# Quantized Yang-Mills $(1+1)$ and the Segal-Bargmann-Hall Transform

Joint with Brian Hall

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Cornell 6th Summer School
Cornell, July 19 - 30, 2010

# Preliminaries

In this talk;

1. We are going to extend results in [Gross & Malliavin, 1996, Hall & Sengupta, 1998] which shows how to get one of Hall's transform introduced in [Hall, 1994].

2. Along the way we will describe the Yang-Mill's quantization problem.

3. Following [Driver & Hall, 1999] (motivated by [Landsman & Wren, 1997]) we will see that a solution to the $YM_2$ –quantization problem $(2 = 1 + 1$ (space+time) dimensions) gives rise to a one parameter family of Hall – transforms which interpolate between his two original transforms.

4. See [Albeverio *et al.*, 1999] for the Segal-Bargmann transform as related to the stochastic quantization of $YM_2$.



# Fock Spaces

**Definition 1** (Bosonic Fock spaces)**.** Given a real Hilbert space, $H$ and $t > 0$, let;

$$\mathrm{Mult}_n(H,\mathbb{C}) = \left\{\alpha : H^n \overset{\text{Multi-Linear}}{\rightarrow} \mathbb{C} : \|\alpha\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} < \infty\right\}$$

where

$$\|\alpha\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} = \sum_{h_1,\dots,h_n\in S} |\alpha(h_1,\dots,h_n)|^2,$$

$$\mathrm{Sym}_n(H,\mathbb{C}) = \{\alpha \in \mathrm{Mult}_n(H,\mathbb{C}) : \alpha \text{ is symmetric}\},$$

and

$$\mathcal{F}(H;t) := \left\{\alpha = (\alpha_n)_{n=0}^{\infty} \in \prod_{n=0}^{\infty} \mathrm{Sym}_n(H,\mathbb{C}) : \|\alpha\|_t^2 < \infty\right\}$$

where

$$\|\alpha\|_t^2 := \sum \frac{t^n}{n!} \|\alpha_n\|^2_{\mathrm{Mult}_n(H,\mathbb{C})}.$$



# Examples

**Example 1.** Suppose $(W, H = H_\mu, \mu)$ is an abstract Wiener space and $f \in \mathcal{P}(W^*)$ then;

$$\alpha_n := D_x^n f \in \mathrm{Sym}_n(H,\mathbb{C})$$

where

$$D_x^n f(h_1,\dots,h_n) := (\partial_{h_1}\dots\partial_{h_n} f)(x),$$

and

$$\alpha = (\alpha_n)_{n=0}^{\infty} \in \cap_{t>0}\mathcal{F}(H;t).$$

Moreover,

$$f(h) = \sum_{n=0}^{\infty} \frac{1}{n!}\alpha_n(h,\dots,h)$$

for all $h \in H$.

**Example 2.** If $H = L^2([0,T], \mathbb{R})$ then

$$L^2([0,T]^n, \mathbb{C}) \ni u \to \alpha_u \in \mathrm{Mult}_n(H, \mathbb{C})$$

where

$$\alpha_u(h_1, \dots, h_n) := \int_{[0,T]^n} u(s_1, \dots, s_n) h_1(s_1) \dots h_n(s_n) d\mathbf{s}.$$

is unitary. ($\mathbf{s} := (s_1, \dots, s_n)$.)

**Example 3.** Similarly if $H = L^2([0,T], \mathbb{R})$ and

$$\Delta_n(T) := \{\mathbf{s} \in [0,T] : 0 \le s_1 \le s_2 \le \dots \le s_n \le T\}$$

then

$$L^2(\Delta_n(T), \mathbb{C}) \ni u \to \alpha_u \in \mathrm{Sym}_n(H, \mathbb{C})$$

is an isomorphism where

$$\alpha_u(h_1, \dots, h_n) := \sum_{\sigma \in \mathrm{Perm}_n} \int_{\Delta_n(T)} u(s_1, \dots, s_n) h_{\sigma 1}(s_1) \dots h_{\sigma n}(s_n) d\mathbf{s}.$$

In this case

$$\|\alpha_u\|^2_{\mathrm{Mult}_n(H,\mathbb{C})} = n! \int_{\Delta_n(T)} |u(\mathbf{s})|^2 d\mathbf{s}.$$



# Summary of Lecture 5

**Theorem 2** (Fock, Itô, Kakutani, Segal, Bargmann)**.** *Let $\mu$ and $\nu$ be non-degenerate Gaussian measures on $(W, \mathcal{B}_W)$ with $H_\mu = H_\nu$ as vector spaces. Then the following diagram of **unitary** maps commute,*

$$\begin{array}{ccc} L^2(W, \mu * \nu) & \xrightarrow{S_{\mu,\nu}} & \mathcal{H}L^2(W_{\mathbb{C}}, \mu \times \nu) \\ F_{\mu*\nu} \downarrow & & \downarrow F_{\mu\times\nu} \\ \mathcal{F}(H_{\mu*\nu}) \ni \alpha & \longrightarrow & \alpha_{\mathbb{C}} \in \mathcal{F}(H_\mu + iH_\nu) \end{array}$$

*where $S_{\mu,\nu}$ is the generalized Segal-Bargmann map;*

$$S_{\mu,\nu} p := \nu_2 * p_{\mathbb{C}} = (\nu_2 * p)_{\mathbb{C}}$$

*and where for any Gaussian measure $\gamma$ on $(W, \mathcal{B}_W)$ $F_\gamma$ is the Fock–Itô-Kakutani isomorphism defined by*

$$L^2(\gamma) \ni f \to F_\gamma f := (D_0^n(\gamma * f))_{n=0}^\infty \in \mathcal{F}(H_\gamma).$$

**Comments:**

1. As vector space $H_\mu + iH_\nu = H_{\mathbb{C}}$ but as real inner product spaces $H_\mu + iH_\nu = H_\mu \times H_\nu$.
2. $\mathcal{F}_{\mathbb{C}}(H_\mu + iH_\nu)$ denotes the Fock space of **complex** multi-linear forms on $H_\mu + iH_\nu$.



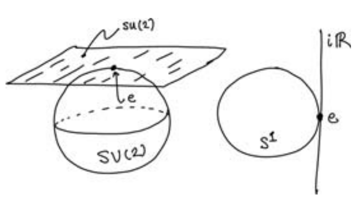
# Yang-Mills set up

- $K = SU(2)$ or $S^1$ or a compact Lie Group

$$SU(2) = \left\{ g := \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a, b \in \mathbb{C} \ni |a|^2 + |b|^2 = 1 \right\}$$

- $\mathfrak{k}$ = Lie$(K)$, e.g. $\mathrm{Lie}(SU(2)) = su(2)$

$$su(2) = \left\{ A := \begin{bmatrix} i\alpha & -\bar{\beta} \\ \beta & -i\alpha \end{bmatrix} : \alpha \in \mathbb{R} \text{ and } \beta \in \mathbb{C} \right\}$$






- **Lie bracket:** $[A, B] = AB - BA =: ad_A B$
- $\langle A, B \rangle = -\mathrm{tr}(AB) = \mathrm{tr}(A^* B)$
  (a fixed Ad – $K$ – invariant inner product)
- $M = \mathbb{R}^d$ or $T^d = (S^1)^d$.
- $\mathcal{A} = L^2(M, \mathfrak{k}^d)$ – the space of connection 1-forms.
- For $A \in \mathcal{A}$ and $1 \le i, k \le d$, let

$$\nabla_k^A := \partial_k + ad_{A_k} \text{ (covariant differential)}$$

and

$$F_{ki}^A := \partial_k A_i - \partial_i A_k + [A_k, A_i] \text{ (Curvature of } A)$$

# Yang – Mills Equations (in the temporal gauge)

For $A(t) \in \mathcal{A}$, i.e. for $(t,x) \in \mathbb{R} \times M$,

$$A(t,x) = (A_1(t,x), A_2(t,x), \ldots, A_d(t,x)) \in \mathfrak{k}^d$$

the Y.M. equations are the Euler Lagrange equations for the action functional,

$$I_T(A) = \frac{1}{4} \int_{[0,T]\times\mathbb{R}^d} F^A(t,x) \cdot F^A(t,x) dx dt$$

where

$$F^A(t,x) \cdot F^A(t,x) = \sum \eta^\mu \eta^\nu \operatorname{tr}\left[F^A_{\mu,\nu}(t,x) F^A_{\mu,\nu}(t,x)\right]$$

and $\eta = (1, -1, -1, \ldots, -1)$.

Using

$$\partial_B F^A_{j,k}(x) = \partial_j B_k - \partial_k B_j + [B_j, A_k] + [A_j, B_k] = \nabla^A_j B_k - \nabla^A_k B_j =: d^A B$$

we find

$$(\partial_B I_T)(A) = \frac{1}{2} \int_{[0,T]\times\mathbb{R}^d} d^A B \cdot F^A(t,x) dx dt = \frac{1}{2} \int_{[0,T]\times\mathbb{R}^d} B \cdot \left(d^A\right)^* F^A(t,x) dx dt.$$



Therefore the Euler Lagrange equations are,

$$\left(d^A\right)^* F^A(t,x) = 0.$$

Writing out these equation explicitly give the Yang – Mills PDE's,

$$\dot{A}(t) = E(t) \quad \text{(i.e. define } E(t) := \dot{A}(t)\text{)} \tag{1}$$

$$\dot{E}_i(t) = \ddot{A}_i = \sum_{k=1}^d \nabla^A_k F^A_{ki} =: Q(A, \partial A) \quad \text{(Dynamical Eqs.)} \tag{2}$$

$$0 = \nabla^A \cdot E = \sum_{k=1}^d \nabla^A_k E_k \quad \text{(Constraint Eqs.)} \tag{3}$$

**Remark 3.** The Yang – Mills equations are invariant under the Gauge group, $\mathcal{G} := C^\infty(M, K)$ which acts on $\mathcal{A}$ by

$$A \in \mathcal{A} \to A^g = g^{-1} A g + g^{-1} \nabla g. \tag{4}$$

This is a group action, namely $(A^g)^k = A^{gk}$ for $g, k \in \mathcal{G}$.



# Maxwell's Equations $(d = 3 \;\; K = S^1)$

If $d = 3$, $K = S^1$ and we set

$$E(t) := \dot{A}(t) \text{ and } B(t) = \nabla \times A(t),$$

then the Yang – Mills equations become Maxwell's Equations:

$$\dot{E} = -\nabla \times B \text{ and } \dot{B} = \nabla \times E$$

$$\nabla \cdot E = 0 \text{ and } \nabla \cdot B = 0.$$



# Newton Form of the Y. M. Equations

Define the potential energy functional, $V(A)$, by

$$V(A) := \frac{1}{2} \int_{\mathbb{R}^d} \sum_{1 \le j < k \le d} |F^A_{j,k}(x)|^2 dx.$$

Then the dynamics equation may be written in Newton form as

$$\ddot{A}(t) = -(\operatorname{grad}_{\mathcal{A}} V)(A).$$

The conserved energy is thus

$$\text{Energy}\left(A, \dot{A}\right) = \frac{1}{2} \left\|\dot{A}\right\|^2_{\mathcal{A}} + V(A). \tag{5}$$

The weak form of the constraint Eq. (3) is, for $h \in C^\infty_c(M, \mathfrak{k})$,

$$0 = \left(\nabla^A \cdot E, h\right)_{L^2(M;\mathfrak{k})} = -\left(E, \nabla^A h\right)_{\mathcal{A}}.$$

# Review Canonical Quantization

| CONCEPT | CLASSICAL | QUANTUM |
|---|---|---|
| **STATE SPACE** | $T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \ni (p,q)$ | $K = PL^2(\mathbb{R}^d, dm)$<br>$\psi \in L^2(\mathbb{R}^d, dm) \ni \|\psi\|_K = 1.$ |
| **OBSERVABLES** | Functions on $T^*\mathbb{R}^d$ | S.A. ops. on $K$ |
| | $p_k$ | $\hat{p}_k = \frac{\hbar}{i}\frac{\partial}{\partial q_k}$ |
| **Examples** | $q_k$ | $\hat{q}_k = M_{q_k}$ |
| | $H(q,p) = \frac{1}{2m} p \cdot p + V(q)$ | $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(q)$ |
| Angular Momentum | $(q \times p)_k = \sum_{l,j} \varepsilon_{kjl} q_j p_l$ | $\frac{1}{i}\sum_{l,j} \varepsilon_{kjl} \hat{q}_j \hat{p}_l$ |
| **DYNAMICS** | Newtons Equations of Motion<br>$\ddot{q}(t) = -\nabla V(q(t))$ | Schrödinger, Eq.<br>$i\hbar\dot{\psi}(t) = \hat{H}\psi(t), \quad \psi(t) \in K$ |
| **MEASUREMENTS** | Evaluation<br>$f(q,p)$ | $\langle \psi, \theta\psi \rangle$ – expected value. |



# Formal Quantization of the Y. M. – Equations

**Open Problem.** When $d = 3$, **"Quantize"** the Yang – Mills equations and show the resulting quantum – mechanical Hamiltonian has a mass gap. See $www.claymath.org$.

Let us explain the **formal** quantization of the Y. M. equations:

Raw quantum Hilbert Space: $\mathbb{H} = L^2(\mathcal{A}, \text{“}\mathcal{D}A\text{”})$

Position: $(A,k) \rightsquigarrow M_{(A,k)}$

Momentum: $(E,k) \rightsquigarrow \frac{1}{i}\partial_k$ for $k \in C_c^\infty(\mathbb{R}^d, \mathfrak{k}^d)$

Energy Function: $K.E. + P.E. \rightsquigarrow H := -\frac{1}{2}\Delta_{\mathcal{A}} + M_V$

Recall that the **P**otential **E**nergy $(V)$ is given by

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1 \le j < k \le d} |F^A_{j,k}(x)|^2 dx.$$



# Constraints

We must also quantize the constraint functionals:

$$\left(E, \nabla^A h\right)_{\mathcal{A}} = \sum_{k \in O.N.B.(\mathcal{A})} \left(k, \nabla^A h\right)_{\mathcal{A}} (E,k)_{\mathcal{A}}$$

$$\rightsquigarrow \sum_{k \in O.N.B.(\mathcal{A})} \left(k, \nabla^A h\right)_{\mathcal{A}} \frac{1}{i}\partial_k = \frac{1}{i}\partial_{\nabla^A h}$$

**Remark 4.** Since

$$\sum_{k \in O.N.B.(\mathcal{A})} \partial_k \left(k, \nabla^A h\right)_{\mathcal{A}} = \sum_{k \in O.N.B.(\mathcal{A})} (k, [k,h])_{\mathcal{A}} = \sum_{k \in O.N.B.(\mathcal{A})} 0 = 0,$$

there is no ordering ambiguity in the quantization of $\left(E, \nabla^A h\right)_{\mathcal{A}}$.



**Definition 5.** For each $h \in C_c^\infty(M, \mathfrak{k})$, let $X^h$ be the vector field on $\mathcal{A}$ defined by:

$$X^h(A) := \nabla^A h = \nabla h + ad_A h.$$

With this notation we want to trim down the raw Hilbert space to:

$$\mathbb{H}_{\text{physical}} = \left\{ F \in \mathbb{H} : \overbrace{X^h F := \partial_{\nabla^A h} F = 0}^{\text{Constraint Conditions}} \; \forall\, h \in C_c^\infty(\mathbb{R}^d, \mathfrak{k}) \right\}.$$

**Theorem 6** (Concrete description of $\mathbb{H}_{\text{physical}}$). *The physical Hilbert space is given by,*

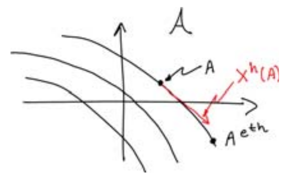
$$\mathbb{H}_{physical} = \{F \in \mathbb{H} : F(A^g) = F(A)\; \forall\, A \in \mathcal{A},\; g \in \mathcal{G}\}.$$

.

**Proof:** First observe that

$$\frac{d}{dt}|_0 A^{e^{th}} = \frac{d}{dt}|_0 \left(Ad_{e^{-th}}A + e^{-th}\nabla e^{th}\right)$$
$$= -[h, A] + \nabla h = ad_A h + \nabla h = X^h(A).$$

Q.E.D.




Hence $X^h$ generates the flow, $A \to A^{e^{th}}$. Therefore the following are equivalent:

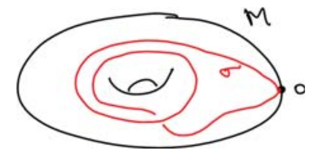
1. $X^h F = 0$ for all $h \in C_c^\infty(M, \mathfrak{k})$
2. $F \circ e^{tX^h} = F$ for all $h \in C_c^\infty(M, \mathfrak{k})$
3. $F\left(A^{e^h}\right) = F(A)$ for all $h \in C_c^\infty(M, \mathfrak{k})$
4. $F(A^g) = F(A)\ \forall\ A \in \mathcal{A},\ g \in \mathcal{G}$.



# Wilson loop variable description of $\mathbb{H}_{\text{physical}}$

**Definition 7** (Restricted Gauge Group). $\mathcal{G}_0 := \{g \in \mathcal{G} : g(0) = id\}$.

Let $\mathcal{L} = \mathcal{L}(M)$ loops on $M$ based at $o \in M$.




**Definition 8.** Let $//^A(\sigma) \in K$ be **(left invariant) parallel translation along** $\sigma \in \mathcal{L}$, that is $//^A(\sigma) := //_1^A(\sigma)$, where

$$\frac{d}{dt}//_t^A(\sigma) + \sum_{i=1}^d \dot{\sigma}_i(t) A_i(\sigma(t)) //_t^A(\sigma) = 0 \text{ with } //_0^A(\sigma) = id.$$



**Theorem 9** (Loop Variable Theorem). *Suppose $A, B \in \mathcal{A}$. Then $//^A(\sigma) = //^B(\sigma)$ for all $\sigma \in \mathcal{L}$ iff $A = B^g$ for some $g \in \mathcal{G}_0$. We call the function, $A \to //^A(\sigma)$, a “**Loop variables**” on $\mathcal{A}/\mathcal{G}_0$.*

**Proof:**

- If $A = B^g$ for some $g \in \mathcal{G}_0$ and $\sigma : [0,1] \to M$ such that $\sigma(0) = o$, then
$$//^A(\sigma) = //^{B^g}(\sigma) = g(\sigma(1))^{-1} //^B(\sigma).$$
- Hence if $A = B^g$ and $\sigma \in \mathcal{L}$, then $//^A(\sigma) = //^B(\sigma)$.
- If $//^A(\sigma) = //^B(\sigma)$ for all $\sigma \in \mathcal{L}$, define $g(\sigma(1)) = //^B(\sigma) //^A(\sigma)^{-1}$ for all $\sigma : [0,1] \to M$ such that $\sigma(0) = o$.
- Then $g$ is well defined and $A = B^g$.

Q.E.D.

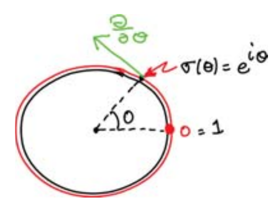
**Corollary 10.**

$$\mathbb{H}_{\textit{physical}} = \{F \in \mathbb{H} : F(A^g) = F(A)\ \forall\ A \in \mathcal{A},\ g \in \mathcal{G}\}$$
$$\text{“}\cong\text{”}\left\{F \in L^2(\mathcal{A}, \mathcal{D}A) : F = F\left(\left\{//^A(\sigma) : \sigma \in \mathcal{L}\right\}\right)\right\}.$$



# Restriction to $d = 1$ (general $K$)

$S^1 = [0,1]/(0 \sim 1) \ni \theta$ and write $\partial_\theta = \frac{\partial}{\partial\theta}$




In this case,

- $\mathcal{A} = L^2(S^1, \mathfrak{k})$, `Configuration space`
- $\mathcal{G}_0 = \{g \in H^1(S^1 \to K) : g(0) = g(1) = id \in K\}$, `Gauge Group`
- $A^g = Ad_{g^{-1}}A + g^{-1}g'$

- $\mathbb{H} =$"$L^2(\mathcal{A}, \mathcal{D}A)$"    `Raw Hilbert Space`

- $\mathbb{H}_{\text{physical}} = \{F \in \mathbb{H} : F_\phi(A) = \phi(//_1(A)),\ \phi : K \to \mathbb{C}\}$, where $//_\theta(A) \in K$ is the solution to
$$\frac{d}{d\theta}//_\theta(A) + A(\theta)//_\theta(A) = 0 \text{ with } //_0(A) = id \in K.$$
$//_1(A) \in K$ is the **holonomy** of $A$.

- $F^A \equiv 0$ when $d = 1$ and therefore, $V(A) \equiv 0$.    `No curvature in 1d`

- $H = -\frac{1}{2}\Delta_{\mathcal{A}}$    (Quantum Hamiltonian)    `Raw Hamiltonian`



# A Physics Idea

**Theorem 11** (Heuristic: c.f. Witten 1991, CMP 141.). *Suppose $K$ is simply connected and for $\phi$ let $F_\phi(A) := \phi(//_1(A))$, then*
$$\phi \in L^2(K, d\text{Haar}) \to F_\phi \in \mathbb{H}_{\text{physical}}$$
*is a "Unitary" map which intertwines $\Delta_{\mathcal{A}}$ and $\Delta_K$, i.e.*
$$\Delta_{\mathcal{A}}[\phi \circ //_1] = \Delta_{\mathcal{A}} F_\phi = F_{\Delta_K \phi} = (\Delta_K \phi) \circ //_1. \tag{6}$$

**Goal:** Give a precise meaning to the previous idea.

To do this we will "regularize" $\mathcal{D}A$ by the Gaussian measure
$$d\tilde{P}_s(A) = \frac{1}{Z_s} \exp\left(-\frac{1}{2s}|A|^2_{\mathcal{A}}\right)\mathcal{D}A$$
with the idea of letting $s \to \infty$ at the end to "recover" $\mathcal{D}A$.

The measure $\tilde{P}_s$ is a Gaussian measure living on a certain completion, $\bar{\mathcal{A}}$, of $\mathcal{A}$.



# A Realization of $\bar{\mathcal{A}}$ as $W(\mathfrak{k})$

- $W(\mathfrak{k}) := \{\omega \in C([0,1] \to \mathfrak{k}) : \omega(0) = 0\}$

- $W(K) := \{g \in C([0,1] \to K : g(0) = e \in K\}$

- $H(\mathfrak{k}) := \{h \in W(\mathfrak{k}) : \int_0^1 |h'(s)|^2 ds < \infty\}$

- Note that $\partial_\theta : H(\mathfrak{k}) \to \mathcal{A} = L^2(S^1; \mathfrak{k})$ is isometric.

- Define $\bar{\mathcal{A}} := \partial_\theta W(\mathfrak{k})$.    `Completed Connection Forms`

- $\tilde{P}_s \to P_s$ – Wiener measure on $W(\mathfrak{k})$ with variance $s$.

- $//_\theta(A) \to //_\theta(a)$ where for $a \in W(\mathfrak{k})$,
$$d//_\theta(a) + a'(\theta)//_\theta(a) = 0 \text{ with } //_0(a) = id \in K.$$

- The action of gauge group, $A \to A^g$ goes over to
$$a \to a^g_s = \int_0^s \left(g^{-1}(\sigma)\, da(\sigma)\, g(\sigma) + g^{-1}(\sigma)\, dg(\sigma)\right).$$



# Gross' Ergodicity Theorem

The following theorem is a stochastic version of the Loop Variable Theorem, item 2. of Theorem 9.

**Theorem 12** ([Gross, 1993]). *Let*
$$\mathbb{H}^s_{\textit{physical}} := \left[L^2(W(\mathfrak{k}), P_s)\right]^{\mathcal{G}_0} = \left\{F \in L^2(W(\mathfrak{k}), P_s) : F(a^g) = F(a) \text{ for } P_s \text{ a.e. } a\right\}.$$
*Then*
$$\mathbb{H}^s_{\textit{physical}} = \left\{F = f(//_1) : f \in L^2(K, p_s(x)dx)\right\}.$$
*where*
$$p_s(x)dx = P_s\text{–Law}(//_1).$$

**Remark 13.** The action, $F(a) \to F(a^g)$ is not unitary except in the limit as $s \to 0$. The unitarized action has no non-trivial fixed elements in $L^2(W(\mathfrak{k}), P_s)$, see [Driver & Hall, 2000] for a proof using the Fourier Wiener transform. Hence it would be a **BAD** idea to unitarize this action.

**Corollary 14.** *The function, $p_s$, is the convolution heat kernel on $K$. Since* $\lim_{s\to\infty} p_s(x) = 1$,
$$\lim_{s\to\infty} \mathbb{H}^s_{\textit{physical}} \cong L^2(K, dx).$$

# An Explanation for Eq. (6)

Recall Eq. (6) states $\Delta_{\mathcal{A}}[\phi \circ //_1] = (\Delta_K \phi) \circ //_1$

- If we let $S_0$ be an orthonormal basis of $H(\mathfrak{k})$ and

$$\Delta_{H(\mathfrak{k})} = \sum_{h \in S_0} \partial_h^2, \tag{7}$$

then the assertion in Eq. (6) becomes:

$$\Delta_{H(\mathfrak{k})} (\phi \circ //_1) \stackrel{?}{=} (\Delta_K \phi) \circ //_1. \tag{8}$$

**Proof:** (Heursitic explanation.)

- Use $\langle \cdot, \cdot \rangle$ on $\mathfrak{k}$ to construct a bi-invariant metric on $TK$.
- Let $H(K)$ be the space of finite energy paths on $K$ starting at $e \in K$.
- Equip $H(K)$ with the right invariant metric induced from the metric on

$$H(\mathfrak{k}) := \mathrm{Lie}(H(K)).$$



Then it is a **fact** that the "Cartan Rolling Map, $\psi : H(\mathfrak{k}) \to H(K)$ defined by

$$\psi(a) := //.(a)$$

is an isometric isomorphism of Riemannian manifolds. Consequently we may "conclude" that $\psi$ intertwines the Laplacian, $\Delta_{H(\mathfrak{k})}$ on $H(\mathfrak{k})$ with the Laplacian, $\Delta_{H(K)}$ on $H(K)$, i.e.

$$\Delta_{H(\mathfrak{k})} (f \circ \psi) = \left(\Delta_{H(K)} f\right) \circ \psi. \tag{9}$$

When $f(g) = \varphi(g(1))$, one can show

$$\Delta_{H(K)} f(g) = (\Delta_K \varphi)(g(1))$$

and therefore Eq. (9) implies,

$$\Delta_{H(\mathfrak{k})} (\phi \circ //_1) = (\Delta_K \phi) \circ //_1.$$

Q.E.D.



# Why is this Explanation not Satisfactory

- The operator $\Delta_{H(\mathfrak{k})}$ makes sense on smooth cylinder functions.
- However, $\phi \circ //_1$ is not a cylinder function.
- **Problematic Theorem:** The densely defined operator $\Delta_{H(\mathfrak{k})}$ on $L^2(W(\mathfrak{k}), P_s)$ is **not** closable.

**Proof.** Consider the case $\mathfrak{k} = \mathbb{R}$ and $s = 1$, so that $\mu = P_1$ is standard Wiener measure. Let

$$f(a) = 2 \int_0^1 a_\theta da_\theta = a_1^2 - 1$$

a cylinder function. One computes

$$\Delta_{H(\mathfrak{k})} f(a) = \sum_{h \in S_0} 2h_1^2 = 2.$$

On the other hand, we have $f(a) = \lim_{|\mathcal{P}| \to 0} f_{\mathcal{P}}(a)$ where $f_{\mathcal{P}}(a)$ is the cylinder function

$$f_{\mathcal{P}}(a) = 2 \sum_{s_i \in \mathcal{P}} a_{s_i}(a_{s_{i+1}} - a_{s_i}).$$



But

$$\Delta_{H(\mathfrak{k})} f_{\mathcal{P}}(a) = 0!$$

(Compare with the harmonic function

$$(x_1 + x_2 + \cdots + x_n) x_{n+1} \text{ on } \mathbb{R}^{n+1}.)$$

Therefore $\lim_{|\mathcal{P}| \to 0} f_{\mathcal{P}} = f$ while

$$0 = \lim_{|\mathcal{P}| \to 0} \Delta_{H(\mathfrak{k})} f_{\mathcal{P}}(a) \neq \Delta_{H(\mathfrak{k})} f = 2.$$

# Segal - Bargmann Theory

- Let $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$ be the complexification of $\mathfrak{k}$
- $K_{\mathbb{C}} =$ the complexification of $K$, e.g. $SU(2)_{\mathbb{C}} = SL(2,\mathbb{C})$.
- For $s > t/2$, let $M_{s,t}$ be the Gaussian measure on $W(\mathfrak{k}_{\mathbb{C}})$,

$$M_{s,t} = \mathrm{Law}\left(\sqrt{s-t/2}\,\alpha + i\sqrt{t/2}\,\beta\right)$$

where $\alpha$ and $\beta$ are independent standard $(\mathfrak{k}, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$ – valued Brownian motions.

**Theorem 15** (Segal- Bargmann)**.** *There exists an isometry*

$$S_t : L^2(W(\mathfrak{k}), P_s) \to L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t})$$

*such that*

$$(S_t f)(c) = \int f_{\mathbb{C}}(c+a)dP_t(a) = (e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} f)_{\mathbb{C}}(c).$$

*For all polynomial cylinder functions* $f$*. Moreover* $\mathrm{Ran}\,(S_t) =$ *closure of Holomorphic cylinder functions.*



**Proof:** Apply our generalized Segal-Bargmann theorem with

$$\mu := P_{s-t/2} = \mathrm{Law}\left(\sqrt{s-t/2}\alpha\right)$$
$$\nu := P_{t/2} = \mathrm{Law}\left(\sqrt{t/2}\beta\right)$$

so that

$$S_t = P_t * (\cdot) = \nu_2 * (\cdot)\,,$$
$$\mu \times \nu = M_{s,t}, \quad \text{and}$$
$$\mu * \nu = \mathrm{Law}\left(\sqrt{s-t/2}\alpha + \sqrt{t/2}\beta\right) = P_s.$$

Q.E.D.

**Theorem 16** (Stochastic Representation Theorem)**.** $S_t$ *is also characterized by*

$$S_t \int_{\triangle_n} \langle \alpha(\tau), da^{\otimes^n}(\tau)\rangle = \int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n}(\tau)\rangle$$

*where* $\alpha : \Delta_n \to \mathfrak{k}$ *is a deterministic function.*



**Proof:** Suppose that

$$\alpha(\tau) = 1_{J_1\times J_2\times\cdots\times J_n}(\tau)\eta$$

with $\eta \in \mathfrak{k}^{\otimes n}$ and $J_i = (s_i, t_i]$ are intervals such that $J_i < J_k$ for all $i < k$, i.e. $t_i < s_k$. Let $a\,(J_i) := a_{t_i} - a_{s_i}$ and

$$f(a) = \int_{\triangle_n} \langle \alpha(\tau), da^{\otimes^n}(\tau)\rangle = \langle \eta, a(J_1)\otimes\cdots\otimes a(J_n)\rangle$$

then

$$f_{\mathbb{C}}(c) = \int_{\triangle_n} \langle \alpha(\tau), dc^{\otimes^n}(\tau)\rangle = \langle \eta, c(J_1)\otimes\cdots\otimes c(J_n)\rangle,$$

where $c\,(J_i) := c_{t_i} - c_{s_i}$. Q.E.D.



Since $a \to \langle \eta, (c(J_1)+a(J_1))\otimes\cdots\otimes(c(J_n)+a(J_n))\rangle$ is a Harmonic polynomial of $a$;

$$\begin{aligned} S_t f\,(c) &:= \int_{W(\mathfrak{k})} f_{\mathbb{C}}(c+a)dP_t(a) \\ &= \int_{W(\mathfrak{k})} \langle \eta, (c(J_1)+a(J_1))\otimes\cdots\otimes(c(J_n)+a(J_n))\rangle dP_t(a) \\ &= \langle \eta, c(J_1)\otimes\cdots\otimes c(J_n)\rangle \\ &= f_{\mathbb{C}}(c) \end{aligned}$$

By a limiting argument one then shows in geneal that

$$S_t\left(\int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), d(c+a)^{\otimes^n}(\tau)\rangle\right) = \int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n}(\tau)\rangle.$$

# The Main Theorem

**Theorem 17** ([Gross & Malliavin, 1996, Driver & Hall, 1999]). *Let*

$$d//_\theta + da_\theta \circ //_\theta = 0 \text{ with } //_0 = Id.$$

*relative to* $P_s$ *and*

$$d//_\theta^{\mathbb{C}} + dc_\theta \circ //_\theta^{\mathbb{C}} = 0 \text{ with } //_0^{\mathbb{C}} = Id.$$

*relative to* $M_{s,t}$. *Then for all* $f \in L^2(K, dx)$,

$$S_t f(//_1) = F(//_1^{\mathbb{C}})$$

*where* $F$ *is the unique Holomorphic function on* $K_{\mathbb{C}}$ *such that*

$$F|_K = e^{\frac{t}{2}\triangle_K} f.$$

**Morally speaking:**

$$S_t H = (e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} H)_{\mathbb{C}} \in \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}))$$

$$(e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} f(//_1))_{\mathbb{C}} = (e^{\frac{t}{2}\triangle_K} f)_{\mathbb{C}}(//_1^{\mathbb{C}})$$

so on "restricting" to $W(\mathfrak{k})$

$$e^{\frac{t}{2}\triangle_{H(\mathfrak{k})}} f(//_1) = (e^{\frac{t}{2}\triangle_K} f)(//_1)$$

which we interpret as a rigorous version of the statement that

$$\triangle_{H(\mathfrak{k})}[f(//_1)] = (\triangle_K f)(//_1).$$



# The generators of $//_\theta \in K$ & $//_\theta^{\mathbb{C}} \in K_{\mathbb{C}}$

**Proposition 18.** *Let*

- $\{X_k : k = 1, \ldots, \dim \mathfrak{k}\}$ *be an orthonormal basis for* $\mathfrak{k}$
- $Y_k = JX_k$, *where* $J$ *is the complex structure on* $\mathfrak{k}_{\mathbb{C}}$.

*Then*

1. *The generator of the diffusion,* $//_\theta \in K$, *is*

$$\Delta_K = \sum X_k^2.$$

2. *The generator of the diffusion,* $//_\theta^{\mathbb{C}} \in K_{\mathbb{C}}$, *is*

$$A_{s,t} = (s - t/2) \sum X_k^2 + \frac{t}{2} \sum Y_k^2$$



# Corollary: Hall's Transform

Let $\rho_s(dx) = \mathrm{Law}(//_1)$ and $m_{s,t}(dg) = \mathrm{Law}(//_1^{\mathbb{C}})$, i.e.

$$\rho_s(x) = \left(e^{s\Delta_K/2}\delta_e\right)(x) \text{ for } x \in K \quad \&$$

$$m_{s,t}(g) = \left(e^{A_{s,t}/2}\delta_e\right)(g) \text{ for } g \in K_{\mathbb{C}}.$$

**Corollary 19** (A One Parameter family of Hall's Transforms). *The map*

$$L^2(K, \rho_s) \ni f \to \left(e^{t\Delta_K/2} f\right)_{\mathbb{C}} \in \mathcal{H}L^2(K_{\mathbb{C}}, m_{s,t})$$

*is unitary.*

This theorem interpolates between the two previous versions of Hall's transform corresponding to $s = \infty$ and $s = \frac{t}{2}$. **(END)**



# Proof Sketch of Main Theorem 17.

For the proof we will need the following notation and facts:

- $\{X_k : k = 1, \ldots, \dim \mathfrak{k}\}$ be an orthonormal basis for $\mathfrak{k}$
- $Y_k = JX_k$, where $J$ is the complex structure on $\mathfrak{k}_{\mathbb{C}}$.
- Let $\Delta_K$ be the generator of $//_\theta$, $\quad \Delta_K = \sum X_k^2$.
- Let $A_{s,t}$ be the generator of $//_\theta^{\mathbb{C}}$,

$$A_{s,t} = (s - t/2) \sum X_k^2 + \frac{t}{2} \sum Y_k^2$$

- Notice that if $\Phi$ is a holomorphic function, then $Y_k \Phi = i X_k \Phi$ so that

$$A_{s,t}\Phi = (s - t)\Delta_K \Phi.$$

- The $X_k$ and $Y_k$ commute with $\Delta_K$.

**Proof.** (Proof of Main Theorem 17.) Let $\Phi = \left(e^{t\Delta_K/2} f\right)_{\mathbb{C}}$ denote the analytic continuation of $e^{t\Delta_K/2} f$ to $K_{\mathbb{C}}$. Using $[\Delta_K, X_k] = 0$ and the **Veretennikov and Krylov** formula,

$$f(//_1) = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle \alpha_n, da^{\otimes n}(\tau) \rangle$$

where $\alpha_n = \left(D^n e^{s\Delta_K/2} f\right)(e)$. Therefore

$$S_t\left[f(//_1)\right] = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle (\alpha_n)_{\mathbb{C}}, dc^{\otimes n}(\tau) \rangle.$$

Similarly,

$$\Phi(//_1^{\mathbb{C}}) = \sum_{n=0}^{\infty} \int_{\Delta_n} \langle \beta_n, dc^{\otimes n}(\tau) \rangle$$



where

$$\begin{aligned} \beta_n &= \left(D^n e^{A_{s,t}/2} \Phi\right)(e) = \left(D^n e^{(s-t)\Delta_K/2} \Phi\right)(e) \\ &= \left(D^n e^{(s-t)\Delta_K/2} \left(e^{t\Delta_K/2} f\right)_{\mathbb{C}}\right)(e) \\ &= \left[D^n \left(e^{s\Delta_K/2} f\right)_{\mathbb{C}}(e)\right]_{\mathbb{C}} = (\alpha_n)_{\mathbb{C}}. \end{aligned}$$

This shows,

$$S_t\left[f(//_1)\right] = \Phi(//_1^{\mathbb{C}}) = \left(e^{t\Delta_K/2} f\right)_{\mathbb{C}}(//_1^{\mathbb{C}})$$

as was to be shown.

**Remark 20.** See Dimock 1996, and Landsman and Wren ( $\cong 1998$) for other approaches to “canonical quantization” of $YM_2$.

(**END NOW FOR SURE!**)



# Related and Further Reading

Here are some references of related along with some more recent developments.

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# Extension to the Path Space

Now let $\tilde{\rho}_t := //_* P_t =$ the law of $//$ relative to $P_t$.

**Theorem 21.** *There exists an isometry*

$$B_t : L^2(W(K), \mathrm{Law}(//)) \to L^2(W(K_{\mathbb{C}}), \mathrm{Law}(//^{\mathbb{C}}))$$

*such that for all cylinder functions $f \in L^2(W(K), P_s)$, $B_t f$ is a Holomorphic cylinder function on $W(K_{\mathbb{C}})$ such that*

$$(B_t f)(y) = \text{“}\left(e^{\frac{t}{2}\Delta_{H(K)}} f\right)(y)\text{”} = \int f(xy)\tilde{\rho}_t(dx) \; \forall \, y \in H(K).$$

*Moreover, Ran $(B_t))$ is the closure of the holomorphic cylinder functions and the following diagram commutes*

$$\begin{array}{ccc} L^2(W(\mathfrak{k}), P_s) & \xrightarrow{S_t} & \mathcal{H}L^2(W(\mathfrak{k}_{\mathbb{C}}), M_{s,t}) \\ \updownarrow // & \circlearrowleft & \updownarrow //^{\mathbb{C}} \\ L^2(W(K), //_* P_s) & \xrightarrow{B_t} & \mathcal{H}L^2(W(K_{\mathbb{C}}), //_*^{\mathbb{C}} M_{s,t}) \end{array}$$

*i.e.*

$$S_t(f \circ //) = (B_t f) \circ //^{\mathbb{C}}$$





# Path Space Result Explanation

**Proof.** (An explanation rather than a proof.) Let $y \in H(K)$ and consider

$$\int_{W(K)} f(xy)\tilde{\rho}_t(dx) = \int_{W(\mathfrak{k})} f(//(a) \cdot y) P_t(da).$$

Notice that

$$//^{-1}(z) = \int_0^{\cdot} z^{-1}\delta z. \quad \text{(Inverse of the Itô Map.)}$$

so that

$$\begin{aligned} //^{-1}\left(//(a)y\right) &= \int \left(//(a)y\right)^{-1} \delta\left(//(a)y\right) \\ &= \int y^{-1}//(a)^{-1}\delta\left(//(a)y\right) \\ &= \int Ad_{y^{-1}}\delta a + \int y^{-1}\delta y \\ &= \int Ad_{y^{-1}}\delta a + //^{-1}(y). \end{aligned}$$





Therefore,

$$//(a)y = //\left(\int Ad_{y^{-1}}da + //^{-1}(y)\right).$$

Noting that

$$\text{Law}\left(\int Ad_{y^{-1}}da\right) = \text{Law}\,(a)\,,$$

we learn that

$$\begin{aligned} (B_t f)(y) &= \int_{W(K)} f(xy)\tilde{\rho}_t(dx) = \int_{W(\mathfrak{k})} f(//(a) \cdot y) P_t(da) \\ &= \int_{W(\mathfrak{k})} f(//\left(\int Ad_{y^{-1}}da + //^{-1}(y)\right)) P_t(da) \\ &= \int_{W(\mathfrak{k})} f(//\left(a + //^{-1}(y)\right)) P_t(da) \\ &= S_t\left(f \circ //\right)\left(//^{-1}(y)\right). \end{aligned}$$

Now replace $y \to //\,(a)$ in the above identity to find

$$(B_t f)(//\,(a)) = S_t\left(f \circ //\right)(a)\,,$$

i.e.

$$(B_t f) \circ // = S_t\left(f \circ //\right).$$





# Isometry Property

By the way one checks the isometry property from this result as follows. On one hand

$$\int_{W(\mathfrak{k})} \left| \int_{\triangle_n} \langle \alpha(\tau), da^{\otimes^n}(\tau) \rangle \right|^2 dP_s(a) = s^n \int_{\triangle_n} |\alpha(\tau)|^2 d\tau,$$

while on the other

$$\int_{W(\mathfrak{k}_{\mathbb{C}})} \left| \int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n} \rangle \right|^2 dM_{s,t}(c) = s^n \int_{\triangle_n} |\alpha(\tau)|^2 d\tau.$$

To prove this last assertion, consider the expectation of the stochastic integral:

$$\begin{aligned}\mathbb{E}\left|\int_\alpha^\beta f(\tau)dc(\tau)\right|^2 &= \mathbb{E}\left|\int_\alpha^\beta f(\tau)da(\tau) + i\, f(\tau)db(\tau)\right|^2 \\ &= \mathbb{E}\int_\alpha^\beta |f(\tau)|^2 d\tau\,(s - \frac{t}{2}) + \frac{t}{2}\mathbb{E}\int_\alpha^\beta |f(\tau)|^2 d\tau \\ &= s\int_\alpha^\beta |f(\tau)|^2 d\tau,\end{aligned}$$

where $f(\tau)$ is assumed to be adapted. Hence the result follows by writing





$\int_{\triangle_n} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^n}\rangle$ as an iterated integral. For example if $n = 2$,

$$\begin{aligned}&\mathbb{E}\left|\int_{\triangle_2} \langle \alpha_{\mathbb{C}}(\tau), dc^{\otimes^2}\rangle\right|^2 \\ &= \mathbb{E}\left|\int_{0\le\tau_1\le\tau_2\le 1} \langle \alpha_{\mathbb{C}}(\tau_1,\tau_2), dc(\tau_1)\otimes dc(\tau)\rangle\right|^2 \\ &= \int_0^1 d\tau_2 \sum_\xi s\mathbb{E}\int_{0\le\tau_1\le\tau_2} |\langle \alpha_{\mathbb{C}}(\tau_1,\tau_2), dc(\tau_1)\otimes\xi\rangle|^2 \\ &= \int_0^1 d\tau_2 \sum_{\xi,\eta} s^2 \int_{0\le\tau_1\le\tau_2} d\tau_1\, |\langle \alpha_{\mathbb{C}}(\tau_1,\tau_2), \eta\otimes\xi\rangle|^2 \\ &= s^2\int_{0\le\tau_1\le\tau_2\le 1} |\alpha_{\mathbb{C}}(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2 \\ &= s^2\int_{\triangle_2} |\alpha_{\mathbb{C}}(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2\end{aligned}$$

where $\xi$ and $\eta$ in the above expression is running over an orthonormal basis of $\mathfrak{k}$.





# A gradient computation

We would like to compute the gradient of $V(A)$ where

$$V(A) := \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j<k\le d} |F^A_{j,k}(x)|^2 dx.$$

To this end, we recall that $F^A_{j,k}(x) = \partial_j A_k - \partial_k A_j + [A_j, A_k]$ and therefore,

$$\begin{aligned}\partial_B F^A_{j,k}(x) &= \partial_j B_k - \partial_k B_j + [B_j, A_k] + [A_j, B_k] \\ &= \nabla^A_j B_k - \nabla^A_k B_j\end{aligned}$$





and hence

$$\begin{aligned}\partial_B V(A) &= \int_{\mathbb{R}^d} \sum_{1\le j<k\le d} \left\langle F^A_{j,k}(x), \partial_B F^A_{j,k}(x)\right\rangle dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left\langle F^A_{j,k}(x), \partial_B F^A_{j,k}(x)\right\rangle dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left\langle F^A_{j,k}, \nabla^A_j B_k - \nabla^A_k B_j\right\rangle dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left(\left\langle -\nabla^A_j F^A_{j,k}, B_k\right\rangle + \left\langle \nabla^A_k F^A_{j,k}, B_j\right\rangle\right) dx \\ &= -\int_{\mathbb{R}^d} \sum_{1\le j,k\le d} \left\langle \nabla^A_j F^A_{j,k}, B_k\right\rangle dx.\end{aligned}$$

Therefore we learn that

$$[\mathrm{grad} V(A)]_k(x) = -\sum_{j=1}^d \nabla^A_j F^A_{j,k}$$

as claimed.