \alpha_n)$ with $\{\alpha_1, \ldots, \alpha_n\} \subset W_{\mathbb{C}}^{*}$, we can choose S_ν so that $h \perp \{J_\nu\alpha_1|_W, \ldots, J_\nu\alpha_n|_W\}$ for all but at most n element of S_ν. This is equivalent to
$$\#\{h \in S_\nu : \alpha_1(h) = \cdots = \alpha_n(h) = 0\} \le n.$$
With such a choice the sum appearing in Eq. (11.5) is really a finite sum.

The computations now go as before, namely
$$\begin{aligned}\dot{u}(t) &= \frac{1}{2}e^{t\tilde{B}/2}\left[\tilde{B}\left[e^{tB/2}f\cdot e^{tB/2}\bar{g}\right] + Be^{tB/2}f\cdot e^{tB/2}\bar{g} + e^{tB/2}f\cdot Be^{tB/2}\bar{g}\right]\\ &= e^{t\tilde{B}/2}\left[\sum_{h\in S_\nu}\partial_h e^{tB/2}f\cdot\partial_h e^{tB/2}\bar{g}\right]\\ &= \sum_{h\in S_\nu}e^{t\tilde{B}/2}\left[e^{tB/2}\partial_h f\cdot e^{tB/2}\partial_h\bar{g}\right].\end{aligned}$$
Hence by induction we learn that
$$u^{(n)}(t) = \sum_{h_1,\ldots,h_n\in S_\nu}e^{t\tilde{B}/2}\left[e^{tB/2}\partial_{h_1}\ldots\partial_{h_n}f\cdot e^{tB/2}\partial_{h_1}\ldots\partial_{h_n}\bar{g}\right]$$
and therefore,
$$\begin{aligned}e^{\tilde{B}/2}\left[e^{B/2}f\cdot e^{B/2}\bar{g}\right] &= u(1) = \sum_{n=0}^{\infty}\frac{1}{n!}u^{(n)}(0)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{h_1,\ldots,h_n\in S_\nu}\left[\partial_{h_1}\ldots\partial_{h_n}f\cdot\partial_{h_1}\ldots\partial_{h_n}\bar{g}\right]\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(D^n f, D^n g\right)_{\mathrm{Mult}_n(H_\nu,\mathbb{C})}.\end{aligned}$$
■

Notation 11.4 *Associated to a probability measure μ on $(W, \mathcal{B}_W)$ are two the two probability measures $\mu\times\delta_0$ and $\delta_0\times\mu$ on $W_{\mathbb{C}} = W + iW \cong W\times W$. We will continue to denote $\mu\times\delta_0$ by μ and we will denote $\delta_0\times\mu$ by $\tilde{\mu}$. Thus the measures μ and $\tilde{\mu}$ satisfy;*
$$\int_{W_{\mathbb{C}}} f d\mu = \int_W f d\mu \text{ and } \int_{W_{\mathbb{C}}} f d\tilde{\mu} = \int_W f(ix)\, d\mu(x)$$
for all bounded measurable functions f on $W_{\mathbb{C}}$.

Lemma 11.5. *Suppose that μ and ν are two probability measures on $(W, \mathcal{B}_W)$.*

1. *If $\mu(\|\cdot\|_W^n) + \nu(\|\cdot\|_W^n) < \infty$[2] for some $n\in\mathbb{N}$ then $\mu*\nu(\|\cdot\|_W^n) < \infty$.*
2. *If there exists $\varepsilon > 0$ such that $\mu\left(e^{\varepsilon\|\cdot\|_W}\right) + \nu\left(e^{\varepsilon\|\cdot\|_W}\right) < \infty$ then $\mu * \nu\left(e^{\varepsilon\|\cdot\|_W}\right) < \infty$ as well.*
3. *If $\mu(\|\cdot\|_W^n) < \infty$ and $f : W\to\mathbb{R}$ satisfies $|f| \le C(1+\|\cdot\|_W^n)$ then there exists $C' < \infty$ such that $|\mu * f| \le C'(1+\|\cdot\|_W^n)$.*

Exercise 11.1. Prove Lemma 11.5.

Corollary 11.6 (Key identity 3). *Continuing the notation in Theorem 11.3, for all $f, g\in\mathcal{P}(W_{\mathbb{C}}^{*})$,*
$$e^{B/2}[f\cdot\bar{g}] = e^{\tilde{B}/2}\left[e^B f\cdot e^B\bar{g}\right]. \tag{11.6}$$
Alternatively we may write this as,
$$\nu * [f\cdot\bar{g}] = \tilde{\nu} * [\nu_2 * f\cdot\nu_2 * \bar{g}] \tag{11.7}$$
where $\nu_2 := \nu*\nu$ and for a measurable function $u : W_{\mathbb{C}}\to\mathbb{C}$ with polynomial growth,
$$(\nu * u)(z) := \int_W u(z-x)\, d\nu(x) = \int_W u(z+x)\, d\nu(x)$$
and
$$(\tilde{\nu} * u)(z) := \int_W u(z-ix)\, d\nu(x) = \int_W u(z+ix)\, d\nu(x)$$

[2] I will often write $\mu(f)$ for $\int_W f d\mu$.

First proof. By Theorem 11.1 and 11.3 we know that

$$e^{B/2}\left[e^{-B/2}f\cdot e^{-B/2}\bar{g}\right]=e^{\tilde{B}/2}\left[e^{B/2}f\cdot e^{B/2}\bar{g}\right].$$

Replacing f and g by $e^{B/2}f$ and $e^{B/2}g$ respectively in this equality proves Eq. (11.6). ∎

Second proof. In this direct proof we will adapt the argument in Hall [19, p. 820]. For $h\in H_\nu$ we define two differential operators on $\mathcal{P}\left(W_{\mathbb{C}}^{\dagger}\right)$;

$$Z_h:=\frac{1}{2}\left[\partial_h-i\partial_{ih}\right]\text{ and }\bar{Z}_h:=\frac{1}{2}\left[\partial_h+i\partial_{ih}\right].$$

For $f\in\mathcal{P}\left(W_{\mathbb{C}}^{*}\right)$ we have $Z_hf=\partial_hf$, $\bar{Z}_hf=0$, $Z_h\bar{f}=0$, $\bar{Z}_h\bar{f}=\overline{Z_hf}=\partial_h\bar{f}$, and

$$Z_h^2+\bar{Z}_h^2=\frac{1}{2}\left(\partial_h^2-\partial_{ih}^2\right).$$

From these observations it follows that

$$\begin{aligned}e^{B}f\cdot e^{B}\bar{g}&=e^{\sum_{h\in S_\nu}Z_h^2}f\cdot e^{\sum_{h\in S_\nu}\bar{Z}_h^2}\bar{g}\\&=e^{\sum_{h\in S_\nu}\left(Z_h^2+\bar{Z}_h^2\right)}\left[f\cdot\bar{g}\right]=e^{\frac{1}{2}\sum_{h\in S_\nu}\left(\partial_h^2-\partial_{ih}^2\right)}\left[f\cdot\bar{g}\right]\\&=e^{\frac{1}{2}\left(B-\tilde{B}\right)}\left[f\bar{g}\right].\end{aligned}$$

Applying $e^{B/2}$ to both sides of this identity completes the proof. ∎

Corollary 11.7. *Let μ be **any** measure on $(W,\mathcal{B}_W)$ such that $\mu\left(\|\cdot\|_W^n\right)<\infty$ for all $n\in\mathbb{N}$. Then for all $f,g\in\mathcal{P}\left(W_{\mathbb{C}}^{*}\right)$ we have*

$$\int_W f\cdot\bar{g}\,d\left(\mu*\nu\right)=\int_{W_{\mathbb{C}}}\left[\nu_2*f\cdot\nu_2*\bar{g}\right]d\left(\mu\times\nu\right).$$

Proof. Integrate Eq. (11.7) relative to μ. In doing so we make use the fact that ν is symmetric and therefore,

$$\begin{aligned}\int_{W_{\mathbb{C}}}\left[\nu*F\right]d\mu&=\int_W\left[\nu*F\right]d\mu=\int_{W\times W}F\left(x-y\right)d\nu\left(y\right)d\mu\left(x\right)\\&=\int_{W\times W}F\left(x+y\right)d\nu\left(y\right)d\mu\left(x\right)=\int_W Fd\left(\mu*\nu\right)\end{aligned}$$

for $F:W\to[0,\infty]$ measurable and

$$\begin{aligned}\int_{W_{\mathbb{C}}}\tilde{\nu}*Fd\mu&=\int_W\left(\tilde{\nu}*F\right)\left(x\right)d\mu\left(x\right)\\&=\int_W\left(\int_W F\left(x-iy\right)d\nu\left(y\right)\right)d\mu\left(x\right)\\&=\int_{W\times W}F\left(x+iy\right)d\mu\left(x\right)d\nu\left(y\right)=\int_{W\times W}Fd\left(\mu\times\nu\right).\end{aligned}$$

∎

11.2 The Segal-Bargmann Transform

Definition 11.8. *Suppose that W is a complex Banach space and μ is any probability measure on $\mathcal{B}_W$ such that $\mu\left(\|\cdot\|_W^n\right)<\infty$ for all $n\in\mathbb{N}$. We let $\dot{\mathcal{H}}L^2\left(\mu\right)$ denote the $L^2\left(\mu\right)$ – closure of $\mathcal{P}\left(W^*\right)$ – the holomorphic polynomial functions on W.*

Remark 11.9. If $\dim W<\infty$ and μ is a non-degenerate (i.e. $H_\mu=W$) Gaussian probability measure on $\left(W,\mathcal{B}_W\right)$, then we will see in Theorem 12.14 below that $\mathcal{P}\left(W^*\right)$ is a dense subspace of the Hilbert space (see Lemma 2.8) $\mathcal{H}L^2\left(W,\mu\right)$. Thus $\dot{\mathcal{H}}L^2\left(W,\mu\right)=\mathcal{H}L^2\left(W,\mu\right)$ in this case.

Theorem 11.10 (Generalized Segal-Bargmann transform). *Let μ be **any** measure on $\left(W,\mathcal{B}_W\right)$ such that $\mu\left(e^{\varepsilon\|\cdot\|_W}\right)<\infty$ for some $\varepsilon>0$. Then there exists a unique unitary map*

$$S_{\mu,\nu}:L^2\left(W,\mu*\nu\right)\to\dot{\mathcal{H}}L^2\left(W_{\mathbb{C}},\mu\times\nu\right)$$

such that for all $p\in\mathcal{P}\left(W^*\right)$,

$$S_{\mu,\nu}p=\nu_2*p_{\mathbb{C}}=\left(e^{B}p\right)_{\mathbb{C}}.$$

Proof. In light of Lemma 11.5, Fernique's Theorem 3.4, and Theorem 2.2, we know that $\mathcal{P}\left(W^*\right)$ is a dense subspace of $L^2\left(\mu*\nu\right)$. Therefore it follows that the isometric map in Corollary 11.7 extends uniquely to a unitary map from $L^2\left(W,\mu*\nu\right)$ to $\dot{\mathcal{H}}L^2\left(W_{\mathbb{C}},\mu\times\nu\right)$. ∎

In the case where μ and ν are both non-degenerate Gaussian measures we can compute $S_{\mu,\nu}f$ more explicitly.

Corollary 11.11. *If μ and ν are both Gaussian measures on W with full support (i.e. $H_\mu=W=H_\nu$) and $f\in L^2\left(\mu*\nu\right)$, then;*

1. *for all $x\in W$ the following integral exists,*

$$\left(\nu_2*f\right)\left(x\right)=\int_W f\left(x-y\right)d\nu_2\left(y\right).$$

2. *$\nu_2*f:H_{\mu*\nu}=W\to\mathbb{C}$ is smooth and even admits a unique analytic continuation, $\left(\nu_2*f\right)_{\mathbb{C}}$ to all of $W_{\mathbb{C}}$.*
3. *$S_{\mu,\nu}f=\left(\nu_2*f\right)_{\mathbb{C}}$ – $\mu\times\nu$ a.s.*
4. *The resulting Segal - Bargmann map,*

$$S_{\mu,\nu}:L^2\left(W,\mu*\nu\right)\to\mathcal{H}L^2\left(W_{\mathbb{C}},\mu\times\nu\right),$$

is unitary.

Proof. 1. Let us write H for H_ν and $x \cdot y$ for $(x,y)_{H_\nu}$. Further let C be the unique positive operator (matrix) on H such that $(x,y)_{H_\mu} = Cx \cdot y$ for all $x, y \in H$. Further let $\lambda := \mu * \nu$ and observe that

$$\frac{d\nu_2(x)}{dx} \propto \exp\left(-\frac{1}{4}x \cdot x\right) \text{ and}$$
$$\frac{d\lambda(x)}{dx} \propto \exp\left(-\frac{1}{2}(I+C)^{-1} x \cdot x\right) = \exp\left(-\frac{1}{2}(I-D)\, x \cdot x\right)$$

where $D = C(I+C)^{-1}$ and $0 < D < I$ as can be seen by the spectral theorem. Using this notation it follows that

$$\begin{aligned}
&\int_W f(x-y)\, d\nu_2(y) \\
&\quad \propto \int_W f(y) \exp\left(-\frac{1}{4}(x-y)\cdot(x-y)\right) dy \\
&\quad \propto \int_W f(y) \frac{\exp\left(-\frac{1}{4}(x-y)\cdot(x-y)\right)}{\exp\left(-\frac{1}{2}(I-D)\,y\cdot y\right)} d\lambda(y) \\
&\quad = e^{-\frac{1}{4}x\cdot x} \int_W f(y) \exp\left(\frac{1}{2}x\cdot y - \frac{1}{4}y\cdot y + \frac{1}{2}(I-D)\,y\cdot y\right) d\lambda(y).
\end{aligned}$$

To verify the latter integral is well defined it suffices to show

$$\exp\left(\frac{1}{2}x\cdot y - \frac{1}{4}y\cdot y + \frac{1}{2}(I-D)\,y\cdot y\right) \in L^2(d\lambda(y))$$

which is the case since,

$$\begin{aligned}
-\frac{1}{2}(I-D)\,x\cdot x + 2\left(\frac{1}{2}x\cdot y - \frac{1}{4}y\cdot y + \frac{1}{2}(I-D)\,y\cdot y\right) \\
= x\cdot y - \frac{1}{2}y\cdot y + \frac{1}{2}(I-D)\,y\cdot y = x\cdot y - \frac{1}{2}Dy\cdot y
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\int_W \left|\exp\left(\frac{1}{2}x\cdot y - \frac{1}{4}y\cdot y + \frac{1}{2}(I-D)\,y\cdot y\right)\right|^2 d\lambda(y) \\
&\quad \propto \int_W \exp\left(x\cdot y - \frac{1}{2}Dy\cdot y\right) dy < \infty
\end{aligned}$$

as $D > 0$.

2. The analytic continuation, $F(z)$, of $x \to \int_W f(x-y)\, d\nu_2(y)$ is given by

$$F(z) = (\text{const.}) \cdot e^{-\frac{1}{4}z\cdot z} \int_W f(y)\, K(z,y)\, d\lambda(y)$$

where

$$K(z,y) := \exp\left(\frac{1}{2}z\cdot y - \frac{1}{4}y\cdot y + \frac{1}{2}(I-D)\,y\cdot y\right)$$

The fact that

$$z \to \int_W f(y)\, K(z,y)\, d\lambda(y)$$

is analytic follows either by Morera's Theorem 2.5 or by differentiating past the integral. (See the proof of Lemma 12.11 below for more details on this type of argument.) The details are left to the reader.

3. As $\mu \times \nu$ is a positive smooth measure on $W_{\mathbb{C}}$ it follows from Lemma 2.8 that the evaluation maps,

$$\mathcal{H}L^2(W_{\mathbb{C}}, \mu\times\nu) \ni F \to F(z) \in \mathbb{C}$$

are continuous linear functionals on $\mathcal{H}L^2(W_{\mathbb{C}}, \mu\times\nu)$. Therefore if $p_n \in \mathcal{P}(W^*)$ with $p_n \to f$ in $L^2(\lambda)$ then

$$(S_{\mu,\nu} f)(z) = \lim_{n\to\infty} (S_{\mu,\nu} p_n)(z).$$

On the other hand since $K(z,\cdot) \in L^2(\lambda)$ it also follows that

$$\lim_{n\to\infty} \int_W p_n(y)\, K(z,y)\, d\lambda(y) = \int_W f(y)\, K(z,y)\, d\lambda(y).$$

Putting these two observations together allows us to conclude that

$$(S_{\mu,\nu} f)(z) = (\text{const.}) \cdot e^{-\frac{1}{4}z\cdot z} \int_W f(y)\, K(z,y)\, d\lambda(y) = (\nu_2 * f)_{\mathbb{C}}(z).$$

4. The unitarity of $S_{\mu,\nu}$ from $L^2(W, \mu*\nu)$ to $\mathcal{H}L^2(W_{\mathbb{C}}, \mu\times\nu)$ now follows from Theorem 11.10 and Remark 11.9.

■

11.2.1 Examples

Bargmann [3] introduces three forms of the Segal-Bargmann transform. I will describe the two most interesting forms of the transform in the next two examples – the third form is rather trivially obtained from one of these forms.

Example 11.12 (Standard form 1). In Corollary 11.11 take $W = \mathbb{R}^d$ and $\mu = \nu = P_{t/2}$ where $P_t = e^{t\Delta/2}\delta_0$, i.e.

$$P_t(dx) = p_t(x)\, dx := (2\pi t)^{-d/2} e^{-\frac{1}{2t}x\cdot x} dx. \tag{11.8}$$

Then we have shown

$$\int_{\mathbb{R}^d} |f(x)|^2 dP_t(x) = \int_{\mathbb{C}^d} \left| \left(e^{t\Delta/2} f\right)_{\mathbb{C}} (x+iy) \right|^2 dP_{t/2}(x)\, dP_{t/2}(y)$$
$$= \left(\frac{1}{\pi t}\right)^d \int_{\mathbb{C}^d} \left| \left(e^{t\Delta/2} f\right)_{\mathbb{C}} (x+iy) \right|^2 e^{-\frac{1}{t}(x\cdot x + y \cdot y)} dx dy$$
$$= \left(\frac{1}{\pi t}\right)^d \int_{\mathbb{C}^d} |(P_t * f)_{\mathbb{C}}(z)|^2 \exp\left(-\frac{1}{t} z \cdot \bar{z}\right) dx dy. \quad (11.9)$$

where

$$(P_t * f)_{\mathbb{C}}(z) = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(x) \exp\left(-\frac{1}{2t}(x-z)\cdot(x-z)\right) dx.$$

Example 11.13 (Standard form 2). In Corollary 11.11 take $W = \mathbb{R}^d$ and $\nu = P_{t/2}$ and $\mu = m$, where m is Lebesgue measure. Technically this is not allowed since m is not a probability measure and certainly does not integrate polynomials. Nevertheless blindly going ahead using $m * P_t = m$ suggests that we should expect

$$\int_{\mathbb{R}^d} |f(x)|^2 dm(x) = \left(\frac{1}{\pi t}\right)^{d/2} \int_{\mathbb{C}^d} |(P_t * f)_{\mathbb{C}}(z)|^2 \exp\left(-\frac{1}{t} y^2\right) dx dy \quad (11.10)$$

where we are now writing y^2 for $y \cdot y$. As it turns out we may derive this formula rigorously from Eq. (11.9).

Following Hall [16, p. 149], for $f \in L^2(m)$ apply Eq. (11.9) with f by $f/\sqrt{p_t} \in L$ in Eq. (11.9) shows,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \left(\frac{1}{\pi t}\right)^d \int_{\mathbb{C}^d} \left| \left(P_t * \frac{f}{\sqrt{p_t}}\right)_{\mathbb{C}} (z) \right|^2 \exp\left(-\frac{1}{t}|z|^2\right) dx dy \quad (11.11)$$

where

$$\left(P_t * \frac{f}{\sqrt{p_t}}\right)_{\mathbb{C}} (z) = \left(\frac{1}{2\pi t}\right)^{d/4} \int_{\mathbb{R}^d} \frac{e^{-\frac{1}{2t}(z-x)^2}}{e^{-\frac{1}{4t}x^2}} f(x)\, dx.$$

Using the identity,

$$-\frac{1}{2t}(z-x)^2 + \frac{1}{4t}x^2 = -\frac{1}{4t}(x-2z)^2 + \frac{1}{2t}z^2$$

we learn that

$$\left(P_t * \frac{f}{\sqrt{p_t}}\right)_{\mathbb{C}} (z) = \left(\frac{1}{2\pi t}\right)^{d/4} e^{\frac{1}{2t}z^2} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{4t}(x-2z)^2\right) f(x)\, dx$$
$$= \left(\frac{1}{2\pi t}\right)^{d/4} e^{\frac{1}{2t}z^2} \int_{\mathbb{R}^d} (4\pi t)^{d/2} p_{2t}(x-2z) f(x)\, dx$$
$$= \left(\frac{1}{2\pi t}\right)^{d/4} (4\pi t)^{d/2} e^{\frac{1}{2t}z^2} (P_{2t} * f)_{\mathbb{C}} (2z).$$

Using this result in Eq. (11.11) then gives,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = C_t \int_{\mathbb{C}^d} \left| e^{\frac{1}{2t}z^2} \left(e^{t\Delta} f\right)_{\mathbb{C}} (2z) \right|^2 \exp\left(-\frac{1}{t}|z|^2\right) dx dy$$
$$= C_t \int_{\mathbb{C}^d} \left| \left(e^{t\Delta} f\right)_{\mathbb{C}} (2z) \right|^2 e^{\frac{1}{t}\operatorname{Re}(z^2)} \exp\left(-\frac{1}{t}|z|^2\right) dx dy$$

where

$$C_t = \left(\frac{1}{\pi t}\right)^d \left(\frac{1}{2\pi t}\right)^{d/2} (4\pi t)^d = 4^d \left(\frac{1}{2\pi t}\right)^{d/2}.$$

Now

$$\operatorname{Re}(z^2) - |z|^2 = x^2 - y^2 - (x^2 + y^2) = -2y^2$$

and so we have shown

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = 4^d \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{C}^d} \left| \left(e^{t\Delta} f\right)_{\mathbb{C}} (2z) \right|^2 \exp\left(-\frac{2}{t}y^2\right) dx dy.$$

Making the change of variables, $z \to \frac{1}{2}z$ then implies,

$$\int_{\mathbb{R}^d} |f(x)|^2 dx = \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{C}^d} \left| \left(e^{t\Delta} f\right)_{\mathbb{C}} (z) \right|^2 \exp\left(-\frac{1}{2t}y^2\right) dx dy$$
$$= \int_{\mathbb{C}^d} \left| \left(e^{t\Delta} f\right)_{\mathbb{C}} (z) \right|^2 dx P_t(dy)$$

which is Eq. (11.10).

Example 11.14 (Interpolating forms). In this example we wish to "interpolate" between the two standard forms. In order to do so we apply Corollary 11.11 with $W = \mathbb{R}^d$ and $\mu = P_\sigma$ and $\nu = P_s$ to arrive at the identity,

$$\int_{\mathbb{R}^d} |f(x)|^2 dP_{\sigma+s}(x)$$
$$= \int_{\mathbb{C}^d} |(P_{2s} * f)_{\mathbb{C}}(x+iy)|^2 dP_\sigma(x)\, dP_s(y)$$
$$= \left(\frac{1}{2\pi\sqrt{\sigma s}}\right)^d \int_{\mathbb{C}^d} |(P_{2s} * f)_{\mathbb{C}}(x+iy)|^2 \exp\left(-\left[\frac{x^2}{2\sigma} + \frac{y^2}{2s}\right]\right) dx dy. \quad (11.12)$$

This gives the results in Example 11.12 when $\sigma = s = t/2$. Moreover for $f \in L^2(m)$ we may multiply Eq. (11.12) by $(2\pi\sigma)^{d/2}$ and then let $\sigma \to \infty$ in order to **formally** arrive at Eq. (11.10) with $t = s$. In a little more detail,

$$(2\pi\sigma)^{d/2}\int_{\mathbb{R}^d}|f(x)|^2\,dP_{\sigma+s}(x)=\left[\frac{\sigma}{\sigma+s}\right]^{d/2}\int_{\mathbb{R}^d}|f(x)|^2\exp\left(-\left[\frac{|x|^2}{2(\sigma+s)}\right]\right)dx$$
$$\to\int_{\mathbb{R}^d}|f(x)|^2\,dx\text{ as }\sigma\to\infty$$

and similarly

$$\begin{aligned}(2\pi\sigma)^{d/2}&\cdot\mathrm{RHS}\,(11.12)\\&=(2\pi s)^{-d/2}\int_{\mathbb{C}^d}|(P_{2s}*f)_{\mathbb{C}}(x+iy)|^2\exp\left(-\left[\frac{x^2}{2\sigma}+\frac{y^2}{2s}\right]\right)dxdy\\&\to(2\pi s)^{-d/2}\int_{\mathbb{C}^d}|(P_{2s}*f)_{\mathbb{C}}(x+iy)|^2\exp\left(-\frac{y^2}{2s}\right)dxdy\\&=\int_{\mathbb{C}^d}|(P_{2s}*f)_{\mathbb{C}}(x+iy)|^2\,dxP_s(dy)\end{aligned}$$

In the next two examples we explore what happens if μ or ν degenerate to δ_0.

Example 11.15. Let $W=\mathbb{R}^d$, $W_{\mathbb{C}}=\mathbb{C}^d$, $\nu=\delta_0$ and μ be a probability measure on $(W,\mathcal{B}_W)$ as in Theorem 11.10. We then have , $\mu*\nu=\delta_0$, $\nu_2=\delta_0*\delta_0=\delta_0$and $\nu_2*f=f$. Therefore Theorem 11.10 states that

$$\begin{aligned}\int_W|p(x)|^2\,d\mu(x)&=\int_{W_{\mathbb{C}}}|p_{\mathbb{C}}(x+iy)|\,d\mu(x)\,\delta_0(dy)\\&=\int_W|p_{\mathbb{C}}(x)|\,d\mu(x)\end{aligned}$$

which gives no new information.

Incidentally, notice that

$$\dot{\mathcal{H}}L^2(\mu\otimes\delta_0):=\overline{\mathcal{HP}(W_{\mathbb{C}})}^{L^2(\mu\otimes\delta_0)}\cong\overline{\mathcal{P}(W)}^{L^2(\mu)}=L^2(W,\mu)$$

since for any holomorphic polynomial p on $\mathbb{C}^d$,

$$\|p\|^2_{L^2(\mu\otimes\delta_0)}=\int_{W_{\mathbb{C}}}|p(x+iy)|^2\,\mu(dx)\,\delta_0(dy)=\int_W|p(x)|^2\,\mu(dx)\,.$$

The following example can essentially be found in Hall [19, Theorem 2.2].

Example 11.16 (Fourier Wiener Transform). Let $W=\mathbb{R}^d$, $W_{\mathbb{C}}=\mathbb{C}^d$, $\mu=\delta_0$, $\nu=P_t$ (see Eq. (11.8)), and $S=S_{\delta_0,P_t}$. By Theorem 11.10,

$$\begin{aligned}\int_{\mathbb{R}^d}|f(x)|^2\,p_t(x)\,dx&=\int_{\mathbb{C}^d}|Sf(z)|^2\,\delta_0(dx)\,P_t(dy)\\&=\int_{\mathbb{R}^d}|Sf(iy)|^2\,P_t(dy)\end{aligned}\tag{11.13}$$

for all $f\in L^2(P_1)$. Similarly to Example 11.15 we may easily conclude that the map,

$$L^2(\mathbb{R}^d,P_t)\ni f\to(z\to f(i\operatorname{Im}z))\in\dot{\mathcal{H}}L^2(\mathbb{C}^d,\delta_0\times P_t)$$

is unitary.

If f a function on $\mathbb{R}^d$ with (for example) at most exponential growth one easily shows that $Sf(z)$ is given explicitly as,

$$Sf(z)=(4\pi t)^{-d/2}\int_{\mathbb{R}^d}f(x)\exp\left(-\frac{1}{4t}(x-z)^2\right)dx.$$

So if $g\in C_c(\mathbb{R}^d,\mathbb{C})$ we may take $f:=g/\sqrt{p_t}$ in the above formulas in order to find,

$$\begin{aligned}S(g/\sqrt{p_t})(z)&=(4\pi t)^{-d/2}\int_{\mathbb{R}^d}\frac{g(x)}{\sqrt{p_t(x)}}e^{-\frac{1}{4t}(x-z)^2}dx\\&=C_t\int_{\mathbb{R}^d}g(x)\,e^{\frac{1}{4t}x\cdot x}e^{-\frac{1}{4t}(x-z)^2}dx\\&=C_te^{-\frac{1}{4t}z^2}\int_{\mathbb{R}^d}g(x)\,e^{\frac{1}{2t}x\cdot z}dx,\end{aligned}$$

where

$$C_t:=(2\pi t)^{d/4}(4\pi t)^{-d/2}.$$

Taking $z=iy$ then implies,

$$S(g/\sqrt{p_t})(iy)=C_te^{\frac{1}{4t}y^2}\hat{g}(y/2t)$$

where

$$\hat{g}(y):=\int_{\mathbb{R}^d}g(x)\,e^{iy\cdot x}dx$$

is the Fourier transform of g. Therefore Eq. (11.13) with $f:=g/\sqrt{p_t}$ becomes

$$\begin{aligned}\int_{\mathbb{R}^d}|g(x)|^2\,dx&=C_t^2\int_{\mathbb{R}^d}|\hat{g}(y/2t)|^2\,e^{\frac{1}{2t}y^2}P_t(dy)\\&=(4\pi t)^{-d}\int_{\mathbb{R}^d}|\hat{g}(y/2t)|^2\,dy\\&=\left(\frac{1}{2\pi}\right)^d\int_{\mathbb{R}^d}|\hat{g}(y)|^2\,dy=\int_{\mathbb{R}^d}|\hat{g}(2\pi y)|^2\,dy\end{aligned}$$

which is the isometry property of the Fourier transform. Because the maps

$$S:L^2(\mathbb{R}^d,P_t)\to\dot{\mathcal{H}}L^2(\mathbb{C}^d,\delta_0\times P_t)\cong L^2(\mathbb{R}^d,P_t)\text{ and}$$
$$L^2(\mathbb{R}^d,dx)\ni g\to g/\sqrt{p_t}\in L^2(\mathbb{R}^d,P_t)$$

are unitary we have actually given a proof of the fact that the Fourier transform, $g\to\hat{g}(2\pi(\cdot))$ is a unitary map on $L^2(\mathbb{R}^d,dx)$.

The Kakutani-Itô-Fock space isomorphism

(Some old stuff has now been moved to the bone yard **??**. BRUCE: see Chapter **??** below for some Q.M. interpretation of this stuff.))

12.1 The Real Case

As usual let $(W, \mathcal{B}_W, \mu)$ be a Gaussian probability space. If $H := H_\mu$ is a proper subspace of W it is not true that the restriction map,

$$\mathcal{P}(W^*) \ni p \to p|_H \in \mathcal{P}(H^*),$$

is one to one. which is a bit annoying. However notice that if $p, q \in \mathcal{P}(W^*)$ with $p = q$ (μ a.s.) then $p = q$ on H the conversely. Thus $\mathcal{P}(W^*)/\sim$ and $\mathcal{P}(H^*)$ are isomorphic where $p \sim g$ iff $p = q$ (μ a.s.) iff $p = q$ on H. For the sake of simplicity and with no real loss in generality let us assume in this section that $H = W$, i.e. μ is non-degenerate.

In what follows we will simply write $\mathcal{F}(H)$ for $\mathcal{F}(H;1)$ – see Definition 10.9.

Theorem 12.1 (Fock Space Isomorphism I). *For $f \in L^2(\mu)$ let $F_\mu f := \left(D_0^n(\mu * f)\right)_{n=0}^\infty$, then $F_\mu f \in \mathcal{F}(H)$ and $F_\mu : L^2(\mu) \to \mathcal{F}(H)$ is unitary.*

Proof. First off from Theorem 9.2 we know that $\mu * f$ is smooth on $H_\mu = W$ so that it makes sense to even write $D_0^n(\mu * f)$. Secondly, from Proposition 10.8 and Theorem 11.1 (with $\nu = \mu$) we know that $F_\mu|_{\mathcal{P}(W^*)}$ is an isometry from $\mathcal{P}(W^*)$ onto a dense subset of $\mathcal{F}(H)$ and hence extends uniquely to a unitary transformation, $F_\mu : L^2(\mu) \to \mathcal{F}(H)$. It only remains to show that $F_\mu f := \left(D_0^n(\mu * f)\right)_{n=0}^\infty$.

Let $h_1, \ldots, h_n \in H$ and define $l(f) := [\partial_{h_1} \ldots \partial_{h_n}(\mu * f)](0)$ for all $f \in L^2(\mu)$. According to Theorem 9.2 this is a bounded linear functional. Thus if $f \in L^2(\mu)$ and $p_k \in \mathcal{P}(W^*)$ such that $p_k \to f$ in $L^2(\mu)$ we will have $l(p_k) \to l(f)$. We also have $D_0^n(\mu * p_k) \to D_0^n(\mu * f)$ and therefore,

$$D_0^n(\mu * f)(h_1, \ldots, h_n) = \lim_{k\to\infty} D_0^n(\mu * p_k)(h_1, \ldots, h_n) = \lim_{k\to\infty} l(p_k) = l(f)$$

which completes the proof. ∎

Example 12.2 (Pointwise bounds revisited). As a simple application of Theorem 12.1 we can give another proof of the pointwise bounds in Eq. (9.1) when $p = 2$. Since $\mu * f$ is real analytic for $f \in L^2(\mu)$ we may express $\mu * f$ as a Taylor series,

$$\begin{aligned}\mu * f(h) &= \sum_{n=0}^\infty \frac{1^n}{n!}\left(\frac{d}{dt}\right)^n|_{t=0}[\mu * f(th)] \\ &= \sum_{n=0}^\infty \frac{1}{n!}\partial_h^n f(0) = \sum_{n=0}^\infty \|h\|_H^n \left(\partial_{\hat{h}}^n f\right)(0)\frac{1}{n!}\end{aligned}$$

where $\hat{h} := h/\|h\|_H$. An application of the Cauchy - Schwarz inequality (in $\ell^2\left(\frac{1}{n!}\right)$) shows,

$$\begin{aligned}|\mu * f(h)| &\le \sqrt{\sum_{n=0}^\infty \|h\|_H^{2n}\frac{1}{n!} \cdot \sum_{n=0}^\infty \left|\left(\partial_{\hat{h}}^n f\right)(0)\right|^2 \frac{1}{n!}} \\ &= \sqrt{\sum_{n=0}^\infty \left|\partial_{\hat{h}}^n f(0)\right|^2 \frac{1}{n!}} \cdot \exp\left(\frac{1}{2}\|h\|_H^2\right).\end{aligned}$$

If $S \subset H$ is an orthonormal basis for H such that $\hat{h} \in S$, then

$$\begin{aligned}\|f\|_{L^2(\mu)}^2 &= \sum_{n=0}^\infty \frac{1}{n!}\sum_{h_1,\ldots,h_n \in S}|\partial_{h_1}\ldots\partial_{h_n} f(0)|^2 \\ &\ge \sum_{n=0}^\infty \frac{1}{n!}\left|\partial_{\hat{h}}^n f(0)\right|^2\end{aligned}$$

which combined with the previously displayed inequality gives the pointwise bound in Eq. (9.1) when $p = 2$, namely that

$$|\mu * f(h)| \le \|f\|_{L^2(\mu)}^2 \exp\left(\frac{1}{2}\|h\|_H^2\right).$$

Definition 12.3. *The* n^{th} ***level Hermite (or homogeneous chaos) subspace of*** $L^2(W, \mu)$ *is the space $\mathcal{F}_n(\mu) = e^{-L_\mu/2}\mathcal{P}_n(W^*)$, where $\mathcal{P}_n(W^*)$ denotes the space of homogeneous polynomials of degree n on W.*

Proposition 12.4 (Hermite/Chaos Expansion). *Let $(W, \mathcal{B}_W, \mu)$ be a non-degenerate Gaussian probability space. Then;*

1. $F_\mu[\mathcal{F}_n(\mu)] = \mathrm{Sym}_n(H_\mu, \mathbb{C})$ *and for any* $\alpha \in \mathrm{Sym}_n(H_\mu, \mathbb{C})$,
$$F_\mu^{-1}\alpha = \frac{1}{n!}e^{-L_\mu/2}(x \to \alpha(x, \ldots, x)). \tag{12.1}$$
2. $L^2(W, \mu)$ *is the orthogonal Hilbert space direct sum of the subspaces* $\mathcal{F}_n(\mu)$ *for* $n = 0, 1, 2\ldots,$ *i.e. every* $f \in L^2(W, \mu)$ *has a unique orthogonal direct sum decomposition of the form*
$$f = \sum_{n=0}^{\infty} f_n \text{ with } f_n \in \mathcal{F}_n(\mu). \tag{12.2}$$
3. *Writing* $e^{L_\mu/2}f$ *for* $\mu * f$, *the* f_n *in Eq. (12.2) are computed via*
$$f_n = \frac{1}{n!}e^{-L/2}(x \to \partial_x^n e^{L/2} f)(0). \tag{12.3}$$
4. $\mathcal{F}_n(\mu)$ *is the set of all polynomials on* W *of degree* n *which are orthogonal to all polynomials of degree at most* $n-1$.

Proof. Let us write H for H_μ, F for F_μ, and L for L_μ in this proof.

1. If $f = e^{-L/2}p \in \mathcal{F}_n(\mu)$ for some $p \in \mathcal{P}_n(W^*)$, then
$$\begin{aligned}(Ff)_k &= D_0^k\left[\mu * e^{-L/2}p\right] = D_0^k\left[e^{L/2}e^{-L/2}p\right] \\ &= D_0^k p = \delta_{k,n}D_0^n p\end{aligned}$$
where the last equality is a consequence of the fact that p is homogeneous of degree n. This shows that $F[\mathcal{F}_n(\mu)] \subset \mathrm{Sym}_n(H_\mu, \mathbb{C})$.
Conversely if $f \in \mathcal{F}(H)$ with $Ff = \alpha \in \mathrm{Sym}_n(H, \mathbb{C})$, let
$$g := \frac{1}{n!}e^{-L_\mu/2}(x \to \alpha(x, \ldots, x)) \in \mathcal{F}_n(\mu).$$
Then
$$\begin{aligned}Fg &= \frac{1}{n!}D_0^n\left[\mu * e^{-L_\mu/2}(x \to \alpha(x, \ldots, x))\right] \\ &= \frac{1}{n!}D_0^n\left[e^{L_\mu/2}e^{-L_\mu/2}(x \to \alpha(x, \ldots, x))\right] \\ &= \frac{1}{n!}D_0^n[\alpha(x, \ldots, x)] = \alpha.\end{aligned}$$
By Theorem 12.1, F is unitary and therefore $f = g$ a.s. and we proved Eq. (12.1) and $\mathrm{Sym}_n(H_\mu, \mathbb{C}) \subset F[\mathcal{F}_n(\mu)]$.

2. & 3. These items follow directly from item 1. and Theorem 12.1 as $f_n = F^{-1}[(Ff)_n]$.

4. Noting that $e^{L/2} : \mathcal{P}(W^*) \to \mathcal{P}(W^*)$ is degree preserving we see that $\mathcal{F}_n(\mu) = e^{-L/2}\mathcal{P}_n(W^*)$ is contained inside the degree n – polynomials in $\mathcal{P}(W^*)$. Moreover $\oplus_{k=0}^n \mathcal{F}_k(\mu)$ is equal to the degree n polynomials in $\mathcal{P}(W^*)$. This is because if $p \in \mathcal{P}(W^*)$ is a degree n – polynomial, then by items 2. and 3.,
$$\begin{aligned}p &= \sum_{k=0}^{\infty} \frac{1}{k!}e^{-L/2}(x \to \partial_x^k e^{L/2}p)(0) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}e^{-L/2}(x \to \partial_x^k e^{L/2}p)(0) \in \oplus_{k=0}^n \mathcal{F}_k(\mu).\end{aligned}$$
Hence we may conclude that $\mathcal{F}_n(\mu)$ is perpendicular to $\oplus_{k=0}^{n-1}\mathcal{F}_k(\mu)$ which is precisely the degree $n-1$ polynomials in $\mathcal{P}(W^*)$. Moreover if $p \in \mathcal{P}(W^*)$ is a degree n polynomial (i.e. $p \in \oplus_{k=0}^n \mathcal{F}_k(\mu)$) which is orthogonal to the degree $n-1$ polynomials (i.e. $p \perp \oplus_{k=0}^{n-1}\mathcal{F}_k(\mu)$) we must have $p \in \mathcal{F}_n(\mu)$.

■

Remark 12.5. Combining the results of items 2. and 3. of Proposition 12.4, if $f \in L^2(\mu)$ then
$$f = \sum_{n=0}^{\infty} \frac{1}{n!}e^{-tL/2}(x \to \partial_x^n(\mu * f)(0)) \text{ (orthogonal terms).}$$
We will write this succinctly as
$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}e^{-L/2}(x \to \partial_x^n e^{L/2} f)(0) \tag{12.4}$$
$$\text{“}=\text{”}\ e^{-L/2}\sum_{n=0}^{\infty} \frac{1}{n!}(x \to \partial_x^n e^{L/2} f)(0). \tag{12.5}$$
In words, to find the Hermite decomposition of $f \in L^2(\mu)$ apply $e^{L/2}$ to f, then compute the Taylor expansion of the result, then apply $e^{-L/2}$ to each term in this expansion. So formally the theorem represents the assertion that
$$Id_{L^2(\mu)} = e^{-L/2} \circ \mathrm{Taylor}_0 \circ e^{L/2}.$$

12.2 (Weakly) complex compatible Gaussian measures

Now suppose that W is a complex vector space and let $W^* := \mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$ be the complex dual of W and $W^\dagger := \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$ be the space of real linear forms on W – that is $W^\dagger$ is the real dual space of W where we forget about the complex structure on W. Further let μ be a Gaussian measure on W by which we mean

$$\hat{\mu}(\alpha) = e^{-\frac{1}{2}q(\alpha)} \text{ for all } \alpha \in W^\dagger$$

where $q : W^\dagger \times W^\dagger \to \mathbb{R}$ is a non-negative quadratic form. A priori the a Gaussian measure on W has no knowledge of the complex structure on W. The minimal sort of compatibility we would like here is for the Cameron-Martin space, $H = H_\mu$ of μ to be at complex subspace of W, i.e. of $iH = H$. This need not be the case in general, just take $W = \mathbb{C}$ with $\mu = \gamma \otimes \delta_0$ where γ is the standard normal distribution. In this case $H_\mu = \mathbb{R} \subset \mathbb{C}$ which is a real but not complex subspace.

Lemma 12.6. *Let μ be a Gaussian measure on W, then $iH_\mu = H_\mu$ is equivalent to the condition on q that $q(\alpha) = 0$ iff $q(\alpha \circ M_i) = 0$.*

Proof. Recall that

$$H = \mathrm{Nul}(q)^\perp := \{x \in W : \alpha(x) = 0 \text{ whenever } q(\alpha) = 0\}.$$

Thus $ix \in H$ iff

$$\alpha(ix) = 0 \text{ whenever } q(\alpha) = 0.$$

Thus if $q(\alpha) = 0$ iff $q(\alpha \circ M_i) = 0$ and $x \in H$, then $\alpha(x) = 0$ when $q(\alpha) = 0$ implies $\alpha \circ M_i(x) = 0$ when $q(\alpha \circ M_i) = 0$ which is equivalent to $\alpha(ix) = 0$ when $q(\alpha) = 0$ showing that $ix \in H$.

Conversely suppose that $iH = H$. Then using $H = \mathrm{Nul}(q)^\perp$ or equivalently $\mathrm{Nul}(q) = H^0$ it follows that $\mathrm{Nul}(q) = (iH)^0$. Therefore $\alpha \in \mathrm{Nul}(q)$ iff $\alpha(H) = \{0\}$ iff $\alpha(iH) = \{0\}$ iff $\alpha \circ M_i(H) = \{0\}$ iff $\alpha \circ M_i \in \mathrm{Nul}(q)$. ∎

Definition 12.7. *If W is a finite dimensional complex vector space, we say that a Gaussian measure μ on W is **weakly compatible** with the complex structure iff $\mathrm{Nul}\left((M_i^{\mathrm{tr}})^* q\right) = \mathrm{Nul}(q)$ iff $iH_\mu = H_\mu$, i.e. iff H_μ is a complex subspace of W.*

*We say that μ **compatible** with the complex structure on W if $q(\alpha \circ M_i) = q(\alpha)$ for all $\alpha \in W^\dagger$. This condition is equivalent to μ being invariant under M_i, i.e. $\mu \circ M_i^{-1} = \mu$.*

Example 12.8. If μ is non-degenerate[1] then μ is of course weakly compatible with the complex structure. Conversely if μ is weakly compatible with the complex structure we may replace W by H and assume that μ is non-degenerate with out any real loss of generality. (If μ were not weakly compatible with the complex structure, the measure μ would be supported on a real but not complex subspace of W.)

[1] It should now be clear that μ is non-degenerate if any one of the following equivalent conditions hold; 1) q_μ is positive definite, 2) $H_\mu = W$, or 3) the support of μ is all of W.

Example 12.9. Suppose that $(W, \mathcal{B}_W, \mu)$ is a **real** Gaussian measure space and let $\lambda = \mu \times \mu$ on $W_\mathbb{C} = W + iW \cong W \times W$ (the complexification of W). Then λ is compatible with the complex structure on W. To see this let $\alpha, \beta \in W^*$ and $\psi(x + iy) = \alpha(x) + \beta(y)$ so that ψ is the general element of $W_\mathbb{C}^\dagger$. Then

$$\psi \circ M_i(x + iy) = \psi(ix - y) = -\alpha(y) + \beta(x).$$

Recall that $q_\lambda(\psi) = q_\mu(\alpha) + q_\mu(\beta)$ and therefore,

$$q_\lambda(\psi \circ M_i) = q_\mu(\beta) + q_\mu(-\alpha) = q_\mu(\alpha) + q_\mu(\beta) = q_\lambda(\psi).$$

Proposition 12.10. *Suppose that λ is a compatible Gaussian measure on a complex Banach space, W and let $H := H_\lambda$ be the Cameron-Martin space. Then $\|ix\|_H = \|x\|_H$ for all $x \in W$ and there is a unique complex inner product[2] $\langle \cdot, \cdot \rangle$ on H such that*

$$(h, k)_H = \mathrm{Re}\langle h, k\rangle \text{ for all } h, k \in H.$$

Consequently if S is a orthonormal basis for $(H, \langle \cdot, \cdot \rangle)$ as a complex Hilbert space then $S \cup iS$ is an orthonormal basis[3] for $(H, (\cdot, \cdot)_H)$ as a real Hilbert space. See [8, Theorem 2.3] for some more information along these lines.

Proof. For the first item we have

$$\begin{aligned} \|ix\|_H^2 &= \sup_{\alpha \in W^\dagger} \frac{|\alpha(ix)|^2}{q(\alpha)} = \sup_{\alpha \in W^\dagger} \frac{|\alpha \circ M_i(x)|^2}{q(\alpha \circ M_i)} \\ &= \sup_{\alpha \in W^\dagger} \frac{|\alpha(x)|^2}{q(\alpha)} = \|x\|_H^2 \end{aligned}$$

wherein we have used $q(\alpha \circ M_i) = q(\alpha)$ and the fact that $W^\dagger \ni \alpha \to \alpha \circ M_i \in W^\dagger$ is a bijection. Polarizing the identity $\|M_i h\|_H^2 = \|h\|_H^2$ for all $h \in H$ implies

$$(M_i h, M_i k)_H = (h, k)_H \text{ for all } h, k \in H.$$

Replacing k by $-M_i k$ shows $(M_i h, k)_H = -(h, M_i k)$ so that M_i skew -adjoint. I leave it to the reader to check that we have to define $\langle \cdot, \cdot \rangle$ by

[2] We take our complex inner products to be conjugate linear in the second variable. For example on $\mathbb{C}^n$ the standard complex inner product is given by $\langle z, w\rangle = z \cdot \bar{w}$ where

$$z \cdot w = \sum_{k=1}^n z_k w_k.$$

[3] In the complex compatible case when dealing with holomorphic functions we often use the basis S rather than $S \cup iS$. This convention is responsible for numerous strange looking factors of 2 appearing in the holomorphic theory.

$$\langle h,k\rangle := (h,k) - i\,(ih,k)$$

and that by doing so we arrive at the desired complex inner product on H. ■

Proposition 12.10 loosely states that Example 12.9 is essentially the only class of example of complex compatible Gaussian measures. Take $W_{\mathrm{Re}} = \mathbb{R}\cdot S \subset W$ to be the real span of S and then take $\mu := \mathrm{Law}\left(\sum_{h\in S} Z_h \cdot h\right)$ where $\{Z_h\}_{h\in S}$ are i.i.d. standard normal random variables. Then $W \cong W_{\mathrm{Re}} + iW_{\mathrm{Re}} = (W_{\mathrm{Re}})_{\mathbb{C}}$ and $\lambda = \mu\times\mu$ under this identification. Now back to the general theory.

12.3 Complex (Weakly) Kakutani-Itô-Fock space isomorphism

Let W be a complex finite dimensional Banach space and μ be a non-degenerate Gaussian measure on W and let $H = H_\mu$ – so $H_\mu = W$ as vector spaces. Ignoring the complex structures on H and W we still have Fock and Hermite expansion results in Theorem 12.1 and Proposition 12.4. Our goal here is to describe these expansions on the Hilbert subspace of holomorphic functions inside of $L^2(\mu)$. The key new ingredient is contained in the next lemma.

Lemma 12.11. *Suppose that W is a complex vector space and μ is a non-degenerate Gaussian measure on W. Then $\mu * f$ is a holomorphic function on $H_\mu = W$ for all $f \in \mathcal{H}L^2(\mu)$.*

Proof. Let $f \in \mathcal{H}L^2(\mu)$. The first thing we want to prove is that $\mu * f$ is still holomorphic. So we have to show for each $x, y \in W$ that $F(\lambda) := \mu * f(x+\lambda y)$ is holomorphic for λ near 0 in $\mathbb{C}$. Recall, using the baby Cameron-Martin Theorem 4.8, that

$$\mu * f(x) = \int_W f(x-z)\,\mu(dz) \tag{12.6}$$
$$= \int_W f(x+z)\,\mu(dz)$$
$$= \int_W f(z)\,e^{(x,z)_H - \frac{1}{2}\|x\|_H^2}\mu(dz). \tag{12.7}$$

Replacing f by $|f|$ in this equation and using $2(x,\cdot)_H$ under μ is Gaussian with variance $4\|x\|_H^2$ along with the Cauchy-Schwarz inequality shows

$$\begin{aligned}\mu * |f|(x) &= \int_W |f(x-z)|\,\mu(dz) \le e^{-\frac{1}{2}\|x\|_H^2}\|f\|_{L^2(\mu)} \cdot \left\|e^{(x,\cdot)_H}\right\|_{L^2(\mu)} \\ &= \|f\|_{L^2(\mu)} \cdot e^{-\frac{1}{2}\|x\|_H^2}\cdot\sqrt{e^{\frac{1}{2}4\|x\|_H^2}} = \|f\|_{L^2(\mu)}\cdot e^{\frac{1}{2}\|x\|_H^2}.\end{aligned}$$

Thus if T is a solid triangle contained in $\mathbb{C}$ we have

$$\int_{\partial T}\int_W |f(x+\lambda y - z)|\,\mu(dz)\,|d\lambda| \le \|f\|_{L^2(\mu)}\cdot\int_{\partial T} e^{\frac{1}{2}\|x+\lambda y\|_H^2}\,|d\lambda| < \infty.$$

Since $\lambda \to f(x+\lambda y - z)$ is holomorphic it follows by Fubini's theorem along with Morera's Theorem 2.5 that

$$\begin{aligned}\int_{\partial T} F(\lambda)\,d\lambda &= \int_{\partial T} \mu * f(x+\lambda y - z)\,d\lambda \\ &= \int_W\left[\int_{\partial T} f(x+\lambda y - z)\,d\lambda\right]\mu(dz) = 0.\end{aligned}$$

Since T was arbitrary, another application of Morera's Theorem 2.5 implies F is holomorphic. ■

Definition 12.12. *Given a complex Hilbert space, H, let*

$$\mathrm{Mult}_n^{\mathbb{C}}(H,\mathbb{C}) := \{\alpha \in \mathrm{Mult}_n(H,\mathbb{C}) : \alpha \text{ is complex multi-linear}\}$$
$$\mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C}) := \mathrm{Sym}_n(H,\mathbb{C}) \cap \mathrm{Mult}_n^{\mathbb{C}}(H,\mathbb{C})$$

and

$$\mathcal{F}^{\mathbb{C}}(H) := \left\{\alpha \in \mathcal{F}(H) : \alpha_n \in \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})\ \forall\, n \in \mathbb{N}_0\right\}.$$

(So $\alpha \in \mathrm{Sym}_n(H,\mathbb{C})$ iff $\alpha \in \mathrm{Sym}_n(H,\mathbb{C})$ and

$$\alpha(ih_1, h_2, \dots, h_n) = i\alpha(h_1, h_2, \dots, h_n) \text{ for all } (h_1,\dots,h_n) \in H^n.)$$

Example 12.13. If $\dim H < \infty$ and $f : H \to \mathbb{C}$ is a function which is holomorphic near $0 \in H$, then $D_0^n f \in \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})$ for all $n \in \mathbb{N}_0$, see Theorem 2.6.

Theorem 12.14 (Fock/Hermite Expansions II). *Suppose that W is a complex finite dimensional vector space and μ is a non-degenerate Gaussian measure on W. Let $\mathcal{H}\mathcal{F}_n(\mu) = \mathcal{F}_n(\mu) \cap \mathcal{H}L^2(\mu)$ – the holomorphic polynomials inside of $\mathcal{F}_n(\mu)$. Then;*

1. *$F_\mu\left[\mathcal{H}L^2(\mu)\right] = \mathcal{F}^{\mathbb{C}}(H)$.*
2. *$F_\mu\left[\mathcal{H}\mathcal{F}_n(\mu)\right] = \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C}) = e^{-L/2}\mathcal{H}^{(n)}$ where $\mathcal{H}^{(n)}$ denotes the homogeneous holomorphic polynomials of degree n.*
3. *$\mathcal{H}L^2(\mu) = \oplus_{n=0}^{\infty}\mathcal{H}\mathcal{F}_n(\mu)$ (orthogonal direct sum).*
4. *if $f \in \mathcal{H}L^2(\mu)$ and $f = \sum_{n=0}^{\infty} f_n$ is the Hermite expansion from Proposition 12.4, then $f_n \in \mathcal{H}\mathcal{F}_n(\mu)$ for all n.*
5. *$\mathcal{H}\mathcal{F}_n(\mu)$ is the set of all holomorphic polynomials on W of degree n which are orthogonal to all holomorphic polynomials on W of degree $n-1$ or less.*

6. The holomorphic polynomials on W are dense in $\mathcal{H}L^2(W,\mu)$.

Proof. To simplify notation let $H = H_\mu$, $L = L_\mu$, and $F = F_\mu$.

1. & 2. By Lemma 12.11, if $f \in \mathcal{H}L^2(\mu)$ then $\mu * f$ is still holomorphic and therefore $[Ff]_n = D_0^n [\mu * f] \in \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})$ for all $n \in \mathbb{N}$. This shows that $F\left[\mathcal{H}L^2(\mu)\right] \subset \mathcal{F}^{\mathbb{C}}(H)$ and that $F[\mathcal{H}\mathcal{F}_n(\mu)] \subset \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})$. If $\alpha \in \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})$, then $p(x) := \alpha(x,\dots,x)$ is complex differentiable and therefore a holomorphic polynomial. Since partial derivations preserve the class of holomorphic functions we may conclude that $F^{-1}\alpha = e^{-L/2}[p]$ is holomorphic and in $\mathcal{F}_n(\mu)$. Therefore $F^{-1}\left[\mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})\right] \subset \mathcal{H}\mathcal{F}_n(\mu)$ and so $\mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C}) \subset F[\mathcal{H}\mathcal{F}_n(\mu)]$ and we have proved the first equality in item 2. If $\alpha = (\alpha_n) \in \mathcal{F}^{\mathbb{C}}(H)$, then $F^{-1}\alpha = \sum_{n=0}^\infty F^{-1}\alpha_n$ is in $\mathcal{H}L^2(\mu)$ as each term is in $\mathcal{H}L^2(\mu)$ and $\mathcal{H}L^2(\mu)$ is a closed subspace of $L^2(\mu)$. This shows that $F^{-1}\left[\mathcal{F}^{\mathbb{C}}(H)\right] \subset \mathcal{H}L^2(\mu)$, i.e. $\mathcal{F}^{\mathbb{C}}(H) \subset F\left[\mathcal{H}L^2(\mu)\right]$ which completes the proof of item 1.
Since
$$\mathcal{H}^{(n)} = \left\{x \to \alpha(x,\dots,x) : \alpha \in \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})\right\}$$
it follows from Eq. (12.1) that
$$\mathcal{H}\mathcal{F}_n(\mu) = F^{-1}\left[\mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})\right] = e^{-L/2}\left[\mathcal{H}^{(n)}\right].$$

3. & 4. Item 3. and 4. follows directly from items 1. and 2. and the fact that $\mathcal{F}^{\mathbb{C}}(H) = \oplus_{n=0}^\infty \mathrm{Sym}_n^{\mathbb{C}}(H,\mathbb{C})$ (orthogonal direct sum).

5. Let $\mathcal{H}_n = \oplus_{k=0}^n \mathcal{H}^{(k)}$ be the holomorphic polynomials of degree less than or equal to n. Since $e^{-L/2}\mathcal{H}_n = \mathcal{H}_n$ we see that
$$\mathcal{H}_n = \oplus_{k=0}^n e^{-L/2}\mathcal{H}^{(k)} = \oplus_{k=0}^n \mathcal{H}\mathcal{F}_k(\mu).$$
Therefore if $p \in \mathcal{H}_n$ is orthogonal to $\mathcal{H}_{n-1}$ then $p \in \oplus_{k=0}^n \mathcal{H}\mathcal{F}_k(\mu)$ and $p \perp \oplus_{k=0}^{n-1}\mathcal{H}\mathcal{F}_k(\mu)$ which implies $p \in \mathcal{H}\mathcal{F}_n(\mu)$. Conversely if $p \in \mathcal{H}\mathcal{F}_n(\mu)$, then $p \in \mathcal{F}_n(\mu)$ and therefore perpendicular to all polynomials of degree less than n and in particular to the holomorphic polynomials of degree less than n.

6. If $f \in \mathcal{H}L^2(\mu)$ and $f = \sum_{n=0}^\infty f_n$ is its Hermite expansion, then $p_N := \sum_{n=0}^N f_n$ is a holomorphic polynomial for each $N \in \mathbb{N}$. Moreover, $p_N \to f$ in $L^2(\mu)$.

∎

12.4 The Segal-Bargmann Action on the Fock Expansions

Notation 12.15 *Let H be a real Hilbert space and $H_{\mathbb{C}} = H + iH$ be its complexification. Given a real multilinear form, $\alpha : H^n \to \mathbb{C}$, we let $\alpha_{\mathbb{C}}$ be the unique complex multi-linear form on $H_{\mathbb{C}}^n$ such that $\alpha_{\mathbb{C}} = \alpha$ on H^n.*

The next theorem shows that the Segal-Bargmann transform acts on the Fock space by this very simple complexification operation.

Theorem 12.16. *Let W be a finite dimensional real Banach space and μ and ν be non-degenerate Gaussian measures on $(W,\mathcal{B}_W)$ so that $H_{\mu*\nu} = H_\mu = H_\nu = W$ as sets. Then the Segal-Bargmann transform $S_{\mu,\nu} : L^2(W,\mu*\nu) \to \mathcal{H}L^2(W_{\mathbb{C}},\mu\times\nu)$ (see Corollary 11.11) satisfies $F_{\mu\times\nu}S_{\mu,\nu}f = (F_{\mu*\nu}f)_{\mathbb{C}}$ for all $f \in L^2(W,\mu*\nu)$, i.e. the following diagram commutes,*

$$\begin{array}{ccc}
L^2(W,\mu*\nu) & \xrightarrow{S_{\mu,\nu}} & \mathcal{H}L^2(W_{\mathbb{C}},\mu\times\nu) \\
\downarrow F_{\mu*\nu} & & \downarrow F_{\mu\times\nu} \\
\mathcal{F}(H_{\mu*\nu}) \ni \alpha & \longrightarrow & \alpha_{\mathbb{C}} \in \mathcal{F}(H_\mu + iH_\nu)
\end{array}$$

Moreover the action of $S_{\mu,\nu} : \mathcal{F}_n(\mu*\nu) \to \mathcal{H}\mathcal{F}_n(\mu\times\nu)$ is given by
$$S_{\mu,\nu}\left[e^{-L_{\mu*\nu}/2}p\right] = e^{-L_{\mu\times\nu}/2}p_{\mathbb{C}}. \tag{12.8}$$
(We write $H_\mu + iH_\nu$ rather than $H_{\mathbb{C}}$ to indicate that as a real Hilbert space $H_\mu + iH_\nu = H_\mu \times H_\nu$.)

Proof. Let $T_{\mu,\nu} := F_{\mu\times\nu}S_{\mu,\nu}F_{\mu*\nu}^{-1}$. Since $L_{\mu*\nu} = L_\mu + L_\nu$, $L_{\mu\times\nu} = L_\mu + \tilde{L}_\nu$, and $S_{\mu,\nu}p = \left[e^{-L_\nu}p\right]_{\mathbb{C}}$ for $p \in \mathcal{P}(W^*)$ we see that
$$\begin{aligned}
S_{\mu,\nu}\left[e^{-L_{\mu*\nu}/2}p\right] &= \left[e^{-L_\nu}e^{-\frac{1}{2}(L_\mu+L_\nu)}p\right]_{\mathbb{C}} = \left[e^{-\frac{1}{2}(L_\mu-L_\nu)}p\right]_{\mathbb{C}} \\
&= e^{-\frac{1}{2}(L_\mu-L_\nu)}p_{\mathbb{C}} = e^{-\frac{1}{2}\left(L_\mu+\tilde{L}_\nu\right)}p_{\mathbb{C}} = e^{-L_{\mu\times\nu}/2}p_{\mathbb{C}}.
\end{aligned}$$
This proves Eq. (12.8). Hence if $\alpha \in \mathrm{Sym}_n(H)$ and $p(x) := \frac{1}{n!}\alpha(x,\dots,x)$, then
$$\begin{aligned}
T_{\mu,\nu}\alpha &= F_{\mu\times\nu}S_{\mu,\nu}\left[e^{-L_{\mu*\nu}/2}p\right] = F_{\mu\times\nu}e^{-L_{\mu\times\nu}/2}p_{\mathbb{C}} \\
&= D_0^n\left[e^{L_{\mu\times\nu}/2}e^{-L_{\mu\times\nu}/2}p_{\mathbb{C}}\right] \\
&= D_0^n p_{\mathbb{C}} = \alpha_{\mathbb{C}} \in \mathrm{Sym}_n^{\mathbb{C}}(H_{\mu\times\nu} \cong H_\mu + iH_\nu)
\end{aligned}$$

∎

Remark 12.17. The theory described above is particularly nice in the special case where $\mu = \nu$ so that $\mu \times \nu = \mu \times \mu$ is compatible with the complex structure on $W_{\mathbb{C}}$. In this case the following properties hold;

1. $q_{\mu\times\mu}(\alpha \circ M_i) = q_{\mu\times\mu}(\alpha)$ so that $q_{\mu\times\mu}$ is invariant under M_i^{tr}.
2. The inner product on $H_\mu \times H_\mu \cong H_\mu + iH_\mu$ is now the real part of a complex inner product and with this complex inner product $H_\mu + iH_\mu = (H_\mu)_{\mathbb{C}}$ as Hilbert spaces.
3. If f is holomorphic then
$$L_{\mu\times\mu}f = \left(L_\mu + \tilde{L}_\mu\right) f = (L_\mu - L_\mu) f = 0,$$
i.e. holomorphic functions are now harmonic. Consequently, $e^{-L_{\mu\times\mu}/2}p = p$ whenever p is a holomorphic polynomial.
4. The Fock-Itô-Kakutani isometry for an element $f \in \mathcal{H}L^2(W_{\mathbb{C}}, \mu \times \mu)$ is now simply given by
$$F_{\mu\times\mu}f = (D_0^n f)_{n=0}^{\infty}$$
which we will simply refer to as a Taylor map.

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