Quantized Yang-Mills $(1+1)$ and the Segal-Bargmann-Hall Transform

Joint with Brian Hall

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Preliminaries

In this talk;

1. We are going to extend results in [Gross & Malliavin, 1996, Hall & Sengupta, 1998] which shows how to get one of Hall's transform introduced in [Hall, 1994].

2. Along the way we will describe the Yang-Mill's quantization problem.

3. Following [Driver & Hall, 1999] (motivated by [Landsman & Wren, 1997]) we will see that a solution to the YM_2 –quantization problem ($2 = 1 + 1$ (space+time) dimensions) gives rise to a one parameter family of Hall – transforms which interpolate between his two original transforms.

4. See [Albeverio *et al.*, 1999] for the Segal-Bargmann transform as related to the stochastic quantization of YM_2.

Fock Spaces

Definition 1 (Bosonic Fock spaces)**.** Given a real Hilbert space, H and $t > 0$, let;

$$\mathrm{Mult}_n(H,\mathbb{C}) = \left\{\alpha : H^n \overset{\text{Multi-Linear}}{\to} \mathbb{C} : \|\alpha\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} < \infty\right\}$$

where

$$\|\alpha\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} = \sum_{h_1,\dots,h_n \in S} |\alpha(h_1,\dots,h_n)|^2,$$

$$\mathrm{Sym}_n(H,\mathbb{C}) = \{\alpha \in \mathrm{Mult}_n(H,\mathbb{C}) : \alpha \text{ is symmetric}\},$$

and

$$\mathcal{F}(H;t) := \left\{\alpha = (\alpha_n)_{n=0}^\infty \in \prod_{n=0}^\infty \mathrm{Sym}_n(H,\mathbb{C}) : \|\alpha\|_t^2 < \infty\right\}$$

where

$$\|\alpha\|_t^2 := \sum \frac{t^n}{n!} \|\alpha_n\|^2_{\mathrm{Mult}_n(H,\mathbb{C})}.$$

Examples

Example 1. Suppose $(W, H = H_\mu, \mu)$ is an abstract Wiener space and $f \in \mathcal{P}(W^*)$ then;

$$\alpha_n := D_x^n f \in \mathrm{Sym}_n(H, \mathbb{C})$$

where

$$D_x^n f(h_1, \ldots, h_n) := (\partial_{h_1} \ldots \partial_{h_n} f)(x),$$

and

$$\alpha = (\alpha_n)_{n=0}^{\infty} \in \cap_{t>0} \mathcal{F}(H; t).$$

Moreover,

$$f(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_n(h, \ldots, h)$$

for all $h \in H$.

Example 2. If $H = L^2([0,T], \mathbb{R})$ then

$$L^2([0,T]^n, \mathbb{C}) \ni u \to \alpha_u \in \mathrm{Mult}_n(H, \mathbb{C})$$

where

$$\alpha_u(h_1, \dots, h_n) := \int_{[0,T]^n} u(s_1, \dots, s_n) h_1(s_1) \dots h_n(s_n) d\mathbf{s}.$$

is unitary. ($\mathbf{s} := (s_1, \dots, s_n)$.)

Example 3. Similarly if $H = L^2([0,T], \mathbb{R})$ and

$$\Delta_n(T) := \{\mathbf{s} \in [0,T] : 0 \le s_1 \le s_2 \le \cdots \le s_n \le T\}$$

then

$$L^2(\Delta_n(T), \mathbb{C}) \ni u \to \alpha_u \in \mathrm{Sym}_n(H, \mathbb{C})$$

is an isomorphism where

$$\alpha_u(h_1, \dots, h_n) := \sum_{\sigma \in \text{Perm}_n} \int_{\Delta_n(T)} u(s_1, \dots, s_n) h_{\sigma 1}(s_1) \dots h_{\sigma n}(s_n) d\mathbf{s}.$$

In this case

$$\|\alpha_u\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} = n! \int_{\Delta_n(T)} |u(\mathbf{s})|^2 d\mathbf{s}.$$

Summary of Lecture 5

Theorem 2 (Fock, Itô, Kakutani, Segal, Bargmann)**.** *Let μ and ν be non-degenerate Gaussian measures on $(W, \mathcal{B}_W)$ with $H_\mu = H_\nu$ as vector spaces. Then the following diagram of **unitary** maps commute,*

$$\begin{array}{ccc} L^2(W, \mu * \nu) & \xrightarrow{S_{\mu,\nu}} & \mathcal{H}L^2(W_{\mathbb{C}}, \mu \times \nu) \\ {\scriptstyle F_{\mu*\nu}}\downarrow & & \downarrow{\scriptstyle F_{\mu\times\nu}} \\ \mathcal{F}(H_{\mu*\nu}) \ni \alpha & \longrightarrow & \alpha_{\mathbb{C}} \in \mathcal{F}(H_\mu + iH_\nu) \end{array}$$

where $S_{\mu,\nu}$ is the generalized Segal-Bargmann map;

$$S_{\mu,\nu} p := \nu_2 * p_{\mathbb{C}} = (\nu_2 * p)_{\mathbb{C}}$$

and where for any Gaussian measure γ on $(W, \mathcal{B}_W)$ F_γ is the Fock–Itô-Kakutani isomorphism defined by

$$L^2(\gamma) \ni f \to F_\gamma f := (D_0^n(\gamma * f))_{n=0}^\infty \in \mathcal{F}(H_\gamma).$$

Comments:

1. As vector space $H_\mu + iH_\nu = H_{\mathbb{C}}$ but as real inner product spaces $H_\mu + iH_\nu = H_\mu \times H_\nu$.
2. $\mathcal{F}_{\mathbb{C}}(H_\mu + iH_\nu)$ denotes the Fock space of **complex** multi-linear forms on $H_\mu + iH_\nu$.

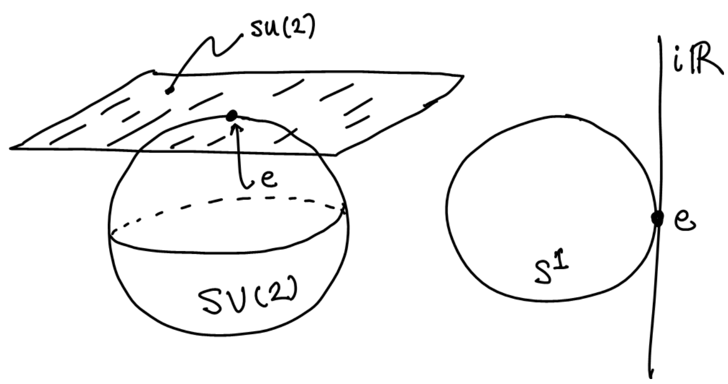
Yang-Mills set up

- $K = SU(2)$ or S^1 or a compact Lie Group

$$SU(2) = \left\{ g := \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \ni \ |a|^2 + |b|^2 = 1 \right\}$$

- $\mathfrak{k}$ = Lie(K), e.g. $\mathrm{Lie}(SU(2)) = su(2)$

$$su(2) = \left\{ A := \begin{bmatrix} i\alpha & -\bar{\beta} \\ \beta & -i\alpha \end{bmatrix} : \alpha \in \mathbb{R} \text{ and } \beta \in \mathbb{C} \right\}$$



- **Lie bracket:** $[A, B] = AB - BA =: ad_A B$

- $\langle A, B\rangle = -\mathrm{tr}(AB) = \mathrm{tr}(A^* B)$
  (a fixed Ad – K – invariant inner product)

- $M = \mathbb{R}^d$ or $T^d = \left(S^1\right)^d$.

- $\mathcal{A} = L^2(M, \mathfrak{k}^d)$ – the space of connection 1-forms.

- For $A \in \mathcal{A}$ and $1 \le i, k \le d$, let

$$\nabla_k^A := \partial_k + ad_{A_k} \text{ (covariant differential)}$$

and

$$F_{ki}^A := \partial_k A_i - \partial_i A_k + [A_k, A_i] \text{ (Curvature of } A)$$

Yang – Mills Equations (in the temporal gauge)

For $A(t) \in \mathcal{A}$, i.e. for $(t,x) \in \mathbb{R} \times M$,

$$A(t,x) = (A_1(t,x), A_2(t,x), \ldots, A_d(t,x)) \in \mathfrak{k}^d$$

the Y.M. equations are the Euler Lagrange equations for the action functional,

$$I_T(A) = \frac{1}{4}\int_{[0,T]\times\mathbb{R}^d} F^A(t,x)\cdot F^A(t,x)\,dx\,dt$$

where

$$F^A(t,x)\cdot F^A(t,x) = \sum \eta^\mu\eta^\nu \operatorname{tr}\left[F^A_{\mu,\nu}(t,x)F^A_{\mu,\nu}(t,x)\right]$$

and $\eta = (1,-1,-1,\ldots,-1)$.

Using

$$\begin{aligned}\partial_B F^A_{j,k}(x) &= \partial_j B_k - \partial_k B_j + [B_j, A_k] + [A_j, B_k]\\ &= \nabla^A_j B_k - \nabla^A_k B_j =: d^A B\end{aligned}$$

we find

$$\begin{aligned}(\partial_B I_T)(A) &= \frac{1}{2}\int_{[0,T]\times\mathbb{R}^d} d^A B\cdot F^A(t,x)\,dx\,dt\\ &= \frac{1}{2}\int_{[0,T]\times\mathbb{R}^d} B\cdot\left(d^A\right)^* F^A(t,x)\,dx\,dt.\end{aligned}$$

Therefore the Euler Lagrange equations are,

$$\left(d^A\right)^* F^A(t,x) = 0.$$

Writing out these equation explicitly give the Yang – Mills PDE's,

$$\dot{A}\left(t\right) = E\left(t\right) \qquad \text{(i.e. define } E\left(t\right) := \dot{A}\left(t\right)\text{)} \tag{1}$$

$$\dot{E}_i\left(t\right) = \boxed{\ddot{A}_i = \sum_{k=1}^{d} \nabla_k^A F_{ki}^A =: Q\left(A, \partial A\right)} \text{ (Dynamical Eqs.)} \tag{2}$$

$$0 = \nabla^A \cdot E = \sum_{k=1}^{d} \nabla_k^A E_k \text{ (Constraint Eqs.)} \tag{3}$$

Remark 3. The Yang – Mills equations are invariant under the Gauge group, $\mathcal{G} := C^\infty\left(M, K\right)$ which acts on $\mathcal{A}$ by

$$A \in \mathcal{A} \rightarrow A^g = g^{-1} A g + g^{-1} \nabla g. \tag{4}$$

This is a group action, namely $\left(A^g\right)^k = A^{gk}$ for $g, k \in \mathcal{G}$.

Maxwell's Equations $(d = 3\ K = S^1)$

If $d = 3$, $K = S^1$ and we set

$$E(t) := \dot{A}(t) \text{ and } B(t) = \nabla \times A(t),$$

then the Yang – Mills equations become Maxwell's Equations:

$$\dot{E} = -\nabla \times B \text{ and } \dot{B} = \nabla \times E$$

$$\nabla \cdot E = 0 \text{ and } \nabla \cdot B = 0.$$

Newton Form of the Y. M. Equations

Define the potential energy functional, $V(A)$, by

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j<k\le d} |F_{j,k}^A(x)|^2 dx.$$

Then the dynamics equation may be written in Newton form as

$$\ddot{A}(t) = -(\mathrm{grad}_{\mathcal{A}} V)(A).$$

The conserved energy is thus

$$\text{Energy}\left(A, \dot{A}\right) = \frac{1}{2}\left\|\dot{A}\right\|_{\mathcal{A}}^2 + V(A). \tag{5}$$

The weak form of the constraint Eq. (3) is, for $h \in C_c^\infty(M, \mathfrak{k})$,

$$0 = \left(\nabla^A \cdot E, h\right)_{L^2(M;\mathfrak{k})} = -\left(E, \nabla^A h\right)_{\mathcal{A}}.$$

Review Canonical Quantization

CONCEPT	CLASSICAL	QUANTUM
STATE SPACE	$T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \ni (p,q)$	$K = PL^2(\mathbb{R}^d, dm)$ $\psi \in L^2(\mathbb{R}^d, dm) \;\ni \|\psi\|_K = 1.$
OBSERVABLES	Functions on $T^*\mathbb{R}^d$	S.A. ops. on K
	p_k	$\hat{p}_k = \frac{\hbar}{i}\frac{\partial}{\partial q_k}$
Examples	q_k	$\hat{q}_k = M_{q_k}$
	$H(q,p) = \frac{1}{2m}p \cdot p + V(q)$	$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(q)$
Angular Momentum	$(q \times p)_k = \sum_{l,j} \varepsilon_{kjl} q_j p_l$	$\frac{1}{i}\sum_{l,j} \varepsilon_{kjl} \hat{q}_j \hat{p}_l$
DYNAMICS	Newtons Equations of Motion $\ddot{q}(t) = -\nabla V(q(t))$	Schrödinger, Eq. $i\hbar\dot{\psi}(t) = \hat{H}\psi(t), \;\; \psi(t) \in K$
MEASUREMENTS	Evaluation $f(q,p)$	$\langle \psi, \theta\psi \rangle$ – expected value.

Formal Quantization of the Y. M. – Equations

Open Problem. When $d = 3$, **"Quantize"** the Yang – Mills equations and show the resulting quantum – mechanical Hamiltonian has a mass gap. See $www.claymath.org$.

Let us explain the **formal** quantization of the Y. M. equations:

$$\text{Raw quantum Hilbert Space: } \mathbb{H} = L^2\left(\mathcal{A}, \text{“}\mathcal{D}A\text{”}\right)$$

$$\text{Position: } (A, k) \rightsquigarrow M_{(A,k)}$$

$$\text{Momentum: } (E, k) \rightsquigarrow \frac{1}{i}\partial_k \text{ for } k \in C_c^\infty(\mathbb{R}^d, \mathfrak{k}^d)$$

$$\text{Energy Function: } K.E. + P.E. \rightsquigarrow H := -\frac{1}{2}\Delta_{\mathcal{A}} + M_V$$

Recall that the **P**otential **E**nergy (V) is given by

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1 \le j < k \le d} |F_{j,k}^A(x)|^2 dx.$$

Constraints

We must also quantize the constraint functionals:

$$\left(E, \nabla^A h\right)_{\mathcal{A}} = \sum_{k \in O.N.B.(\mathcal{A})} \left(k, \nabla^A h\right)_{\mathcal{A}} (E, k)_{\mathcal{A}}$$

$$\rightsquigarrow \sum_{k \in O.N.B.(\mathcal{A})} \left(k, \nabla^A h\right)_{\mathcal{A}} \frac{1}{i} \partial_k = \frac{1}{i} \partial_{\nabla^A h}$$

Remark 4. Since

$$\sum_{k \in O.N.B.(\mathcal{A})} \partial_k \left(k, \nabla^A h\right)_{\mathcal{A}} = \sum_{k \in O.N.B.(\mathcal{A})} (k, [k, h])_{\mathcal{A}} = \sum_{k \in O.N.B.(\mathcal{A})} 0 = 0,$$

there is no ordering ambiguity in the quantization of $\left(E, \nabla^A h\right)_{\mathcal{A}}$.

Definition 5. For each $h \in C_c^\infty(M, \mathfrak{k})$, let X^h be the vector field on $\mathcal{A}$ defined by:

$$X^h(A) := \nabla^A h = \nabla h + ad_A h.$$

With this notation we want to trim down the raw Hilbert space to:

$$\mathbb{H}_{\text{physical}} = \left\{ F \in \mathbb{H} : \overbrace{X^h F := \partial_{\nabla^A h} F = 0}^{\text{Constraint Conditions}} \,\forall\, h \in C_c^\infty(\mathbb{R}^d, \mathfrak{k}) \right\}.$$

Theorem 6 (Concrete description of $\mathbb{H}_{\text{physical}}$). *The physical Hilbert space is given by*,

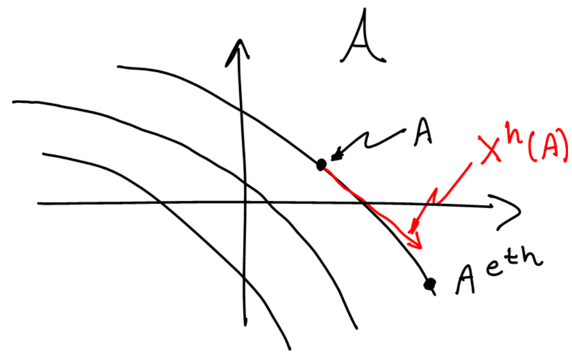
$$\mathbb{H}_{\textit{physical}} = \{F \in \mathbb{H} : F(A^g) = F(A)\ \forall\ A \in \mathcal{A},\ g \in \mathcal{G}\}.$$

.

Proof: First observe that

$$\frac{d}{dt}|_0 A^{e^{th}} = \frac{d}{dt}|_0 \left(Ad_{e^{-th}} A + e^{-th} \nabla e^{th} \right)$$
$$= -[h, A] + \nabla h = ad_A h + \nabla h = X^h(A).$$

Q.E.D.



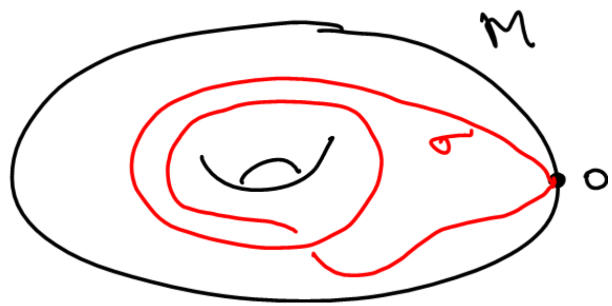
Hence X^h generates the flow, $A \to A^{e^{th}}$.Therefore the following are equivalent:

1. $X^h F = 0$ for all $h \in C_c^\infty(M, \mathfrak{k})$
2. $F \circ e^{tX^h} = F$ for all $h \in C_c^\infty(M, \mathfrak{k})$
3. $F\left(A^{e^h}\right) = F(A)$ for all $h \in C_c^\infty(M, \mathfrak{k})$
4. $F(A^g) = F(A) \; \forall \; A \in \mathcal{A}, \, g \in \mathcal{G}.$

Wilson loop variable description of $\mathbb{H}_{\text{physical}}$

Definition 7 (Restricted Gauge Group). $\mathcal{G}_0 := \{g \in \mathcal{G} : g(0) = id\}$.

Let $\mathcal{L} = \mathcal{L}(M)$ loops on M based at $o \in M$.



Definition 8. Let $//^A(\sigma) \in K$ be (**left invariant) parallel translation along** $\sigma \in \mathcal{L}$, that is $//^A(\sigma) := //^A_1(\sigma)$, where

$$\frac{d}{dt}//^A_t(\sigma) + \sum_{i=1}^{d} \dot{\sigma}_i(t) A_i(\sigma(t)) //^A_t(\sigma) = 0 \text{ with } //^A_0(\sigma) = id.$$

Theorem 9 (Loop Variable Theorem)**.** *Suppose* $A, B \in \mathcal{A}$*. Then* $//^A(\sigma) = //^B(\sigma)$ *for all* $\sigma \in \mathcal{L}$ *iff* $A = B^g$ *for some* $g \in \mathcal{G}_0$*. We call the function,* $A \to //^A(\sigma)$ *, a "**Loop variables**" on* $\mathcal{A}/\mathcal{G}_0$*.*

Proof:

- If $A = B^g$ for some $g \in \mathcal{G}_0$ and $\sigma : [0,1] \to M$ such that $\sigma(0) = o$, then
$$//^A(\sigma) = //^{B^g}(\sigma) = g(\sigma(1))^{-1} //^B(\sigma).$$
- Hence if $A = B^g$ and $\sigma \in \mathcal{L}$, then $//^A(\sigma) = //^B(\sigma)$.
- If $//^A(\sigma) = //^B(\sigma)$ for all $\sigma \in \mathcal{L}$, define $g(\sigma(1)) = //^B(\sigma) //^A(\sigma)^{-1}$ for all $\sigma : [0,1] \to M$ such that $\sigma(0) = o$.
- Then g is well defined and $A = B^g$.

Q.E.D.

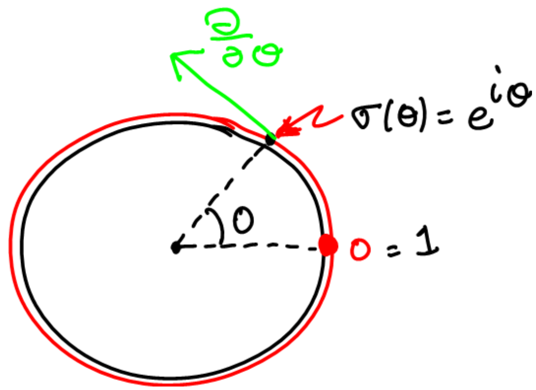
Corollary 10.

$$\mathbb{H}_{\textit{physical}} = \{F \in \mathbb{H} : F(A^g) = F(A) \ \forall\, A \in \mathcal{A},\ g \in \mathcal{G}\}$$

$$\text{“}\cong\text{”} \left\{F \in L^2(\mathcal{A}, \mathcal{D}A) : F = F\left(\left\{//^A(\sigma) : \sigma \in \mathcal{L}\right\}\right)\right\}.$$

Restriction to $d = 1$ (general K)

$S^1 = [0,1]/(0 \sim 1) \ni \theta$ and write $\partial_\theta = \frac{\partial}{\partial\theta}$



In this case,

- $\mathcal{A} = L^2(S^1, \mathfrak{k})$, Configuration space
- $\mathcal{G}_0 = \{g \in H^1(S^1 \to K) : g(0) = g(1) = id \in K\}$, Gauge Group
- $A^g = Ad_{g^{-1}} A + g^{-1} g'$

- $\mathbb{H} =$"$L^2(\mathcal{A}, \mathcal{D}A)$" Raw Hilbert Space

- $\mathbb{H}_{\text{physical}} = \{F \in \mathbb{H} : F_\phi(A) = \phi(//_1(A)),\ \phi : K \to \mathbb{C}\}$, where $//_\theta(A) \in K$ is the solution to
$$\frac{d}{d\theta}//_\theta(A) + A(\theta)//_\theta(A) = 0 \text{ with } //_0(A) = id \in K.$$
$//_1(A) \in K$ is the **holonomy** of A.

- $F^A \equiv 0$ when $d = 1$ and therefore, $V(A) \equiv 0$. No curvature in 1d

- $H = -\frac{1}{2}\Delta_{\mathcal{A}}$ (Quantum Hamiltonian) Raw Hamiltonian

A Physics Idea

Theorem 11 (Heuristic: c.f. Witten 1991, CMP 141.). *Suppose* K *is simply connected and for* ϕ *let* $F_\phi(A) := \phi(//_1(A))$, *then*

$$\phi \in L^2(K, d\text{Haar}) \to F_\phi \in \mathbb{H}_{\text{physical}}$$

is a "Unitary" map which intertwines $\Delta_{\mathcal{A}}$ *and* Δ_K, *i.e.*

$$\Delta_{\mathcal{A}}[\phi \circ //_1] = \Delta_{\mathcal{A}} F_\phi = F_{\Delta_K \phi} = (\Delta_K \phi) \circ //_1. \tag{6}$$

Goal: Give a precise meaning to the previous idea.

To do this we will "regularize" $\mathcal{D}A$ by the Gaussian measure

$$d\tilde{P}_s(A) = \frac{1}{Z_s} \exp\left(-\frac{1}{2s}|A|^2_{\mathcal{A}}\right) \mathcal{D}A$$

with the idea of letting $s \to \infty$ at the end to "recover" $\mathcal{D}A$.

The measure $\tilde{P}_s$ is a Gaussian measure living on a certain completion, $\bar{\mathcal{A}}$, of $\mathcal{A}$.

A Realization of $\bar{\mathcal{A}}$ as $W(\mathfrak{k})$

- $W(\mathfrak{k}) := \{\omega \in C([0,1] \to \mathfrak{k}) : \omega(0) = 0\}$
- $W(K) := \{g \in C([0,1] \to K : g(0) = e \in K\}$
- $H(\mathfrak{k}) := \{h \in W(\mathfrak{k}) : \int_0^1 |h'(s)|^2 ds < \infty\}$
- Note that $\partial_\theta : H(\mathfrak{k}) \to \mathcal{A} = L^2(S^1; \mathfrak{k})$ is isometric.
- Define $\bar{\mathcal{A}} := \partial_\theta W(\mathfrak{k})$. Completed Connection Forms
- $\tilde{P}_s \to P_s$ – Wiener measure on $W(\mathfrak{k})$ with variance s.
- $//_\theta(A) \to //_\theta(a)$ where for $a \in W(\mathfrak{k})$,

$$d//_\theta(a) + a'(\theta)\, //_\theta(a) = 0 \text{ with } //_0(a) = id \in K.$$

- The action of gauge group, $A \to A^g$ goes over to

$$a \to a_s^g = \int_0^s \left(g^{-1}(\sigma)\, da(\sigma)\, g(\sigma) + g^{-1}(\sigma)\, dg(\sigma)\right).$$

Gross' Ergodicity Theorem

The following theorem is a stochastic version of the Loop Variable Theorem, item 2. of Theorem 9.

Theorem 12 ([Gross, 1993]). *Let*

$$\begin{aligned}\mathbb{H}^s_{\textit{physical}} &:= \left[L^2\left(W\left(\mathfrak{k}\right), P_s\right)\right]^{\mathcal{G}_0} \\ &= \left\{F \in L^2\left(W\left(\mathfrak{k}\right), P_s\right) : F\left(a^g\right) = F\left(a\right) \text{ for } P_s \text{ a.e. } a\right\}.\end{aligned}$$

Then

$$\mathbb{H}^s_{\textit{physical}} = \left\{F = f\left(//_1\right) : f \in L^2\left(K, p_s(x)dx\right)\right\}.$$

where

$$p_s(x)dx = P_s\text{–Law}(//_1).$$

Remark 13. The action, $F\left(a\right) \to F\left(a^g\right)$ is not unitary except in the limit as $s \to 0$. The unitarized action has no non-trivial fixed elements in $L^2\left(W\left(\mathfrak{k}\right), P_s\right)$, see [Driver & Hall, 2000] for a proof using the Fourier Wiener transform. Hence it would be a **BAD** idea to unitarize this action.

Corollary 14. *The function, p_s, is the convolution heat kernel on K. Since* $\lim_{s\to\infty} p_s(x) = 1$,

$$\lim_{s\to\infty} \mathbb{H}^s_{\textit{physical}} \cong L^2(K, dx).$$

An Explanation for Eq. (6)

Recall Eq. (6) states $\Delta_{\mathcal{A}}[\phi \circ //_1] = (\Delta_K \phi) \circ //_1$

- If we let S_0 be an orthonormal basis of $H(\mathfrak{k})$ and

$$\boxed{\Delta_{H(\mathfrak{k})} = \sum_{h \in S_0} \partial_h^2,} \tag{7}$$

then the assertion in Eq. (6) becomes:

$$\boxed{\Delta_{H(\mathfrak{k})} (\phi \circ //_1) \overset{?}{=} (\Delta_K \phi) \circ //_1.} \tag{8}$$

Proof: (Heursitic explanation.)

- Use $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ to construct a bi-invariant metric on TK.
- Let $H(K)$ be the space of finite energy paths on K starting at $e \in K$.
- Equip $H(K)$ with the right invariant metric induced from the metric on

$$H(\mathfrak{k}) := \mathrm{Lie}(H(K)).$$

Then it is a **fact** that the “Cartan Rolling Map, $\psi : H(\mathfrak{k}) \to H(K)$ defined by

$$\psi(a) := //_{\cdot}(a)$$

is an isometric isomorphism of Riemannian manifolds. Consequently we may “conclude” that ψ intertwines the Laplacian, $\Delta_{H(\mathfrak{k})}$ on $H(\mathfrak{k})$ with the Laplacian, $\Delta_{H(K)}$ on $H(K)$, i.e.

$$\Delta_{H(\mathfrak{k})}(f \circ \psi) = \left(\Delta_{H(K)} f\right) \circ \psi. \tag{9}$$

When $f(g) = \varphi(g(1))$, one can show

$$\Delta_{H(K)} f(g) = (\Delta_K \varphi)(g(1))$$

and therefore Eq. (9) implies,

$$\boxed{\Delta_{H(\mathfrak{k})}(\phi \circ //_1) = (\Delta_K \phi) \circ //_1.}$$

Q.E.D.

Why is this Explanation not Satisfactory

- The operator $\Delta_{H(\mathfrak{k})}$ makes sense on smooth cylinder functions.
- However, $\phi \circ //_1$ is not a cylinder function.
- **Problematic Theorem:** The densely defined operator $\Delta_{H(\mathfrak{k})}$ on $L^2(W(\mathfrak{k}), P_s)$ is **not** closable.

Proof. Consider the case $\mathfrak{k} = \mathbb{R}$ and $s = 1$, so that $\mu = P_1$ is standard Wiener measure. Let

$$f(a) = 2\int_0^1 a_\theta da_\theta = a_1^2 - 1$$

a cylinder function. One computes

$$\Delta_{H(\mathfrak{k})} f(a) = \sum_{h \in S_0} 2h_1^2 = 2.$$

On the other hand, we have $f(a) = \lim_{|\mathcal{P}| \to 0} f_{\mathcal{P}}(a)$ where $f_{\mathcal{P}}(a)$ is the cylinder function

$$f_{\mathcal{P}}(a) = 2\sum_{s_i \in \mathcal{P}} a_{s_i}(a_{s_{i+1}} - a_{s_i}).$$

But

$$\Delta_{H(\mathfrak{k})} f_{\mathcal{P}}(a) = 0!$$

(Compare with the harmonic function

$$(x_1 + x_2 + \cdots + x_n)x_{n+1} \text{ on } \mathbb{R}^{n+1}.)$$

Therefore $\lim_{|\mathcal{P}|\to 0} f_{\mathcal{P}} = f$ while

$$0 = \lim_{|\mathcal{P}|\to 0} \Delta_{H(\mathfrak{k})} f_{\mathcal{P}}(a) \neq \Delta_{H(\mathfrak{k})} f = 2.$$

Segal - Bargmann Theory

- Let $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$ be the complexification of $\mathfrak{k}$
- $K_{\mathbb{C}} =$ the complexification of K, e.g. $SU(2)_{\mathbb{C}} = SL(2, \mathbb{C})$.
- For $s > t/2$, let $M_{s,t}$ be the Gaussian measure on $W(\mathfrak{k}_{\mathbb{C}})$,

$$M_{s,t} = \mathrm{Law}\left(\sqrt{s - t/2}\,\alpha + i\sqrt{t/2}\,\beta\right)$$

where α and β are independent standard $(\mathfrak{k}, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ – valued Brownian motions.

Theorem 15 (Segal- Bargmann)**.** *There exists an isometry*

$$S_t : L^2(W(\mathfrak{k}), P_s) \to L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$$

such that

$$(S_t f)(c) = \int f_{\mathbb{C}}(c + a) dP_t(a) = (e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} f)_{\mathbb{C}}(c).$$

For all polynomial cylinder functions f. Moreover $\mathrm{Ran}\,(S_t) =$ *closure of Holomorphic cylinder functions.*

Proof: Apply our generalized Segal-Bargmann theorem with

$$\mu := P_{s-t/2} = \mathrm{Law}\left(\sqrt{s-t/2}\alpha\right)$$
$$\nu := P_{t/2} = \mathrm{Law}\left(\sqrt{t/2}\beta\right)$$

so that

$$S_t = P_t * (\cdot) = \nu_2 * (\cdot),$$
$$\mu \times \nu = M_{s,t}, \quad \text{and}$$
$$\mu * \nu = \mathrm{Law}\left(\sqrt{s-t/2}\alpha + \sqrt{t/2}\beta\right) = P_s.$$

Q.E.D.

Theorem 16 (Stochastic Representation Theorem). *S_t is also characterized by*

$$S_t \int_{\triangle_n} \langle \alpha(\tau), da^{\otimes^n}(\tau)\rangle = \int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n}(\tau)\rangle$$

where $\alpha : \Delta_n \to \mathfrak{k}$ is a deterministic function.

Proof: Suppose that

$$\alpha(\tau) = 1_{J_1 \times J_2 \times \cdots \times J_n}(\tau)\eta$$

with $\eta \in \mathfrak{k}^{\otimes n}$ and $J_i = (s_i, t_i]$ are intervals such that $J_i < J_k$ for all $i < k$, i.e. $t_i < s_k$. Let $a(J_i) := a_{t_i} - a_{s_i}$ and

$$f(a) = \int_{\triangle_n} \langle \alpha(\tau), da^{\otimes^n}(\tau) \rangle = \langle \eta, a(J_1) \otimes \cdots \otimes a(J_n) \rangle$$

then

$$f_{\mathbb{C}}(c) = \int_{\triangle_n} \langle \alpha(\tau), dc^{\otimes^n}(\tau) \rangle = \langle \eta, c(J_1) \otimes \cdots \otimes c(J_n) \rangle,$$

where $c(J_i) := c_{t_i} - c_{s_i}$. Q.E.D.

Since $a \to \langle \eta, (c(J_1) + a(J_1)) \otimes \cdots \otimes (c(J_n) + a(J_n)) \rangle$ is a Harmonic polynomial of a;

$$\begin{aligned} S_t f\,(c) &:= \int_{W(\mathfrak{k})} f_{\mathbb{C}}(c+a) dP_t(a) \\ &= \int_{W(\mathfrak{k})} \langle \eta, (c(J_1) + a(J_1)) \otimes \cdots \otimes (c(J_n) + a(J_n)) \rangle dP_t(a) \\ &= \langle \eta, c(J_1) \otimes \cdots \otimes c(J_n) \rangle \\ &= f_{\mathbb{C}}(c) \end{aligned}$$

By a limiting argument one then shows in geneal that

$$S_t \left(\int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), d(c+a)^{\otimes^n}(\tau) \rangle \right) = \int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n}(\tau) \rangle.$$

The Main Theorem

Theorem 17 ([Gross & Malliavin, 1996, Driver & Hall, 1999])**.** *Let*

$$d//_\theta + da_\theta \circ //_\theta = 0 \text{ with } //_0 = Id.$$

relative to P_s *and*

$$d//_\theta^{\mathbb{C}} + dc_\theta \circ //_\theta^{\mathbb{C}} = 0 \text{ with } //_0^{\mathbb{C}} = Id.$$

relative to $M_{s,t}$*. Then for all* $f \in L^2(K, dx)$,

$$S_t f(//_1) = F(//_1^{\mathbb{C}})$$

where F *is the unique Holomorphic function on* $K_{\mathbb{C}}$ *such that*

$$F|_K = e^{\frac{t}{2}\triangle_K} f.$$

Morally speaking:

$$S_t H = (e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} H)_{\mathbb{C}} \in \mathcal{H}L^2 (W(\mathfrak{k}_{\mathbb{C}}))$$

$$(e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} f(//_1))_{\mathbb{C}} = (e^{\frac{t}{2}\triangle_K} f)_{\mathbb{C}}(//_1^{\mathbb{C}})$$

so on "restricting" to $W(\mathfrak{k})$

$$e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} f(//_1) = (e^{\frac{t}{2}\triangle_K} f)(//_1)$$

which we interpret as a rigorous version of the statement that

$$\triangle_{H(\mathfrak{k})} [f(//_1)] = (\triangle_K f)(//_1).$$

The generators of $//_\theta \in K$ & $//_\theta^{\mathbb{C}} \in K_{\mathbb{C}}$

Proposition 18. *Let*

- $\{X_k : k = 1, \ldots, \dim \mathfrak{k}\}$ *be an orthonormal basis for* $\mathfrak{k}$
- $Y_k = JX_k$, *where* J *is the complex structure on* $\mathfrak{k}_{\mathbb{C}}$.

Then

1. *The generator of the diffusion,* $//_\theta \in K$, *is*
$$\Delta_K = \sum X_k^2.$$
2. *The generator of the diffusion,* $//_\theta^{\mathbb{C}} \in K_{\mathbb{C}}$, *is*
$$A_{s,t} = (s - t/2) \sum X_k^2 + \frac{t}{2} \sum Y_k^2$$

Corollary: Hall's Transform

Let $\rho_s(dx) = \mathrm{Law}(//_1)$ and $m_{s,t}(dg) = \mathrm{Law}(//_1^{\mathbb{C}})$, i.e.

$$\rho_s(x) = \left(e^{s\Delta_K/2}\delta_e\right)(x) \text{ for } x \in K \quad \&$$

$$m_{s,t}(g) = \left(e^{A_{s,t}/2}\delta_e\right)(g) \text{ for } g \in K_{\mathbb{C}}.$$

Corollary 19 (A One Parameter family of Hall's Transforms). *The map*

$$L^2(K, \rho_s) \ni f \to \left(e^{t\Delta_K/2} f\right)_{\mathbb{C}} \in \mathcal{H}L^2(K_{\mathbb{C}}, m_{s,t})$$

is unitary.

This theorem interpolates between the two previous versions of Hall's transform corresponding to $s = \infty$ and $s = \frac{t}{2}$. **(END)**

Proof Sketch of Main Theorem 17.

For the proof we will need the following notation and facts:

- $\{X_k : k = 1, \ldots, \dim \mathfrak{k}\}$ be an orthonormal basis for $\mathfrak{k}$
- $Y_k = JX_k$, where J is the complex structure on $\mathfrak{k}_{\mathbb{C}}$.
- Let Δ_K be the generator of $//_\theta$, $\quad \Delta_K = \sum X_k^2$.
- Let $A_{s,t}$ be the generator of $//_\theta^{\mathbb{C}}$,

$$\boxed{A_{s,t} = (s - t/2) \sum X_k^2 + \frac{t}{2} \sum Y_k^2}$$

- Notice that if Φ is a holomorphic function, then $Y_k \Phi = i X_k \Phi$ so that

$$A_{s,t}\Phi = (s - t)\,\Delta_K \Phi.$$

- The X_k and Y_k commute with Δ_K.

Proof. (Proof of Main Theorem 17.) Let $\Phi = \left(e^{t\Delta_K/2}f\right)_{\mathbb{C}}$ denote the analytic continuation of $e^{t\Delta_K/2}f$ to $K_{\mathbb{C}}$. Using $[\Delta_K, X_k] = 0$ and the **Veretennikov and Krylov** formula,

$$f(//_1) = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle \alpha_n, da^{\otimes n}(\tau) \rangle$$

where $\alpha_n = \left(D^n e^{s\Delta_K/2} f\right)(e)$. Therefore

$$S_t\left[f(//_1)\right] = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle (\alpha_n)_{\mathbb{C}}, dc^{\otimes n}(\tau) \rangle.$$

Similarly,

$$\Phi(//_1^{\mathbb{C}}) = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle \beta_n, dc^{\otimes n}(\tau) \rangle$$

where

$$\beta_n = \left(D^n e^{A_{s,t}/2}\Phi\right)(e) = \left(D^n e^{(s-t)\Delta_K/2}\Phi\right)(e)$$
$$= \left(D^n e^{(s-t)\Delta_K/2}\left(e^{t\Delta_K/2}f\right)_{\mathbb{C}}\right)(e)$$
$$= \left[D^n\left(e^{s\Delta_K/2}f\right)_{\mathbb{C}}(e)\right]_{\mathbb{C}} = (\alpha_n)_{\mathbb{C}}.$$

This shows,

$$S_t\left[f(//_1)\right] = \Phi(//_1^{\mathbb{C}}) = \left(e^{t\Delta_K/2}f\right)_{\mathbb{C}}\left(//_1^{\mathbb{C}}\right)$$

as was to be shown.

Remark 20. See Dimock 1996, and Landsman and Wren $(\cong 1998)$ for other approaches to “canonical quantization” of YM_2.

(**END NOW FOR SURE!**)

Related and Further Reading

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Extension to the Path Space

Now let $\tilde{\rho}_t := //_* P_t =$ the law of $//$ relative to P_t.

Theorem 21. *There exists an isometry*

$$B_t : L^2(W(K), \mathrm{Law}(//)) \to L^2(W(K_{\mathbb{C}}), \mathrm{Law}(//^{\mathbb{C}}))$$

such that for all cylinder functions $f \in L^2(W(K), P_s)$, $B_t f$ is a Holomorphic cylinder function on $W(K_{\mathbb{C}})$ such that

$$(B_t f)(y) = \text{“}\left(e^{\frac{t}{2}\Delta_{H(K)}} f\right)(y)\text{”} = \int f(xy)\tilde{\rho}_t(dx)\ \forall\, y \in H(K).$$

Moreover, Ran $(B_t))$ is the closure of the holomorphic cylinder functions and the following diagram commutes

$$\begin{array}{ccc} L^2(W(\mathfrak{k}), P_s) & \xrightarrow{S_t} & \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t}) \\ \updownarrow // & \circlearrowleft & \updownarrow //^{\mathbb{C}} \\ L^2(W(K), //_* P_s) & \xrightarrow{B_t} & \mathcal{H}L^2(W(K_{\mathbb{C}}), //_*^{\mathbb{C}} M_{s,t}) \end{array}$$

i.e.

$$S_t(f \circ //) = (B_t f) \circ //^{\mathbb{C}}$$

Path Space Result Explanation

Proof. (An explanation rather than a proof.) Let $y \in H(K)$ and consider

$$\int_{W(K)} f(xy)\tilde{\rho}_t(dx) = \int_{W(\mathfrak{k})} f(//(a) \cdot y)P_t(da).$$

Notice that

$$//^{-1}(z) = \int_0^{\cdot} z^{-1}\delta z. \quad \text{(Inverse of the Itô Map.)}$$

so that

$$\begin{aligned} //^{-1}\left(//(a)y\right) &= \int \left(//(a)y\right)^{-1} \delta\left(//(a)y\right) \\ &= \int y^{-1}//(a)^{-1}\delta\left(//(a)y\right) \\ &= \int Ad_{y^{-1}}\delta a + \int y^{-1}\delta y \\ &= \int Ad_{y^{-1}}\delta a + //^{-1}(y). \end{aligned}$$

Therefore,

$$//(a)y = //\left(\int Ad_{y^{-1}}da + //^{-1}(y)\right).$$

Noting that

$$\text{Law}\left(\int Ad_{y^{-1}}da\right) = \text{Law}\,(a)\,,$$

we learn that

$$\begin{aligned}(B_tf)(y) &= \int_{W(K)} f(xy)\tilde{\rho}_t(dx) = \int_{W(\mathfrak{k})} f(//(a)\cdot y)P_t(da)\\ &= \int_{W(\mathfrak{k})} f(//\left(\int Ad_{y^{-1}}da + //^{-1}(y)\right))P_t(da)\\ &= \int_{W(\mathfrak{k})} f(//\left(a + //^{-1}(y)\right))P_t(da)\\ &= S_t\,(f\circ //)\left(//^{-1}(y)\right).\end{aligned}$$

Now replace $y \to //\,(a)$ in the above identity to find

$$(B_tf)(//\,(a)) = S_t\,(f\circ //)\,(a)\,,$$

i.e.

$$(B_tf)\circ // = S_t\,(f\circ //)\,.$$

Isometry Property

By the way one checks the isometry property from this result as follows. On one hand

$$\int_{W(\mathfrak{k})} \left| \int_{\triangle_n} \langle \alpha(\tau), da^{\otimes^n}(\tau) \rangle \right|^2 dP_s(a) = s^n \int_{\triangle_n} |\alpha(\tau)|^2 d\tau,$$

while on the other

$$\int_{W(\mathfrak{k}_{\mathbb{C}})} \left| \int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n} \rangle \right|^2 dM_{s,t}(c) = s^n \int_{\triangle_n} |\alpha(\tau)|^2 d\tau.$$

To prove this last assertion, consider the expectation of the stochastic integral:

$$\begin{aligned}
\mathbb{E}\left|\int_\alpha^\beta f(\tau)dc(\tau)\right|^2 &= \mathbb{E}\left|\int_\alpha^\beta f(\tau)da(\tau) + i\ f(\tau)db(\tau)\right|^2 \\
&= \mathbb{E}\int_\alpha^\beta |f(\tau)|^2 d\tau\ (s - \frac{t}{2}) + \frac{t}{2}\mathbb{E}\int_\alpha^\beta |f(\tau)|^2 d\tau \\
&= s\int_\alpha^\beta |f(\tau)|^2 d\tau,
\end{aligned}$$

where $f(\tau)$ is assumed to be adapted. Hence the result follows by writing

$\int\limits_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n} \rangle$ as an iterated integral. For example if $n = 2$,

$$\begin{aligned}
&\mathbb{E}\left|\int\limits_{\triangle_2} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^2} \rangle\right|^2 \\
&= \mathbb{E}\left|\int_{0\le\tau_1\le\tau_2\le1} \langle \alpha_{\mathbb{C}}(\tau_1,\tau_2), dc(\tau_1)\otimes dc(\tau)\rangle\right|^2 \\
&= \int_0^1 d\tau_2 \sum_{\xi} s\mathbb{E}\int_{0\le\tau_1\le\tau_2} \left|\langle \alpha_{\mathbb{C}}(\tau_1,\tau_2), dc(\tau_1)\otimes \xi\rangle\right|^2 \\
&= \int_0^1 d\tau_2 \sum_{\xi,\eta} s^2 \int_{0\le\tau_1\le\tau_2} d\tau_1 \left|\langle \alpha_{\mathbb{C}}(\tau_1,\tau_2), \eta\otimes \xi\rangle\right|^2 \\
&= s^2 \int_{0\le\tau_1\le\tau_2\le1} \left|\alpha_{\mathbb{C}}(\tau_1,\tau_2)\right|^2 d\tau_1 d\tau_2 \\
&= s^2 \int\limits_{\triangle_2} \left|\alpha_{\mathbb{C}}(\tau_1,\tau_2)\right|^2 d\tau_1 d\tau_2
\end{aligned}$$

where ξ and η in the above expression is running over an orthonormal basis of $\mathfrak{k}$.

A gradient computation

We would like to compute the gradient of $V(A)$ where

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\leq j<k\leq d} |F^A_{j,k}(x)|^2 dx.$$

To this end, we recall that $F^A_{j,k}(x) = \partial_j A_k - \partial_k A_j + [A_j, A_k]$ and therefore,

$$\begin{aligned}\partial_B F^A_{j,k}(x) &= \partial_j B_k - \partial_k B_j + [B_j, A_k] + [A_j, B_k] \\ &= \nabla^A_j B_k - \nabla^A_k B_j\end{aligned}$$

and hence

$$\begin{aligned}\partial_B V(A) &= \int_{\mathbb{R}^d} \sum_{1\le j<k\le d} \left\langle F^A_{j,k}(x), \partial_B F^A_{j,k}(x)\right\rangle dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left\langle F^A_{j,k}(x), \partial_B F^A_{j,k}(x)\right\rangle dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left\langle F^A_{j,k}, \nabla^A_j B_k - \nabla^A_k B_j\right\rangle dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left(\left\langle -\nabla^A_j F^A_{j,k}, B_k\right\rangle + \left\langle \nabla^A_k F^A_{j,k}, B_j\right\rangle\right) dx \\ &= -\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left\langle \nabla^A_j F^A_{j,k}, B_k\right\rangle dx.\end{aligned}$$

Therefore we learn that

$$[\mathrm{grad} V(A)]_k(x) = -\sum_{j=1}^{d} \nabla^A_j F^A_{j,k}$$

as claimed.