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Path Integrals on a Compact Manifold with Non-negative Curvature

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by

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The dissertation of Poon Chuan Adrian Lim is approved, and it is acceptable in quality and form for publication on microfilm:

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To Corraine, my family and friends

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ABSTRACT OF THE DISSERTATION

Path Integrals on a Compact Manifold with Non-negative Curvature

by

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Professor Bruce K. Driver, Chair

A typical path integral on a manifold, M is an informal expression of the form

$$\frac{1}{Z}\int_{\sigma\in H(M)} f(\sigma)e^{-E(\sigma)}\mathcal{D}\sigma,$$

where $H(M)$ is a Hilbert manifold of paths with energy $E(\sigma)<\infty$, f is a real valued function on $H(M)$, $\mathcal{D}\sigma$ is a "Lebesgue measure " and Z is a normalization constant. For a compact Riemannian manifold M, we wish to interpret $\mathcal{D}\sigma$ as a Riemannian "volume form " over $H(M)$, equipped with its natural G^1 metric. Given an equally spaced partition, $\mathcal{P}$ of $[0,1]$, let $H_{\mathcal{P}}(M)$ be the finite dimensional Riemannian submanifold of $H\left(M\right)$ consisting of piecewise geodesic paths adapted to $\mathcal{P}$. Under certain curvature restrictions on M, it is shown that

$$\frac{1}{Z_{\mathcal{P}}}e^{-\frac{1}{2}E(\sigma)}dVol_{H_{\mathcal{P}}}(\sigma)\to\rho(\sigma)d\nu(\sigma)\text{ as mesh}(\mathcal{P})\to 0,$$

where $Z_{\mathcal{P}}$ is a "normalization" constant, $E:H\left(M\right)\to[0,\infty)$ is the energy functional, $Vol_{H_{\mathcal{P}}}$ is the Riemannian volume measure on $H_{\mathcal{P}}\left(M\right)$, ν is Wiener measure on continuous paths in M, and ρ is a certain density determined by the curvature tensor of M.

1

Introduction

Suppose we have a Riemannian manifold $(M,\ g)$ of dimension d with metric g. We will only consider M to be compact or $\mathbb{R}^d$. Fix a point o on the manifold M and let $V : M \to \mathbb{R}$ be a potential function. In classical mechanics, the path $\sigma : [0,T] \to M,\ \sigma(0) = o$, subject to the potential V, can be obtained by solving Newton's equation of motion

$$\frac{\nabla}{dt}\sigma'(t) = -\mathrm{grad}\ V(\sigma(t)),$$

where given any vector field $X(s)$ on $\sigma(s)$, define

$$\frac{\nabla X(s)}{ds} := //_s(\sigma)\frac{d}{ds}\left\{//_s^{-1}(\sigma)X(s)\right\}, \tag{1.1}$$

and $//_s(\sigma) : T_oM \to T_{\sigma(s)}M$ is parallel translation along σ relative to the Levi Civita covariant derivative ∇. The Hamiltonian of the system, H is then given by

$$H(\sigma(t),\sigma'(t)) = \frac{1}{2} \parallel \sigma'(t) \parallel^2 + V(\sigma(t)),$$

where $\parallel v \parallel^2 := g(v,v)$ and mass is set to be 1.

In Quantum Mechanics, observables are no longer functions, but rather Hermitian operators on some Hilbert space. Let $q = (q_1, q_2, \ldots q_d)$ be the cartesian coordinates on $\mathbb{R}^d$ and p_i be the momentum corresponding to q_i. In canonical quantization on $\mathbb{R}^d$, the quantum mechanical operator $\widehat{H}$ corresponding to the classical Hamiltonian, $H = \frac{1}{2}\sum_i p_i^2 + V(q)$, is given by

$$\widehat{H} = -\frac{\hbar^2}{2}\sum_i \frac{\partial^2}{\partial q_i^2} + M_V$$

where $\hbar = \frac{h}{2\pi}$, h is Planck's constant and M_V is multiplication operator by V.

However, on a manifold, one aims to quantize the Hamiltonian $H = \frac{1}{2}g^{ij}(q)p_i p_j + V(q)$ where q is some coordinate system on M. Using the "Feynman's (Kac) path integral prescription ", one defines the operator $e^{-T\hat{H}}$ via an integral

$$(e^{-T\widehat{H}}f)(o) := \text{“}\frac{1}{Z_T}\int_{H_T(M)} e^{-\int_0^T H(\sigma(t),\sigma'(t))\,dt} f(\sigma(T))\ \mathcal{D}\sigma,\text{”} \tag{1.2}$$

where $H(\sigma(t), \sigma'(t)) = \frac{1}{2} \parallel \sigma'(t) \parallel^2 + V(\sigma(t))$ is the classical Hamiltonian. $H_T(M)$ is the space of finite energy paths, Z_T is some normalization constant and $\mathcal{D}\sigma$ is to be interpreted as a " Lebesque type measure". The operator $\widehat{H}$ can then be obtained by differentiating the operator $e^{-T\hat{H}}$ with respect to T at 0.

Without loss of generality, assume that $T = 1$ and $V = 0$.

Definition 1.1. *Define $H(M)$, a Hilbert manifold of absolutely continuous paths with finite energy,*

$$H(M) = \{\sigma : [0,1] \to M |\ \sigma(0) = o \in M \text{ and } E(\sigma) < \infty\} \tag{1.3}$$

where the energy E is given by

$$E(\sigma) := \int_0^1 g(\sigma'(s),\ \sigma'(s))\ ds. \tag{1.4}$$

The tangent space $T_\sigma H(M)$ to $H(M)$ at σ may be identified with the space of absolutely continuous vector fields along σ. On this Hilbert manifold $H(M)$, we can define a metric G^1 as follows.

Definition 1.2. *Let $T_\sigma H(M)$ be the space of absolutely continuous vector fields X along σ* (i.e. $X(s) \in T_{\sigma(s)}M\ \forall s \in [0,1]$) *such that $G^1(X,X) < \infty$ where*

$$G^1(X,\ X) := \int_0^1 g\left(\frac{\nabla X(s)}{ds},\ \frac{\nabla X(s)}{ds}\right)\ ds, \tag{1.5}$$

and $\frac{\nabla}{ds}$ is defined as in Equation (1.1).

See [22, 23, 26, 32, 37] for more details. By polarization, Equation (1.5) defines a Riemannian metric on $H(M)$.

The integral over $H(M)$, defined in Equation (1.2) however, is highly heuristic. Firstly, the normalization constant Z_1 maybe interpreted to be 0 or ∞. Secondly, $\mathcal{D}\sigma$ which is to be interpreted as "Lebesgue measure", fails to exist in an infinite dimensional space.

We would like to make sense out of the RHS of Equation (1.2), by writing it as a limit of a sequence of integrals over finite dimensional spaces $H_{\mathcal{P}}(M)$.

Definition 1.3. *Let*

$$\mathcal{P} = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\} \tag{1.6}$$

be a partition of $[0, 1]$. *Define* $H_{\mathcal{P}}(M)$ *as a set of piecewise geodesics paths in* $H(M)$ *which changes directions only at the partition points.*

$$H_{\mathcal{P}}(M) = \left\{ \sigma \in H(M) \cap C^2(I \backslash \mathcal{P}) \;\middle|\; \frac{\nabla \sigma'(s)}{ds} = 0 \; for \; s \notin \mathcal{P} \right\}.$$

It will be shown later that $H_{\mathcal{P}}(M)$ is a finite dimensional submanifold of $H(M)$. In fact, $H_{\mathcal{P}}(M)$ is diffeomorphic to $(\mathbb{R}^d)^n$. For $\sigma \in H_{\mathcal{P}}(M)$, the tangent space $T_\sigma H_{\mathcal{P}}(M)$ can be identified with elements $X \in T_\sigma H_{\mathcal{P}}(M)$ satisfying the Jacobi equations on $I \backslash \mathcal{P}$. As a submanifold of $H(M)$, $H_{\mathcal{P}}(M)$ inherits the induced metric $G^1|_{TH_{\mathcal{P}}(M)}$ by restricting the G^1 metric on $H_{\mathcal{P}}(M)$.

If N^p is any manifold with a metric G, define a volume density Vol_G on $T_n N$ by

$$Vol_G\,(v_1, v_2, \ldots, v_p) = \sqrt{\det\, \{G(v_i, v_j)\}_{i,j=1}^p} \tag{1.7}$$

where $\{v_1, v_2, \ldots, v_p\} \subset T_n N$ is a basis and $n \in N$.

Theorem 1.4. *Given a density of the form* ρVol_G, *where* $\rho : N \to [0, \infty)$, *there exists a unique measure* m_G *on* N *such that*

$$\int_{\mathcal{D}(y)} f \; dm_G = \int_{\mathcal{D}(y)} f\rho \; Vol_G \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots, \frac{\partial}{\partial y_p} \right) dy_1 \ldots dy_p$$

for any local coordinates $y = (y_1, \ldots, y_p) : \mathcal{D}(y) \to \mathbb{R}^p$ *and measurable function* $f : N \to [0, \infty)$. *If* $\rho = 1$, *the associated measure will be called Riemann volume measure.*

Definition 1.5. *Let $Vol_{\mathcal{P}}$ denote the density on $H_{\mathcal{P}}(M)$ determined by $G^1|_{TH_{\mathcal{P}}(M)\otimes TH_{\mathcal{P}}(M)}$ using Equation (1.7).*

Given the above definition, we can now define a measure on $H_{\mathcal{P}}(M)$.

Definition 1.6. *For each partition $\mathcal{P}$ of $[0,1]$ as in Equation (1.6), let $\nu_{\mathcal{P}}$ denote the unique measure on $H_{\mathcal{P}}(M)$ as in Theorem 1.4, defined by the density*

$$\frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E} Vol_{\mathcal{P}}$$

where $E : H(M) \to [0,\infty)$ is the energy functional defined in Equation (1.4) and $Z^1_{\mathcal{P}}$ is a normalization constant given by

$$Z^1_{\mathcal{P}} = (2\pi)^{\frac{dn}{2}}. \tag{1.8}$$

We can now write the RHS of Equation (1.2) as a limit of a sequence of integrals over the finite dimensional space $H_{\mathcal{P}}(M)$, equipped with the measure $\nu_{\mathcal{P}}$. Our result shows that this limit can be written as an integral over the Wiener space of M, with Wiener measure ν. (See Definition 1.8 below.)

Definition 1.7. *Define $\Delta_i s = s_i - s_{i-1}$ and $|\mathcal{P}| = \bigvee_{i=1,\ldots,n} \Delta_i s = \max\{\Delta_i s : i = 1, 2, \ldots, n\}$ be the norm of the partition and $J_i := (s_{i-1}, s_i]$ for $i = 1, 2, \ldots, n$.*

Let $\triangle = \mathrm{tr}\, \nabla^2$ denote the Laplacian acting on $C^\infty(M)$ and $p_s(x,y)$ be the fundamental solution to the heat equation,

$$\frac{\partial u}{\partial s} = \frac{1}{2}\triangle u.$$

In the case when $M = \mathbb{R}^d$,

$$p_s(x,y) = \left(\frac{1}{2\pi s}\right)^{\frac{d}{2}} e^{-\frac{1}{2s}\|x-y\|^2}.$$

Definition 1.8. *The Wiener space $W(M)$ is the path space*

$$W(M) = \{\sigma :\ [0,1] \to M : \sigma(0) = o \text{ and } \sigma \text{ is continuous}\}.$$

The Wiener measure ν associated to $(M,\ g,\ o)$ is the unique probability measure on $W(M)$ such that

$$\int_{W(M)} f(\sigma)\ d\nu(\sigma) = \int_{M^n} F(x_1, \ldots, x_n)\ \prod_{i=1}^{n} p_{\Delta_i s}(x_{i-1},\ x_i)\ dm(x_1)\cdots dm(x_n) \quad (1.9)$$

for all functions f of the form $f(\sigma) = F(\sigma(s_1), \ldots, \sigma(s_n))$, for all $\mathcal{P}$, a partition of $[0,1]$ as in Equation (1.6) and $F:\ M^n \to \mathbb{R}$ is a bounded measurable function. In Equation (1.9), $dm(x)$ denotes the Riemann volume measure on M as in Theorem 1.4 and by convention $x_0 := o$.

It is known that there exists a unique probability measure ν on $W(M)$ satisfying Equation (1.9). The measure ν is concentrated on continuous but nowhere differentiable paths.

Notation 1.9. *When $M = \mathbb{R}^d$, $g(\cdot,\cdot)$ is the usual dot product and $o = 0$, the measure ν defined in Definition 1.8 is the standard Wiener measure on $W(\mathbb{R}^d)$. We will denote this standard Wiener measure by μ rather than ν. We will also let $b(s) : W(\mathbb{R}^d) \to \mathbb{R}^d$ be the coordinate map such that*

$$b(s)(\omega) := \omega(s)$$

for all $\omega \in W(\mathbb{R}^d)$.

Remark 1.10. *The process $\{b(s)\}_{s\in[0,1]}$ is a standard $\mathbb{R}^d$-valued Brownian motion on the probability space $(W(\mathbb{R}^d), \mu)$.*

Suppose we now view M as an imbedded submanifold of $\mathbb{R}^d$ with the induced Riemannian structure. Let $P(m)$ be the projection on the tangent space T_mM and $v \in T_mM$. Then for a vector field $X(m)$,

$$\nabla_v X = P(\sigma(0))\frac{d}{dt}X(\sigma(t))\Big|_{t=0}$$

where σ is a path in M such that $\sigma(0) = m$ and $\sigma'(0) = v$. Define a projection Q on the orthogonal complement of T_mM by $Q = I - P$, where I is the identity. With this definition, for any vector $v \in T_mM$, one can define parallel translation along σ by $//_s(\sigma)v := w(s)v$ where w solves the following differential equation

$$w'(t) + dQ(\sigma'(t))w(t) = 0,\ w(0) = P(\sigma(0)).$$

Theorem 1.11. *Let Σ be an M-valued semi-martingale and $V_0(m)$ be a measurable vector field on M, then there is a unique parallel TM-valued semi-martingale V such that $V_0 = V_0(\Sigma_0)$ and $V_s \in T_{\Sigma_s}M$ for all s. More explicitly, $V_s = w_s V(\Sigma_0)$ where w solves the following Stratonovich stochastic differential equation*

$$\delta w + dQ(\delta\Sigma)w = 0, \;\; w_0 = P(\Sigma_0). \tag{1.10}$$

For a proof of this theorem, the reader should refer to [40]. Thus we can now define a "stochastic" extension of parallel translation.

Definition 1.12. (Stochastic Parallel Translation) *Given $v \in T_{\Sigma_0}M$ and M-valued semi-martingale Σ, define stochastic parallel translation $\widetilde{//}$ by*

$$\widetilde{//}_s v := w_s v$$

where w solves Equation (1.10). This is going to be used for the particular semi-martingale $\Sigma_s(\sigma) := \sigma(s)$ on $(W(M), \nu)$.

Definition 1.13. *The curvature tensor R of ∇ is*

$$R(X,\ Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,\ Y]} Z$$

for all vector fields X, Y and Z on M. The sectional curvature $S(V)$ where $V \subseteq T_m M$ with $dim(V) = 2$, is defined by

$$S(V) = \frac{g(R(u,v)u,v)}{\| u \|^2 \| v \|^2 - g(u,v)^2},$$

where $\{u, v\}$ is a basis for V. It can be shown that this definition is independent of the basis used. Let $\{e_i\}_{i=1}^d \subseteq T_m M$ be an orthornormal frame at $m \in M$. The Ricci tensor of $(M,\ g)$ is $Ric\ v = \sum_{i=1}^d R(v,\ e_i)e_i$, the scalar curvature $Scal$ is $Scal = \sum_{i=1}^d g(Ric\ e_i,\ e_i)$ and Γ_m is given by

$$\Gamma_m = \sum_{i,j=1}^{d} \Big(R(e_i, R(e_i, \cdot)e_j)e_j + R(e_i, R(e_j, \cdot)e_i)e_j + R(e_i, R(e_j, \cdot)e_j)e_i \Big).$$

Define for any $\sigma \in W(M)$, $K_\sigma : L^2([0,1] \to T_o M) \longrightarrow L^2([0,1] \to T_o M)$ by

$$(K_\sigma f)(s) = \int_0^1 (s \wedge t)\ \widetilde{//}_t^{-1}(\sigma) \left(\Gamma_{\sigma(t)} \widetilde{//}_t(\sigma) f(t) \right)\ dt$$

where $\widetilde{//}$ is stochastic parallel translation.

We can now state the main result.

Theorem 1.14. *Let $(M,\ g)$ be a compact Riemannian manifold with dimension d. Let $\mathcal{P} = \left\{0,\ \frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n-1}{n},\ 1\right\}$ be an equally spaced partition. Suppose that $f:\ W(M) \to \mathbb{R}$ is bounded and continuous and that $0 \leq S < \frac{3}{17d}$, then*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\ d\nu_{\mathcal{P}}(\sigma)$$
$$= \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 Scal(\sigma(s))\ ds} \sqrt{\det\left(I + \frac{1}{12}K_\sigma\right)}\ d\nu(\sigma). \tag{1.11}$$

1.1 Known Results

Using $H_{\mathcal{P}}(M)$ to approximate the Wiener space $W(M)$ was done in [1]. However, the choice of metrics used on $TH_{\mathcal{P}}(M)$ in [1] are different from $G^1|_{TH_{\mathcal{P}}(M)}$.

Definition 1.15. *For each partition $\mathcal{P}$ of $[0,1]$ as in Equation (1.6), let $G^1_{\mathcal{P}}$, $G^0_{\mathcal{P}}$ be the metrics on $TH_{\mathcal{P}}(M)$, given by*

$$G^1_{\mathcal{P}}(X,Y) = \sum_{i=1}^{n} g\left(\ \frac{\nabla X(s_{i-1}+)}{ds},\ \frac{\nabla Y(s_{i-1}+)}{ds}\ \right)\ \Delta_i s,$$
$$G^0_{\mathcal{P}}(X,Y) = \sum_{i=1}^{n} g\left(\ X(s_{i-1}+),\ Y(s_{i-1}+)\ \right)\ \Delta_i s.$$

for all $X, Y \in T_\sigma H_{\mathcal{P}}(M)$ and $\sigma \in H_{\mathcal{P}}(M)$. Note that $\frac{\nabla X(s_{i-1}+)}{ds} = \lim_{s\to s_{i-1}+}\ \frac{\nabla X(s)}{ds}$.

Observe that $G^1_{\mathcal{P}}$ is some sort of Riemann sum approximation to G^1.

Definition 1.16. *For each partition $\mathcal{P}$ of $[0,1]$ as in Equation (1.6), define unique measures $\nu_{G^1_{\mathcal{P}}}$ and $\nu_{G^0_{\mathcal{P}}}$ on $H_{\mathcal{P}}(M)$, as in Theorem 1.4, defined by densities*

$$\frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E} Vol_{G^1_{\mathcal{P}}} \text{ and } \frac{1}{Z^0_{\mathcal{P}}} e^{-\frac{1}{2}E} Vol_{G^0_{\mathcal{P}}}$$

respectively, where $E : H(M) \to [0,\infty)$ is the energy functional defined in Equation (1.4) and $Vol_{G^1_{\mathcal{P}}}$ and $Vol_{G^0_{\mathcal{P}}}$ are determined by $G^1_{\mathcal{P}}$ and $G^0_{\mathcal{P}}$ respectively using Equation (1.7) . $Z^1_{\mathcal{P}}$ is a normalization constant given by Equation (1.8) and

$$Z^0_{\mathcal{P}} = \prod_{i=1}^{n} \left(\sqrt{2\pi}\,(s_i - s_{i-1})\right)^d.$$

The following theorem was proved in [1].

Theorem 1.17. *Let M be a compact Riemannian manifold. Suppose that $f : W(M) \to \mathbb{R}$ is bounded and continuous, then*

$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\ d\nu_{G^1_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma)\ d\nu(\sigma) \ \textit{and}$$
$$\lim_{|\mathcal{P}|\to 0} \int_{H_{\mathcal{P}}(M)} f(\sigma)\ d\nu_{G^0_{\mathcal{P}}}(\sigma) = \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 Scal(\sigma(s))ds}\ d\nu(\sigma).$$

Using the Feynman-Kac formula, $\widehat{H}$ in Equation (1.2) is given by

$$\widehat{H} = -\frac{1}{2}\Delta + \kappa Scal$$

where Δ is the Laplacian operator and $\kappa = 0, \frac{1}{6}$ for the $G^1_{\mathcal{P}}$ and $G^0_{\mathcal{P}}$ metrics respectively. However, the integral in Equation (1.11) is not of the Feynman-Kac form. Hence the interpretation of the operator $\widehat{H}$ corresponding to this integral is unclear at this stage.

2

Finite Dimensional Approximations

A detailed account of this section is given in [1].

Let $\pi : O(M) \to M$ denote the bundle of orthogonal frames on M. An element $u \in O(M)$ is an isometry

$$u : \mathbb{R}^d \to T_{\pi(u)}M.$$

Fix $u_o \in \pi^{-1}(o)$, which identifies T_oM of M at o with $\mathbb{R}^d$.

Define θ, a $\mathbb{R}^d$-valued form on $O(M)$ by $\theta_u(\xi) = u^{-1}\pi_*\xi$ for all $u \in O(M)$, $\xi \in T_uO(M)$ and let ϑ be the $so(d)$-valued connection form on $O(M)$ defined by ∇. Explicitly, if $s \to u(s)$ is a smooth path in $O(M)$ then

$$\vartheta(u'(0)) = u(0)^{-1}\frac{\nabla u(s)}{ds}\Big|_{s=0}$$

where $\frac{\nabla}{ds}$ is defined as in Equation (1.1) with X replaced by u. We define the horizontal lift $\mathcal{H}_u : T_{\pi(u)}M \to T_uO(M)$ by

$$\theta\mathcal{H}_u u = id_{\mathbb{R}^d},\ \vartheta_u\mathcal{H}_u = 0.$$

Explicitly, for $v \in T_{\pi(u)}M$, $\mathcal{H}_u v = \frac{d}{dt}|_{t=0}//_t(\sigma)u$ where $\dot{\sigma}(0) = v$.

Definition 2.1. *For $a,\ c \in \mathbb{R}^d$, let*

$$\Omega_u(a,c) := u^{-1}R(ua,uc)u.$$

Let $H(O(M))$ be the set of finite energy paths $u : [0,1] \to O(M)$ as defined in Equation (1.3) with M replaced by $O(M)$ and o by u_o. For $\sigma \in H(M)$, let u be defined by the ordinary differential equation

$$u'(s) = \mathcal{H}_{u(s)}\sigma'(s),\ u(0) = u_o.$$

This equation implies that $\vartheta(u'(s)) = 0$ or that $\frac{\nabla u(s)}{ds} = 0$. Thus we have

$$u(s) = //_s(\sigma)u_o$$

where $//_s(\sigma)$ is parallel translation along σ. Since u_o is fixed, we will drop it and write $u(s) = //_s$. We will call $u(s)$ a horizontal lift of σ starting at u_o and use it to define ϕ, which associates $\omega \in H(\mathbb{R}^d)$ with a path $\sigma \in H(M)$.

Definition 2.2. (Cartan's Development Map) *The development map, $\phi : H(\mathbb{R}^d) \to H(M)$ is defined for $\omega \in H(\mathbb{R}^d)$ by $\phi(\omega) = \sigma \in H(M)$ where σ solves the functional differential equation*

$$\sigma'(s) = //_s(\sigma)\omega'(s),\ \sigma(0) = o. \tag{2.1}$$

The development map, ϕ is smooth and injective. We can define an anti-development map, $\phi^{-1} : H(M) \to H(\mathbb{R}^d)$ given by $\omega = \phi^{-1}(\sigma)$ where

$$\omega(s) = \int_0^s //_r^{-1}(\sigma)\sigma'(r)\ dr.$$

Again, ϕ^{-1} is smooth and is injective. Therefore, $\phi : H(\mathbb{R}^d) \to H(M)$ is a diffeomorphism of infinite dimensional Hilbert manifolds.

Definition 2.3. *For each $h \in C^\infty(H(M) \to H(\mathbb{R}^d))$ and $\sigma \in H(M)$, let $X^h(\sigma) \in T_\sigma H(M)$ be given by*

$$X_s^h(\sigma) := //_s(\sigma)h_s(\sigma) \tag{2.2}$$

for all $s \in [0,1]$, where we have written $h_s(\sigma)$ as $h(\sigma)(s)$.

Define $H_\mathcal{P}(\mathbb{R}^d) = \{\omega \in H \cap C^2(I\backslash\mathcal{P}) \mid \omega''(s) = 0\ for\ s \notin \mathcal{P}\}$, the set of piecewise linear paths in $H(\mathbb{R}^d)$, which changes directions only at the partition points.

Remark 2.4. *The development map $\phi : H(\mathbb{R}^d) \to H(M)$ has the property that $\phi\left(H_{\mathcal{P}}(\mathbb{R}^d)\right) = H_{\mathcal{P}}(M)$ where $H_{\mathcal{P}}(M)$ has been defined in Definition 1.3. If $\sigma = \phi(\omega)$ with $\omega \in H_{\mathcal{P}}(\mathbb{R}^d)$, then differentiating Equation (2.1) gives*

$$\frac{\nabla \sigma'(s)}{ds} = \frac{\nabla}{ds}\Big(//_s(\sigma)\omega'(s)\Big) = //_s(\sigma)\omega''(s) = 0 \ for \ all \ s \notin \mathcal{P}.$$

Because $\phi : H(\mathbb{R}^d) \to H(M)$ is a diffeomorphism and $H_{\mathcal{P}}(\mathbb{R}^d) \subset H(\mathbb{R}^d)$ is an embedded submanifold, so it follows that $H_{\mathcal{P}}(M)$ is an embedded submanifold of $H(M)$. Therefore for each $\sigma \in H_{\mathcal{P}}(M)$, $T_\sigma H_{\mathcal{P}}(M)$ may be viewed as a subspace of $T_\sigma H(M)$. The next proposition identifies this subspace. See [1] for a proof.

Proposition 2.5. *Let $\sigma \in H_{\mathcal{P}}(M)$, then $X \in T_\sigma H(M)$ is in $T_\sigma H_{\mathcal{P}}(M)$ if and only if*

$$\frac{\nabla^2}{ds^2} X(s) = R(\sigma'(s),\ X(s))\sigma'(s) \tag{2.3}$$

on $I \backslash \mathcal{P}$. Equivalently, letting $\omega = \phi^{-1}(\sigma)$, $u = //(\sigma)$ and $h \in H(\mathbb{R}^d)$, then $X^h \in T_\sigma H(M)$ defined in Equation (2.2) is in $T_\sigma H_{\mathcal{P}}(M)$ if and only if

$$h''(s) = \Omega_{u(s)}(\omega'(s), h(s))\omega'(s) \tag{2.4}$$

on $I \backslash \mathcal{P}$.

2.1 Comparing $\nu_{\mathcal{P}}$ and $\nu_{G^1_{\mathcal{P}}}$ on M

Definition 2.6. *For $\omega \in H_{\mathcal{P}}(\mathbb{R}^d)$, let $\{h_{k,a}\}_{\substack{k=1,2,\dots,n \\ a=1,2,\dots,d}}$ be any basis in $\phi_*^{-1}(T_{\phi(\omega)} H_{\mathcal{P}}(M)) = H_{\mathcal{P}}(T_o M)$ and $\sigma = \phi(\omega)$. Let $Vol_{\mathcal{P}}$ be the density associated to $G^1|_{TH_{\mathcal{P}}(M) \otimes TH_{\mathcal{P}}(M)}$ metric and $Vol_{G^1_{\mathcal{P}}}$ be the density associated to $G^1_{\mathcal{P}}$ metric. Then*

$$X_s^{h_{k,a}}(\sigma) := //_s(\sigma)\, h_{k,a}(s) \ \ for \ k = 1, 2, \dots, n \ and \ a = 1, 2, \dots, d$$

is a basis for $T_\sigma H_{\mathcal{P}}(M)$ and we define

$$\begin{aligned} \rho_{\mathcal{P}} &= \frac{\left| Vol_{\mathcal{P}}\left(\{X^{h_{k,a}}\}_{\substack{i=1,2,\dots,n \\ a=1,2,\dots,d}}\right)\right|}{\left| Vol_{G^1_{\mathcal{P}}}\left(\{X^{h_{k,a}}\}_{\substack{i=1,2,\dots,n \\ a=1,2,\dots,d}}\right)\right|} \\ &= \frac{\sqrt{\det\left(\left\{G^1\left(X^{h_{k,a}}, X^{h_{k',a'}}\right)\right\}_{(k,a),(k',a')}\right)}}{\sqrt{\det\left(\left\{G^1_{\mathcal{P}}\left(X^{h_{k,a}}, X^{h_{k',a'}}\right)\right\}_{(k,a),(k',a')}\right)}}. \end{aligned} \tag{2.5}$$

The relevance of this definition is contained in the next remark.

Remark 2.7. *First off, it is well known (and easily verified) that the* $\rho_{\mathcal{P}}(\sigma)$ *defined in Equation (2.5) is well defined independent of the choice of basis* $\{h_{k,a}\}_{\substack{k=1,2,\ldots,n \\ a=1,2,\ldots,d}}$. *Secondly, if* $\nu_{\mathcal{P}}$ *and* $\nu_{G^1_{\mathcal{P}}}$ *are the measures associated to* $G^1|_{TH_{\mathcal{P}}(M)\otimes TH_{\mathcal{P}}(M)}$ *and* $G^1_{\mathcal{P}}$ *respectively, then* $d\nu_{\mathcal{P}} = \rho_{\mathcal{P}} \cdot d\nu_{G^1_{\mathcal{P}}}$.

From [1], we know the limiting behavior of the measure $\nu_{G^1_{\mathcal{P}}}$. Hence, our proof that $\nu_{\mathcal{P}}$ has a limit will break into two parts. Very roughly speaking we are going to first show that $\{\rho_{\mathcal{P}} : \mathcal{P}\}$ is uniformly integrable and then we will show that $\lim_{|\mathcal{P}|\to 0} \rho_{\mathcal{P}}$ exists in μ-measure. This rough outline will have to be appropriately modified since $\{\rho_{\mathcal{P}} : \mathcal{P}\}$ are functions on different probability spaces for each $\mathcal{P}$. This will be remedied by pulling $\rho_{\mathcal{P}}$ to classical Wiener space $(W(T_oM), \mu)$ using Cartan's rolling map ϕ and the natural piecewise linear approximation map from $W(T_oM)$ to $H_{\mathcal{P}}(T_oM)$. We will identify $T_o(M)$ with $\mathbb{R}^d$.

Before we move on, we would like to point out that $\rho_{\mathcal{P}} \circ \phi$ is only defined on $H_{\mathcal{P}}(\mathbb{R}^d)$, which has μ-measure zero.

Definition 2.8. *Let* $\{b(s)\}_{s\in[0,1]}$ *be the standard* $\mathbb{R}^d$ *Brownian motion on* $(W(\mathbb{R}^d),\ \mu)$ *and* $\mathcal{P}$ *be any partition, i.e.* $b(s) : W(\mathbb{R}^d) \to \mathbb{R}^d$,

$$b(s)(\omega) := b(s,\omega) := \omega(s).$$

By abuse of notation, define $b_{\mathcal{P}} : W(\mathbb{R}^d) \to H_{\mathcal{P}}(\mathbb{R}^d)$ *by*

$$b_{\mathcal{P}}(s) = b(s_{i-1}) + (s - s_{i-1})\frac{\Delta_i b}{\Delta_i s} \quad if\ \ s \in (s_{i-1},\ s_i]$$

where $\Delta_i b := b(s_i) - b(s_{i-1})$. *We will write* $b_n = b_{\mathcal{P}_n}$ *if* $\mathcal{P}_n = \{0 < \frac{1}{n} < \cdots < \frac{n}{n} = 1\}$ *is an equally spaced partition*

Thus by composing with $b_{\mathcal{P}}$, we can now view $\rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}$ as a random variable on $(W(\mathbb{R}^d), \mu)$.

3

Uniform Integrability of $\{\rho_n\}_{n=1}^{\infty}$

We will first show that $\rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}$ is uniformly integrable. But first, we need to write down a formula for $\rho_{\mathcal{P}}$.

3.1 A First Formula for ρ_n

We will now only consider equally spaced partitions $\mathcal{P}_n = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$, such that $\Delta_i s = \frac{1}{n},\ i = 1, \ldots n$ and write $\rho_n = \rho_{\mathcal{P}_n}$. Let $\sigma \in H_{\mathcal{P}_n}(M)$ and consider $\omega = \phi^{-1}(\sigma)$. On each $J_i = (s_{i-1}, s_i],\ i = 1, 2, \ldots, n$, $\omega_i' := \omega'(s_{i-1}+)$ is a constant. Thus $\Delta_i \omega := \omega_i' \Delta_i s$.

From Proposition 2.5, we know that for $s \in J_i$, for each h such that $X^h \in TH_{\mathcal{P}_n}(M)$, $h(\omega, s)$ satisfies the ordinary differential equation

$$\frac{d^2 h(\omega, s)}{ds^2} = \Omega_{u(s)}(\omega_i', h\,(\omega, s))\omega_i'. \tag{3.1}$$

Let $\{e_a\}_{a=1}^{d}$ be the standard basis for $\mathbb{R}^d$ and for $i = 1, 2, \ldots, n$, let

$$e_{i,a} = \left(0, \ldots, 0, \overset{\mathrm{i^{th}\ -spot}}{e_a}, 0, \ldots, 0\right) \in \left(\mathbb{R}^d\right)^n = \mathbb{R}^{nd}.$$

Then $\{e_{i,a}\}_{\substack{i=1,\ldots n \\ a=1,\ldots,d}}$ is an indexing of the standard basis for $\mathbb{R}^{nd}$ such that all the components of $e_{i,a}$ are 0 except at the $a + (i-1)d$ position, which is 1.

Notation 3.1. *Let $h_{i,a}(\omega,s)$ denote the continuous function in $\mathbb{R}^d$ which solves Equation (3.1) on $[0,1]\setminus\mathcal{P}_n$ and satisfies*

$$h_{i,a}(\omega,0)=0,\ \text{and } h'_{i,a}(\omega,s_{j-1}+)=\delta_{ij}e_a \text{ for } j=1,\ldots,n.$$

It is easily seen that $\{h_{i,a}\}_{\substack{i=1,\ldots,n\\ a=1,\ldots,d}}$ forms a basis for $\phi_*^{-1}(T_{\phi(\omega)}H_{\mathcal{P}_n}(M))=TH_{\mathcal{P}_n}(\mathbb{R}^d)\cong H_{\mathcal{P}_n}\left(\mathbb{R}^d\right)$. Further let $\mathcal{Q}^n$ denote the $nd\times nd$ matrix which is given in $d\times d$ blocks, $\mathcal{Q}^n:=\{(\mathcal{Q}^n_{mk})\}_{m,k=1}^n$, with

$$\left(\mathcal{Q}^n_{mk}e_a,e_c\right)(\omega):=\int_0^1\left\langle h'_{ma}\left(\omega,s\right),h'_{kc}\left(\omega,s\right)\right\rangle ds \text{ for } a,c=1,2,\ldots,d.$$

Notation 3.2. *Unless stated otherwise, upper case letters without a superscript n will denote $d\times d$ matrices. Upper case letters and scripted upper case letters with a superscript n will denote $n\times n$ block matrices with entries being $d\times d$ blocks. We will reserve I and $\mathcal{I}^n$ for a $d\times d$ identity matrix and a $nd\times nd$ identity matrix respectively. Matrices with a superscript T will denote the matrix transpose. To avoid confusion, we will use* Tr *and* tr *to denote taking the trace of a $nd\times nd$ matrix and a $d\times d$ matrix respectively.*
b will denote a standard d – dimensional Brownian path in $\mathbb{R}^d$. For a piecewise continuous function on $[0,1]$, we will use the notation $f(s+)=\lim_{r\downarrow s}f(r)$. We also let $\langle\cdot,\cdot\rangle$ denote $g(\cdot,\cdot)|_o$ at base point o.

Remark 3.3. *All norms used for matrices will be the operator norm. Norms used for vectors will be the euclidean norm.*

Lemma 3.4. *The relationship between $\rho_n\circ\phi$ and $\mathcal{Q}^n$ is*

$$\rho_n\circ\phi=\sqrt{\det\left(n\mathcal{Q}^n\right)}. \tag{3.2}$$

Proof. Observe that

$$\begin{aligned}G^1\left(X^{h_{k,a}},\ X^{h_{m,c}}\right)(\omega)&=\int_0^1 g\left(\frac{\nabla X^{h_{k,a}}(\omega,s)}{ds},\ \frac{\nabla X^{h_{m,c}}(\omega,s)}{ds}\right)\ ds\\&=\int_0^1\left\langle h'_{k,a}(\omega,s),\ h'_{m,c}(\omega,s)\right\rangle\ ds\\&=\left\langle\mathcal{Q}^n_{mk}(\omega)\ e_a,\ e_c\right\rangle=\left\langle\mathcal{Q}^n(\omega)\ e_{k,a},\ e_{m,c}\right\rangle\end{aligned}$$

Hence

$$\left|Vol_{\mathcal{P}}\left(\{X^{h_{k,a}}:\ k=1,2,\ldots,n,a=1,2,\ldots,d\}\right)\right| := \sqrt{\det\left\{G^1\left(X^{h_{k,a}},X^{h_{k',c}}\right)\right\}} = \sqrt{\det\ \mathcal{Q}^n}.$$

On the other hand,

$$\begin{aligned} G^1_{\mathcal{P}}\left(X^{h_{k,a}},X^{h_{m,c}}\right)(\omega) &= \sum_{i=1}^{n} g\left(\frac{\nabla X^{h_{k,a}}(\omega,s_{i-1}+)}{ds},\ \frac{\nabla X^{h_{m,c}}(\omega,s_{i-1}+)}{ds}\right)\Delta_i s \\ &= \sum_{i=1}^{n}\langle h'_{k,a}(\omega,s_{i-1}+),h'_{m,c}(\omega,s_{i-1}+)\rangle\ \Delta_i s \\ &= \sum_{i=1}^{n}\delta_{k,i}\delta_{m,i}\langle e_a,e_c\rangle\ \Delta_i s = \delta_{k,m}\delta_{a,c}\Delta_k s, \end{aligned}$$

i.e. $\left\{G^1_{\mathcal{P}}\left(X^{h_{k,a}},X^{h_{m,c}}\right)\right\} = \frac{1}{n}\mathcal{I}^n$ and hence

$$\left|Vol_{\mathcal{P}}\left(\{X^{h_{k,a}}:\ k=1,2,\ldots,n,a=1,2,\ldots,d\}\right)\right| := \sqrt{\det\left\{G^1_{\mathcal{P}}\left(X^{h_{k,a}},X^{h_{k',c}}\right)\right\}} = \sqrt{\prod_{i=1}^{n}\left(\Delta_i s\right)^d} = n^{-dn/2}.$$

Hence it follows that

$$\rho_n\circ\phi = \frac{\sqrt{\det\ \mathcal{Q}^n}}{\sqrt{\prod_{i=1}^{n}\left(\Delta_i s\right)^d}} = \sqrt{\det(n\mathcal{Q}^n)}.$$

□

Our next goal is to estimate the size of $\det(n\mathcal{Q}^n)$. For this we are going to apply Proposition C.2 in the Appendix as follows. For $\alpha\geq 1$, to be chosen later, we have, from Equation (C.4) with $M = n\mathcal{Q}^n$, that

$$\begin{aligned} \det(n\mathcal{Q}^n) &\leq \alpha^{nd}\exp\left(\alpha^{-1}\operatorname{Tr}\left(n\mathcal{Q}^n-\mathcal{I}^n\right)\right) = \alpha^{nd}\exp\left(\alpha^{-1}\sum_{m=1}^{n}\operatorname{tr}\left(n\mathcal{Q}^n_{mm}-I\right)\right) \\ &\leq \alpha^{nd}\exp\left(\alpha^{-1}d\sum_{m=1}^{n}\left\|n\mathcal{Q}^n_{mm}-I\right\|\right), \end{aligned} \tag{3.3}$$

where we have used Equation (C.2) of Proposition C.1 in the last inequality. So according to Equation (3.3) we need to estimate $\|n\mathcal{Q}^n_{mm}-I\|$ for each m. The first step in the process is to record a formula for $\mathcal{Q}^n_{mm}$.

3.2 A Formula for $\mathcal{Q}^n$

Notation 3.5. *Given any partition,* $\mathcal{P} := \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ *of* $[0,1]$, $j \in \{1, 2, \ldots, n\}$, *and* $s \in [0,1]$, *let*

$$[s]_j := [(s - s_{j-1}) \vee 0] \wedge \Delta_j s = \begin{cases} 0 & \text{if } s \le s_{j-1} \\ s - s_{j-1} & \text{if } s \in J_j \\ \Delta_j s & \text{if } s \ge s_j \end{cases}.$$

Definition 3.6. *For* $i = 1, 2, \ldots, n$ *and* $0 \le s \le \Delta_i s$*, let*

$$A_i(\omega, s) := \Omega_{u(s_{i-1}+s)}(\omega_i', \cdot)\omega_i' \tag{3.4}$$

and $C_i(\omega, s)$ *and* $S_i(\omega, s)$ *be the* $\mathrm{End}\left(\mathbb{R}^d\right)$ *– valued functions determined by*

$$\frac{d^2 C_i(\omega, s)}{ds^2} = A_i(\omega, s) C_i(\omega, s) \text{ with } C_i(\omega, 0) = I \text{ and } C_i'(\omega, 0) = 0$$

and

$$\frac{d^2 S_i(\omega, s)}{ds^2} = A_i(\omega, s) S_i(\omega, s) \text{ with } S_i(\omega, 0) = 0 \text{ and } S_i'(\omega, 0) = I.$$

Let $K > 0$ *be a constant such that*

$$\begin{aligned} \| A_i(\omega, s)\Delta_i s^2 \| &= \| \Omega_{u(s_{i-1}+s)}(\omega_i', \cdot)\omega_i' \| \Delta_i s^2 \\ &\le K \| \Delta_i \omega \|^2 . \end{aligned}$$

Refer to Section D of Appendix.

Using this notation it follows that $h(\omega, \cdot)$ is a continuous function which solves Equation (3.1) on $[0,1] \setminus \mathcal{P}_n$, then

$$h(\omega, s) = C_i(\omega, s - s_{i-1})\ h(\omega, s_{i-1}) + S_i(\omega, s - s_{i-1})\ h'(\omega, s_{i-1}+) \text{ when } s \in J_i.$$

Notation 3.7. *For* $m, l \in \{1, 2, \ldots, n\}$ *and* $d \times d$ *matrices,* $\{M_k\}_{k=1}^n$, *let*

$$\prod_{k=l}^{m} M_k := \begin{cases} I & \text{if } m < l \\ M_m M_{m-1} \ldots M_{l+1} M_l & \text{if } m \ge l \end{cases}.$$

With all of this notation, we may write $h_{i,a}(\omega, s)$ as in the following lemma.

Lemma 3.8. *Continuing to use the notation introduced above, we have*

$$h_{m,a}(\omega,s)=\left[\prod_{k=m+1}^{n}C_k\left(\omega,[s]_k\right)\right]S_m\left(\omega,[s]_m\right)e_a \tag{3.5}$$

$$=\begin{cases}0 & \text{if} \quad s\le s_{m-1}\\ S_m(\omega,s-s_{m-1})e_a & \text{if} \quad s\in J_m\\ C_j(\omega,s-s_{j-1})V_{mj}\left(\omega\right)e_a & \text{if} \quad s\in J_j \text{ and } j\ge m+1\end{cases} \tag{3.6}$$

where

$$V_{mj}(\omega):=\left[\prod_{k=m+1}^{j-1}C_k(\omega,\Delta_k s)\right]S_m(\omega,\Delta_m s). \tag{3.7}$$

Differentiating Equation (3.6),

$$h'_{m,a}\left(\omega,s\right)=1_{J_m}\left(s\right)S'_m\left(\omega,s-s_{m-1}\right)e_a+\sum_{j=m+1}^{n}1_{J_j}\left(s\right)C'_j(\omega,s-s_{j-1})V_{mj}e_a. \tag{3.8}$$

From Equation (3.8), we learn that

$$\begin{aligned}\langle\mathcal{Q}^n_{mm}(\omega)e_a,e_c\rangle &= \int_{J_m}\langle S'_m\left(\omega,s-s_{m-1}\right)e_a,S'_m\left(\omega,s-s_{m-1}\right)e_c\rangle\,ds\\ &\quad+\sum_{j=m+1}^{n}\int_{J_j}\left\langle C'_j(\omega,s-s_{j-1})V_{mj}e_a,C'_j(\omega,s-s_{j-1})V_{mj}e_c\right\rangle ds\\ &=\left\langle\int_0^{\Delta_m s}S'_m\left(\omega,s\right)^T S'_m\left(\omega,s\right)ds\;e_a,e_c\right\rangle\\ &\quad+\sum_{j=m+1}^{n}\left\langle V^T_{mj}\left[\int_0^{\Delta_j s}C'_j(\omega,s)^T C'_j(\omega,s)ds\right]V_{mj}e_a,e_c\right\rangle\end{aligned}$$

and hence we have shown,

$$\begin{aligned}&\mathcal{Q}^n_{mm}(\omega)\\ &=\int_0^{\Delta_m s}S'_m\left(\omega,s\right)^T S'_m\left(\omega,s\right)ds+\sum_{j=m+1}^{n}V^T_{mj}\left[\int_0^{\Delta_j s}C'_j(\omega,s)^T C'_j(\omega,s)ds\right]V_{mj}\\ &=\int_0^{1/n}S'_m\left(\omega,s\right)^T S'_m\left(\omega,s\right)ds+\sum_{j=m+1}^{n}V^T_{mj}\left[\int_0^{1/n}C'_j(\omega,s)^T C'_j(\omega,s)ds\right]V_{mj}.\end{aligned} \tag{3.9}$$

Noting that

$$\left\|V_{mj}(\omega)\right\|=\left\|V^T_{mj}(\omega)\right\|\le\left\|S_m(\omega,\Delta_m s)\right\|^2\prod_{k=m+1}^{j-1}\left\|C_k(\omega,\Delta_k s)\right\|^2,$$

it follows from Equation (3.9) that

$$\|n\mathcal{Q}^n_{mm} - I\| \le A_m + B_m \tag{3.10}$$

where

$$A_m(\omega) := n\int_0^{1/n} \left\| S'_m(\omega, s)^T S'_m(\omega, s) - I \right\| ds \tag{3.11}$$

and

$$\begin{aligned} B_m(\omega) &= \sum_{j=m+1}^{n} \|S_m(\omega, \Delta_m s)\|^2 \prod_{k=m+1}^{j-1} \|C_k(\omega, \Delta_k s)\|^2 \left\| n\int_0^{1/n} C'_j(\omega, s)^T C'_j(\omega, s) ds \right\| \\ &\le \sum_{j=m+1}^{n} \|S_m(\omega, \Delta_m s)\|^2 \prod_{k=m+1}^{j-1} \|C_k(\omega, \Delta_k s)\|^2 \, n\int_0^{1/n} \left\|C'_j(\omega, s)\right\|^2 ds. \end{aligned} \tag{3.12}$$

Thus we are now left to estimate the quantities comprising A_m and B_m.

As a warm up, we will first consider the case when M is a symmetric space. The general case will be covered in Section 3.4.

3.3 Symmetric Space Case

Proposition 3.9. *Suppose N is a d – dimensional standard Gaussian normal random variable and $h : [0, \infty) \to \mathbb{R}$ is a C^1 function.*

1. *If $\alpha := \sup h > \frac{1}{2}$, then*
$$\lim_{n\to\infty} \mathbb{E}\left[e^{\|N\|^2 h\left(n^{-1}\|N\|^2\right)}\right] = \infty.$$
2. *If $\alpha := \sup h < \frac{1}{2}$, $h(0) = 0$, and $\beta := \sup |h'| < \infty$, then*
$$\lim_{n\to\infty} \left(\mathbb{E}\left[e^{\|N\|^2 h\left(n^{-1}\|N\|^2\right)}\right]\right)^n = e^{3dh'(0)}.$$

Proof. If $\alpha > \frac{1}{2}$, then there exists $0 \le a < b$ such that $h(t) \ge \frac{1}{2}$ for $a \le t \le b$. Therefore,

$$\begin{aligned} \mathbb{E}\left[e^{\|N\|^2 h\left(n^{-1}\|N\|^2\right)}\right] &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\|x\|^2 h\left(n^{-1}\|x\|^2\right)} e^{-\frac{1}{2}\|x\|^2} dx \\ &= (2\pi)^{-d/2}\, \omega_d \int_0^\infty e^{r^2\left[h\left(n^{-1}r^2\right) - \frac{1}{2}\right]} r^{d-1} dr \\ &\ge (2\pi)^{-d/2}\, \omega_d \int_{a \le n^{-1}r^2 \le b} r^{d-1} dr \sim n^{d/2} \to \infty \text{ as } n \to \infty, \end{aligned}$$

where ω_d is the volume of the d-sphere in $\mathbb{R}^d$. On the other hand, if $\alpha < \frac{1}{2}$, let

$$f(x) := \mathbb{E}\left[e^{\|N\|^2 h\left(x\|N\|^2\right)}\right].$$

Since

$$e^{\|N\|^2 h\left(x\|N\|^2\right)} \leq e^{\alpha\|N\|^2}$$

and

$$e^{\|N\|^2 h\left(x\|N\|^2\right)} \|N\|^4 \left|h'\left(x\|N\|^2\right)\right| \leq \beta \|N\|^4 e^{\alpha\|N\|^2}$$

with the right hand members of these inequalities being integrable functions, it follows that $f(x)$ is a C^1 - function with

$$f'(x) = \mathbb{E}\left[e^{\|N\|^2 h\left(x\|N\|^2\right)} \|N\|^4 h'\left(x\|N\|^2\right)\right].$$

In particular we have $f(0) = 1$ and

$$f'(0) = h'(0)\,\mathbb{E}\left[\|N\|^4\right] = 3dh'(0).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(\mathbb{E}\left[e^{\|N\|^2 h\left(n^{-1}\|N\|^2\right)}\right]\right)^n &= \lim_{n\to\infty}\left(f\left(n^{-1}\right)\right)^n = \lim_{n\to\infty} e^{n\ln f\left(n^{-1}\right)} = \lim_{x\to 0} e^{\frac{1}{x}\ln f(x)} \\ &= \lim_{x\to 0} e^{f'(x)/f(x)} = e^{f'(0)} = e^{3dh'(0)}.\end{aligned}$$

□

Let us assume that M is a symmetric space and in particular suppose that $\nabla R = 0$ with sectional curvature being non-negative. In this case we know that $A_k(\omega, s)$ is constant in s and we may choose an orthonormal basis $\{e_l\}_{l=1}^d$ of eigenvectors of $-A_k(\omega, s)$ with eigenvalues $\{\lambda_l^k\}_{l=1}^d$, such that $0 \leq \lambda_l^k \leq K\|\omega_k'\|^2$ for all l. Note that K is some curvature bound. Since

$$C_k(\omega, s) = \cos\left(\sqrt{-A_k(\omega_k')}s\right) \text{ and } S_k(\omega, s) = \frac{\sin\left(\sqrt{-A_k(\omega_k')}s\right)}{\sqrt{-A_k(\omega_k')}},$$

we have,

$$C_k(\omega,s)e_l = \cos\left(\sqrt{\lambda_l^k}s\right)e_l,$$

$$S_k(\omega,s)\,e_l = \frac{\sin\left(\sqrt{\lambda_l^k}s\right)}{\sqrt{\lambda_l^k}}e_l = \frac{\sin\left(\sqrt{\lambda_l^k}s\right)}{\sqrt{\lambda_l^k}s}se_l, \text{ and}$$

$$C_k'(\omega,s)e_l = -\sin\left(\sqrt{\lambda_l^k}s\right)\sqrt{\lambda_l^k}e_l = -\frac{\sin\left(\sqrt{\lambda_l^k}s\right)}{\sqrt{\lambda_l^k}s}\lambda_l^k s\cdot e_l.$$

In particular we have

$$\|C_k'(\omega,s)\|^2 = \sup_l \sin^2\left(\sqrt{\lambda_l^k}s\right)\lambda_l^k \le K\|\omega_k'\|^2\left\|\sin\left(\sqrt{-A_k(\omega,s)}\right)\right\|^2$$

From these observations it follows that

$$\prod_{k=m+1}^{j-1}\|C_k(\omega,\Delta_k s)\| \le 1, \quad \|S_m(\omega,\Delta_m s)\| \le \Delta_m s \text{ and}$$

$$\|C_k'(\omega,s)\| \le K\|\omega_k'\|^2 s.$$

From these observations we learn that

$$\int_0^{1/n} C_j'(\omega,s)^T C_j'(\omega,s)ds = -A_k(\omega_k')\int_0^{1/n}\sin^2\left(\sqrt{-A_k(\omega_k')}s\right)ds.$$

Using

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

in the previous equation shows

$$\begin{aligned} n\int_0^{1/n} C_j'(\omega,s)^T C_j'(\omega,s)ds &= -\frac{1}{2}A_k(\omega_k')\,n\int_0^{1/n}\left(I-\cos\left(2\sqrt{-A_k(\omega_k')}s\right)\right)ds \\ &= -\frac{1}{2}A_k(\omega_k')\left(I - n\frac{\sin\left(2\sqrt{-A_k(\omega_k')}n^{-1}\right)}{2\sqrt{-A_k(\omega_k')}}\right). \end{aligned}$$

Thus we have

$$\begin{aligned} \left\|n\int_0^{1/n} C_j'(\omega,s)^T C_j'(\omega,s)ds\right\| &\le \frac{1}{2}K\|\omega_k'\|^2\left\|I - \frac{\sin\left(2\sqrt{-A_k(\omega_k')}n^{-1}\right)}{2\sqrt{-A_k(\omega_k')}n^{-1}}\right\| \\ &= \frac{1}{2}K\|\omega_k'\|^2\left\|I - \frac{\sin\left(2\sqrt{\alpha_k}\,\|\Delta_k\omega\|\right)}{2\sqrt{\alpha_k}\,\|\Delta_k\omega\|}\right\| \end{aligned}$$

where $\alpha_k(\omega) := -\left\|\omega_k'\right\|^{-2} A_k\left(\omega_k'\right)$. Thus we have

$$
\begin{aligned}
B_m(\omega) &= \sum_{j=m+1}^{n} \|S_m(\omega, \Delta_m s)\|^2 \left(\prod_{k=m+1}^{j-1} \|C_k(\omega, \Delta_k s)\|^2\right) \left\| n \int_0^{1/n} C_j'(\omega, s)^T C_j'(\omega, s) ds \right\| \\
&\leq \sum_{j=m+1}^{n} (\Delta_m s)^2 \frac{1}{2} K \left\|\omega_j'\right\|^2 \left\| I - \frac{\sin\left(2\sqrt{\alpha_j(\omega)}\,\|\Delta_j \omega\|\right)}{2\sqrt{\alpha_j(\omega)}\,\|\Delta_j \omega\|} \right\| \\
&= \frac{1}{2} K \sum_{j=m+1}^{n} \|\Delta \omega_j\|^2 \left\| I - \frac{\sin\left(2\sqrt{\alpha_j(\omega)}\,\|\Delta_j \omega\|\right)}{2\sqrt{\alpha_j(\omega)}\,\|\Delta_j \omega\|} \right\|. \qquad (3.13)
\end{aligned}
$$

Since

$$
1 - \frac{\sin x}{x} = 1 - \left(1 - \frac{x^2}{3!} + \dots\right) = O\left(x^2\right)
$$

we also see that

$$
\left\| I - \frac{\sin\left(2\sqrt{\alpha_m(\omega)}\,\|\Delta_m \omega\|\right)}{2\sqrt{\alpha_m(\omega)}\,\|\Delta_m \omega\|} \right\| = O\left(\|\Delta_m \omega\|^2\right).
$$

Moreover,

$$
S_m'\left(\omega, s\right) e_l = \cos\left(\sqrt{\lambda_l^m} s\right) e_l
$$

so that

$$
\begin{aligned}
& S_m'\left(\omega, s\right)^T S_m'\left(\omega, s\right) e_l - e_l \\
&= \left(\cos^2\left(\sqrt{\lambda_l^m} s\right) - 1\right) e_l = \left(\frac{1 + \cos\left(2\sqrt{\lambda_l^m} s\right)}{2} - 1\right) e_l \\
&= \left(\frac{\cos\left(2\sqrt{\lambda_l^m} s\right) - 1}{2}\right) e_l = \left(\frac{1}{2} \int_0^s -2\sqrt{\lambda_l^m} \sin\left(2\sqrt{\lambda_l^m} r\right) dr\right) \\
&= -\int_0^s \sqrt{\lambda_l^m} \sin\left(2\sqrt{\lambda_l^m} r\right) dr
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \left\| \left(S_m'\left(\omega, s\right)^T S_m'\left(\omega, s\right) - I\right) e_l \right\| \leq \sqrt{\lambda_l^m} \int_0^s \left|\sin\left(2\sqrt{\lambda_l^m} r\right)\right| dr \\
& \leq 2\lambda_l^m \int_0^s \left| \frac{\sin\left(2\sqrt{\lambda_l^m} r\right)}{2\sqrt{\lambda_l^m} r} \right| r dr \leq \lambda_l^m s \leq K \left\|\omega_m'\right\|^2 s^2 \leq K \left\|\Delta_m \omega\right\|^2.
\end{aligned}
$$

From this computation it follows that

$$A_m(\omega) \leq K \|\Delta_m \omega\|^2 . \tag{3.14}$$

Now for $\omega \in (W(\mathbb{R}^d), \mu)$,

$$\Delta_i b_n(\omega) = \Delta_i b(\omega) = b(s_i)(\omega) - b(s_{i-1})(\omega) = \omega(s_i) - \omega(s_{i-1}) = \Delta_i \omega.$$

Hence from Equations (3.13) and (3.14), with $\alpha_j^n := \alpha_j \circ b_n$

$$\begin{aligned}
&\sum_{m=1}^{n} \|(n\mathcal{Q}^n_{mm} - I)\circ b_n\| \\
&\leq K \sum_{m=1}^{n} \|\Delta_m b\|^2 + \frac{1}{2} K \sum_{m=1}^{n} \sum_{j=m+1}^{n} \|\Delta_j b\|^2 \left\| I - \frac{\sin\left(2\sqrt{\alpha_j^n}\,\|\Delta_j b\|\right)}{2\sqrt{\alpha_j^n}\,\|\Delta_j b\|} \right\| \\
&= K \sum_{m=1}^{n} \|\Delta_m b\|^2 + \frac{1}{2} K \sum_{m=1}^{n} \sum_{j>m}^{n} \|\Delta_j b\|^2 \left\| I - \frac{\sin\left(2\sqrt{\alpha_j^n}\,\|\Delta_j b\|\right)}{2\sqrt{\alpha_j^n}\,\|\Delta_j b\|} \right\| \\
&= K \sum_{m=1}^{n} \|\Delta_m b\|^2 + \frac{1}{2} K \sum_{j=2}^{n} (j-1) \|\Delta_j b\|^2 \left\| I - \frac{\sin\left(2\sqrt{\alpha_j^n}\,\|\Delta_j b\|\right)}{2\sqrt{\alpha_j^n}\,\|\Delta_j b\|} \right\| .
\end{aligned}$$

Combining this with Equation (3.3) implies, with $\alpha = 1$ in this case that

$$\begin{aligned}
&\sqrt{\det\left(n\mathcal{Q}^n \circ b_n\right)} \\
&= \rho_n \circ \phi \circ b_n \leq \exp\left(\frac{dK}{2} \left[\sum_{m=1}^{n} \left(1 + \frac{m-1}{2} \left\| I - \frac{\sin(2\sqrt{\alpha_m^n}\,\|\Delta_m b\|)}{2\sqrt{\alpha_m^n}\,\|\Delta_m b\|} \right\| \right) \|\Delta_m b\|^2 \right] \right)
\end{aligned}$$

and we have

$$\begin{aligned}
&\mathbb{E}\sqrt{\det\left(n\mathcal{Q}^n \circ b_n\right)} \\
&\leq \prod_{m=1}^{n} \mathbb{E}\left[\exp\left(\frac{dK}{2} \left[\left(1 + \frac{1}{2}(m-1) \left\| I - \frac{\sin\left(2\sqrt{\alpha_m^n}\frac{1}{\sqrt{n}}\,\|N_m\|\right)}{2\sqrt{\alpha_m^n}\frac{1}{\sqrt{n}}\,\|N_m\|} \right\| \right) \frac{1}{n} \|N_m\|^2 \right] \right) \right],
\end{aligned}$$

where $\| N_m \| := \sqrt{n} \| \Delta_m b \|$. Now in the case of a sphere the above expression becomes something like

$$\begin{aligned}
&\mathbb{E}\sqrt{\det\left(n\mathcal{Q}^n \circ b_n\right)} \\
&\leq \left[\mathbb{E}\exp\left(\frac{dK}{2} \left[\left(\frac{1}{n} + \frac{1}{2} \left(1 - \frac{\sin\left(2\sqrt{K}\frac{1}{\sqrt{n}}\,\|N\|\right)}{2\sqrt{K}\frac{1}{\sqrt{n}}\,\|N\|} \right) \right) \|N\|^2 \right] \right) \right]^n .
\end{aligned}$$

To understand what happens to this expression as $n \to \infty$, let

$$\psi\left(x^2\right) := 1 - \frac{\sin 2\sqrt{K}x}{2\sqrt{K}x}$$

where ψ is a smooth function such that $0 \le \psi \le 1$ and $\psi\left(t\right) = \frac{4}{6}Kt + O\left(t^2\right)$ in which case the above expression may be written as

$$\mathbb{E}\sqrt{\det(n\mathcal{Q}^n \circ b_n)} \le \left[\mathbb{E}\exp\left(\frac{dK}{2}\left[\frac{1}{n} + \frac{1}{2}\psi\left(\frac{1}{n}\left\|N\right\|^2\right)\right]\left\|N\right\|^2\right)\right]^n.$$

If $K < \frac{1}{d}$ and for $x \in [0, \frac{1}{2})$, by letting

$$\theta\left(x, \left\|N\right\|^2\right) = \frac{dK}{2}\left(x + \frac{1}{2}\psi\left(x\left\|N\right\|^2\right)\right),\ f\left(x\right) := \mathbb{E}\left[\exp\left(\theta\left(x, \left\|N\right\|^2\right)\left\|N\right\|^2\right)\right],$$

we learn that

$$f\left(0\right) = 1$$

and

$$\sup_{x \in [0, \frac{1}{2}]} f'\left(x\right) = \sup_{x \in [0, \frac{1}{2}]} \mathbb{E}\left[\frac{dK}{2}\left[1 + \frac{1}{2}\psi'\left(x\left\|N\right\|^2\right)\left\|N\right\|^2\right]\exp\left(\theta\left(x, \left\|N\right\|^2\right)\left\|N\right\|^2\right)\right] = C.$$

Therefore $f\left(x\right) \le 1 + Cx$ and we learn that

$$\mathbb{E}\sqrt{\det(n\mathcal{Q}^n \circ b_n)} \le \left[f\left(\frac{1}{n}\right)\right]^n \le \left[1 + C\frac{1}{n}\right]^n \le e^C.$$

3.4 Estimates for Solutions to Jacobi's Equation

Remark 3.10. *In what follows we will make use of the following elementary estimates without further comment.*

1. $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \le \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2}$.
2. $\cosh x = \cosh|x| = \frac{e^{|x|} + e^{-|x|}}{2} \le e^{|x|}$ *so that*

$$\cosh x \le \min\left(e^{x^2/2}, e^{|x|}\right),$$

and

3. for $x \geq 0$,

$$\sinh x = \int_0^x \cosh t \; dt \leq \int_0^x \cosh x \; dt = x \cosh x.$$

This estimate is also easily understood using the power series expansions for $\sinh$ *and* $\cosh$.

Lemma 3.11 (Global Estimate)**.** *Let* $A(s)$ *be a* $d \times d$ *matrix for all* $s \geq 0$, $\kappa := \sup_{s \geq 0} \|A(s)\| < \infty$, *and let* $Z(s)$ *be either a* $\mathbb{R}^d$ *or* $d \times d$ *matrix valued solution to the second order differential equation*

$$Z''(s) = A(s)Z(s).$$

Then

$$\|Z(s) - Z(0)\| \leq \|Z(0)\| \left(\cosh \sqrt{\kappa} s - 1\right) + \|Z'(0)\| \frac{\sinh \sqrt{\kappa} s}{\sqrt{\kappa}}. \tag{3.15}$$

Proof. By Taylor's theorem with integral remainder,

$$\begin{aligned} Z(s) &= Z(0) + sZ'(0) + \int_0^s Z''(u)(s-u) \; du \\ &= Z(0) + sZ'(0) + \int_0^s A(u)Z(u)(s-u) \; du \end{aligned} \tag{3.16}$$

and therefore

$$\begin{aligned} \|Z(s) - Z(0)\| &\leq s \|Z'(0)\| + \kappa \int_0^s \|Z(u)\| (s-u) \; du \\ &\leq s \|Z'(0)\| + \kappa \int_0^s \|Z(u) - Z(0)\| (s-u) \; du + \frac{1}{2} s^2 \kappa \|Z(0)\| \\ &:= f(s). \end{aligned} \tag{3.17}$$

Note that $f(0) = 0$,

$$f'(s) = \|Z'(0)\| + \kappa \int_0^s \|Z(u) - Z(0)\| (s-u) \; du + s\kappa \|Z(0)\|,$$

$f'(0) = \|Z'(0)\|$, and

$$f''(s) = \kappa \| \; Z(s) - Z(0) \; \| + \kappa \|Z(0)\| \leq \kappa f(s) + \kappa \|Z(0)\|.$$

That is,

$$f''(s) = \kappa f(s) + \eta(s), \qquad f(0) = 0, \qquad \text{and} \qquad f'(0) = \|Z'(0)\|, \tag{3.18}$$

where $\eta(s) := f''(s) - \kappa f(s) \leq \kappa \|Z(0)\|$. Equation (3.18) may be solved by variation of parameters to find

$$\begin{aligned} f(s) &= \|Z'(0)\| \frac{\sinh \sqrt{\kappa} s}{\sqrt{\kappa}} + \int_0^s \frac{\sinh \sqrt{\kappa}(s-r)}{\sqrt{\kappa}} \eta(r)\ dr \\ &\leq \|Z'(0)\| \frac{\sinh \sqrt{\kappa} s}{\sqrt{\kappa}} + \|Z(0)\| \int_0^s \sqrt{\kappa} \sinh \sqrt{(s-r)}\ dr \\ &= \|Z'(0)\| \frac{\sinh \sqrt{\kappa} s}{\sqrt{\kappa}} + \|Z(0)\| (\cosh \sqrt{\kappa} s - 1). \end{aligned}$$

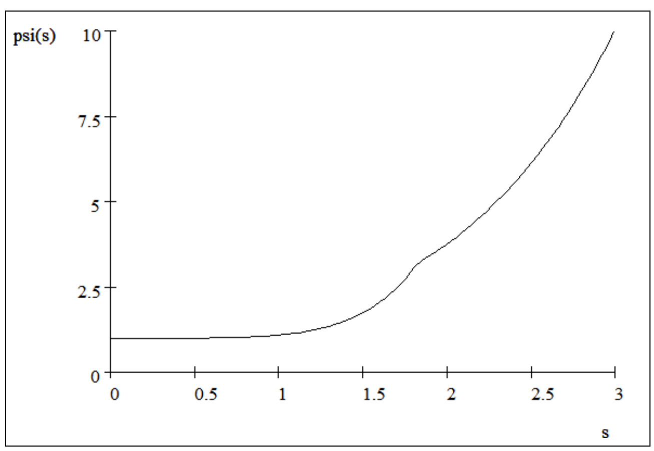
Combining this with Equation (3.17) proves Equation (3.15). □

Theorem 3.12. *Suppose that $A(s)$ above satisfies, $0 \leq -A(s) \leq \kappa I$ for all s or equivalently that $-\kappa I \leq A(s) \leq 0$ for all s. Let*

$$\psi(s) := \min \left(1 + \frac{\cosh(s)}{16} s^4, \cosh(s) \right)$$

whose graph is shown below and $C(s)$ and $S(s)$ be the matrix functions defined by

$$C''(s) = A(s)C(s), \ \textit{with}\ C(0) = I,\ C'(0) = 0,$$
$$S''(s) = A(s)S(s), \ \textit{with}\ S(0) = 0,\ S'(0) = I.$$



3.1 Graph of ψ.

Then

1. $\|C(s)\| \le \psi(\sqrt{\kappa}s)$,
2. $\|S(s)\| \le s\psi(\sqrt{\kappa}s)$,
3. $\|C'(s)\| \le \kappa s\psi(\sqrt{\kappa}s)$,
4. $\|S'(s) - I\| \le \frac{1}{2}\kappa s^2\psi(\sqrt{\kappa}s)$, *and*
5. $\left\|S'(s)^T S'(s) - I\right\| \le \psi(\sqrt{\kappa}s)\kappa s^2 + \frac{1}{3}\psi^2(\sqrt{\kappa}s)\kappa^2 s^4.$

Moreover, if we only assume that $\|A(s)\| \le \kappa$, *all of the above estimates still hold provided that* ψ *is replaced by* $\cosh$.

Proof. If Z solves, $Z''(s) = A(s)Z(s)$, then iterating Equation (3.16) shows

$$\begin{aligned}Z(s) &= Z(0) + sZ'(0) + \int_0^s A(u)\left[Z(0) + uZ'(0) + \int_0^u A(r)Z(r)(u-r)\ dr\right](s-u)\ du\\ &= \left(I + \int_0^s (s-u)A(u)du\right)Z(0) + \left(sI + \int_0^s u(s-u)A(u)\ du\right)Z'(0)\\ &+ \int_{0\le r\le u\le s}(u-r)(s-u)A(u)A(r)Z(r)drdu.\end{aligned}$$

In particular this shows

$$\begin{aligned}C(s) = &\left(I - \int_0^s (s-u)\left[-A(u)\right]du\right)\\ &+ \int_{0\le r\le u\le s}(u-r)(s-u)A(u)A(r)C(r)drdu\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}S(s) = &\left(sI - \int_0^s u(s-u)\left[-A(u)\right]\ du\right)\\ &+ \int_{0\le r\le u\le s}(u-r)(s-u)A(u)A(r)S(r)drdu.\end{aligned} \tag{3.20}$$

From Equation (3.19),

$$\begin{aligned}\|C(s)\| &\le \left\|I + \int_0^s (s-u)A(u)du\right\| + \int_{0\le r\le u\le s}(u-r)(s-u)\left\|A(u)A(r)C(r)\right\|drdu\\ &\le \left\|I + \int_0^s (s-u)A(u)du\right\| + \frac{\cosh(\sqrt{\kappa}s)}{24}\kappa^2 s^4.\end{aligned}$$

Moreover,

$$\left(1-\kappa\frac{s^2}{2}\right)I = I-\kappa\left[\int_0^s (s-u)du\right]I \le I+\int_0^s (s-u)A(u)du \le I$$

from which it follows that

$$\left\|I+\int_0^s (s-u)A(u)du\right\| \le \max\left(1,\kappa\frac{s^2}{2}-1\right) = 1_{s^2\le 4/\kappa}+\left(\kappa\frac{s^2}{2}-1\right)1_{s^2\ge 4/\kappa}$$

and hence we have

$$\|C\left(s\right)\| \le 1_{s^2\le 4/\kappa}+\left(\kappa\frac{s^2}{2}-1\right)1_{s^2\ge 4/\kappa}+\frac{\cosh\left(\sqrt{\kappa}s\right)}{24}\kappa^2 s^4 \le 1+\frac{\cosh\left(\sqrt{\kappa}s\right)}{16}\kappa^2 s^4.$$

This is because $f(s) = s^4\frac{\cosh s}{48}-\left(\frac{s^2}{2}-1\right) = 1+s^2\left[s^2\frac{\cosh s}{48}-\frac{1}{2}\right]$ is an increasing function and for $s\ge 2$, $f(s)\ge f(2) > 0.25$. Recall that we also know that $\|C\left(s\right)\| \le \cosh\left(\sqrt{\kappa}s\right)$ and therefore we have

$$\begin{aligned}\|C\left(s\right)\| &\le \min\left(1_{s^2\le 4/\kappa}+\left(\kappa\frac{s^2}{2}-1\right)1_{s^2\ge 4/\kappa}+\frac{\cosh\left(\sqrt{\kappa}s\right)}{24}\kappa^2 s^4,\cosh\left(\sqrt{\kappa}s\right)\right)\\ &\le \min\left(1+\frac{\cosh\left(\sqrt{\kappa}s\right)}{16}\kappa^2 s^4,\cosh\left(\sqrt{\kappa}s\right)\right).\end{aligned}$$

It will be convenient to define

$$\psi\left(s\right) := \min\left(1+\frac{\cosh\left(s\right)}{16}s^4,\cosh\left(s\right)\right)$$

and hence

$$\|C\left(s\right)\| \le \psi\left(\sqrt{\kappa}s\right).$$

Similarly, from Equation (3.20)

$$\begin{aligned}\|S\left(s\right)\| &\le \left\|sI+\int_0^s u(s-u)A(u)\ du\right\|+\int_{0\le r\le u\le s}(u-r)(s-u)\left\|A(u)A(r)S(r)\right\|drdu\\ &\le s\left\|I+\frac{1}{s}\int_0^s u(s-u)A(u)\ du\right\|+\frac{\kappa^2}{24}s^4\frac{\sinh\left(\sqrt{\kappa}s\right)}{\sqrt{\kappa}}.\end{aligned}$$

In this case,

$$\left(1-\frac{\kappa}{6}s^2\right)I = \left(1-\kappa\frac{1}{s}\int_0^s u(s-u)\ du\right)I \le I+\frac{1}{s}\int_0^s u(s-u)A(u)\ du \le I.$$

Combining this with the previous equation shows

$$\begin{aligned}\|S(s)\| &\le s\left[1_{s^2\le 12/\kappa}+\left(\kappa\frac{s^2}{6}-1\right)1_{s^2\ge 12/\kappa}\right]+\frac{\kappa^2}{24}s^4\frac{\sinh\left(\sqrt{\kappa}s\right)}{\sqrt{\kappa}}\\ &\le s\left[1_{s^2\le 12/\kappa}+\left(\kappa\frac{s^2}{6}-1\right)1_{s^2\ge 12/\kappa}+\frac{\cosh\left(\sqrt{\kappa}s\right)}{24}\kappa^2 s^4\right]\\ &\le s\left[1+\frac{\cosh\left(\sqrt{\kappa}s\right)}{16}\kappa^2 s^4\right].\end{aligned}$$

Since we also have

$$\|S(s)\|\le\frac{\sinh\left(\sqrt{\kappa}s\right)}{\sqrt{\kappa}}\le s\cosh\left(\sqrt{\kappa}s\right),$$

we may conclude that

$$\|S(s)\|\le s\,\min\left(1+\frac{\cosh\left(\sqrt{\kappa}s\right)}{16}\kappa^2 s^4,\cosh\left(\sqrt{\kappa}s\right)\right)=s\psi\left(\sqrt{\kappa}s\right).$$

Furthermore,

$$\|C'(s)\|=\left\|\int_0^s A(r)\,C(r)\,dr\right\|\le\int_0^s\|A(r)\|\,\|C(r)\|\,dr\le\kappa s\psi\left(\sqrt{\kappa}s\right)$$

and

$$\begin{aligned}\|S'(s)-I\| &=\left\|\int_0^s A(r)\,S(r)\,dr\right\|\le\int_0^s\|A(r)\|\,\|S(r)\|\,dr\\ &\le\kappa\int_0^s r\psi\left(\sqrt{\kappa}r\right)dr\le\frac{1}{2}\kappa s^2\psi\left(\sqrt{\kappa}s\right).\end{aligned}$$

Alternatively we have

$$\begin{aligned}S'(s) &= I+\int_0^s A(r)\,S(r)\,dr = I+\int_0^s A(r)\left[\int_0^r S'(u)\,du\right]dr\\ &= I+\int_0^s A(r)\left[\int_0^r\left[I+\int_0^u A(v)\,S(v)\,dv\right]du\right]dr\\ &= I+\int_0^s rA(r)\,dr+\int_{0\le v\le u\le r\le s}A(r)\,A(v)\,S(v)\,dvdudr\\ &= I+\int_0^s rA(r)\,dr+\int_{0\le v\le r\le s}(r-v)\,A(r)\,A(v)\,S(v)\,dvdr \qquad (3.21)\end{aligned}$$

and therefore,

$$\begin{aligned}\|S'(s)\| &\le 1+\kappa^2\psi\left(\sqrt{\kappa}s\right)\int_{0\le v\le r\le s}(r-v)\,vdvdr\\ &= 1+\kappa^2\psi\left(\sqrt{\kappa}s\right)\int_0^s\frac{(s-v)^2}{2}vdv\\ &= 1+\frac{1}{24}\kappa^2 s^4\psi\left(\sqrt{\kappa}s\right).\end{aligned}$$

Finally

$$\frac{d}{ds}\left[S'(s)^T S'(s)\right] = S(s)^T A(s) S'(s) + S'(s)^T A(s) S(s)$$

and therefore,

$$\begin{aligned}\left\|\frac{d}{ds}\left[S'(s)^T S'(s)\right]\right\| &\le 2\|S(s)\|\,\|A(s)\|\,\|S'(s)\| \\ &\le 2\|S(s)\|\,\|A(s)\|\,\|S'(s)\| \\ &\le 2\kappa s\psi\left(\sqrt{\kappa}s\right)\left(1+\frac{1}{2}\kappa s^2\psi\left(\sqrt{\kappa}s\right)\right).\end{aligned}$$

Integrating this equation then implies,

$$\begin{aligned}\left\|S'(s)^T S'(s) - I\right\| &\le \int_0^s 2\kappa r\psi\left(\sqrt{\kappa}r\right)\left(1+\frac{1}{2}\kappa r^2\psi\left(\sqrt{\kappa}r\right)\right)dr \\ &\le \int_0^s 2\kappa\psi\left(\sqrt{\kappa}s\right)\left(r+\frac{1}{2}\kappa r^3\psi\left(\sqrt{\kappa}s\right)\right)dr \\ &= \kappa\psi\left(\sqrt{\kappa}s\right)\left(s^2+\frac{1}{3}\kappa s^4\psi\left(\sqrt{\kappa}s\right)\right) \\ &= \psi\left(\sqrt{\kappa}s\right)\kappa s^2 + \frac{1}{3}\psi^2\left(\sqrt{\kappa}s\right)\kappa^2 s^4.\end{aligned}$$

□

Proposition 3.13. *If* $\|A(s)\| \le \kappa$, *then the following estimates hold:*

$$\left\|S'(s) - \left(I + \int_0^s rA(r)\,dr\right)\right\| \le s^4\kappa^2\cosh\left(\sqrt{\kappa}s\right) \tag{3.22}$$

$$\left\|\frac{S(s)}{s} - \left(I + \frac{1}{s}\int_0^s (s-r)\,rA(r)\,dr\right)\right\| \le s^4\kappa^2\cosh\left(\sqrt{\kappa}s\right) \tag{3.23}$$

$$\left\|C(s) - \left(I + \int_0^s (s-u)A(u)du\right)\right\| \le s^4\kappa^2\cosh\left(\sqrt{\kappa}s\right). \tag{3.24}$$

Proof. From Equation (3.21)

$$\begin{aligned}\left\|S'(s) - \left(I + \int_0^s rA(r)\,dr\right)\right\| &\le \int_{0\le v\le r\le s}(r-v)\,\kappa^2 v\cosh\left(\sqrt{\kappa}v\right)dvdr \\ &\le s^4\kappa^2\cosh\left(\sqrt{\kappa}s\right),\end{aligned}$$

Integrating this estimate implies

$$\left\| S(s) - Is - \int_0^s (s-r) A(r)\, dr \right\| \le s \cdot s^4 \kappa^2 \cosh\left(\sqrt{\kappa} s\right)$$

which is equivalent to Equation (3.23). Similarly from Equation (3.19) we have

$$\begin{aligned}\left\| C(s) - \left(I + \int_0^s (s-u) A(u) du \right) \right\| &= \left\| \int_{0 \le r \le u \le s} (u-r)(s-u) A(u) A(r) C(r) dr du \right\| \\ &\le s^4 \kappa^2 \cosh\left(\sqrt{\kappa} s\right).\end{aligned}$$

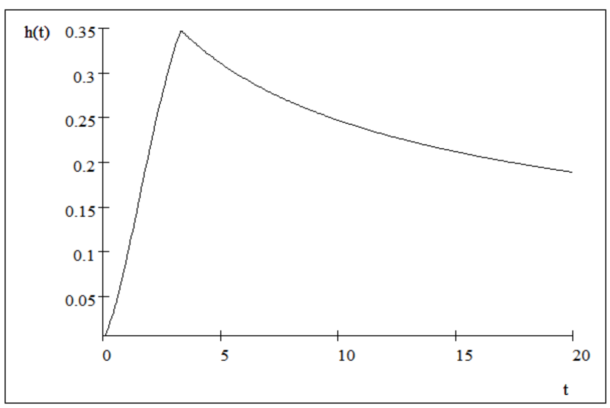
□

Definition 3.14. *Let*

$$h(t) = \frac{1}{t} \ln \psi\left(\sqrt{t}\right) = t^{-1} \ln\left(\min\left(1 + \frac{\cosh\left(\sqrt{t}\right)}{16} t^2, \cosh\left(\sqrt{t}\right) \right) \right)$$

whose graph is given in Figure 3.2, i.e.

$$e^{s^2 h\left(s^2\right)} = \psi(s).$$

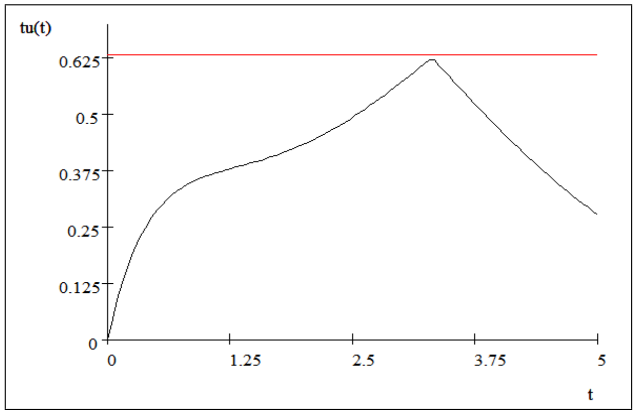


3.2 Here is the graph of $h(t)$.

Let φ be a function, which we will specify shortly, such that $\varphi \ge \psi$. Further let $g(t) := \frac{1}{t} \ln \varphi\left(\sqrt{t}\right)$ so that $\varphi(s) = e^{s^2 g\left(s^2\right)}$ and define

$$u(t) := \frac{\psi^2\left(\sqrt{t}\right)}{\varphi^2\left(\sqrt{t}\right)} = \frac{e^{2th(t)}}{e^{2tg(t)}} = e^{-2t(g(t)-h(t))}.$$

We will specify $\varphi(t)$ by requiring $g(t)$ to be a smooth function such that $g(t) = h(t)$ for t near zero which then rises rapidly to a height of .6 as t increases. For later purposes, let us observe that with this definition $tu(t)$ is bounded by .63 as the graph below indicates.



3.3 Graph of $tu(t)$ and the line $y = .63$. We see that $tu(t) \le .63$.

3.5 Proof of Uniform Integrability

Proposition 3.15. *Suppose N is a d – dimensional standard Gaussian normal random variable and $G(x, \|N\|)$ is a C^1 – function in $x \in (-\varepsilon, \varepsilon)$ such that $G(0, \|N\|) \equiv 0$,*

$$\alpha := \sup\{G(x,t) : x \in (-\varepsilon, \varepsilon) \ \text{and } t \ge 0\} < \frac{1}{2}$$

and there exist constants $C < \infty$ and $\beta < \infty$ such that

$$G_x(x,t) = \frac{\partial G}{\partial x}(x,t) \le C(1+t)^{\beta} \ \text{for } x \in (-\varepsilon, \varepsilon) \ \text{and } t \ge 0.$$

Then

$$\lim_{n\to\infty}\left(\mathbb{E}\left[e^{\|N\|^2 G\left(\frac{1}{n}, \|N\|\right)}\right]\right)^n = e^{\mathbb{E}\left[\|N\|^2 G_x(0,\|N\|)\right]}.$$

Proof. Let

$$f(x) := \mathbb{E}\left[e^{\|N\|^2 G(x, \|N\|)}\right].$$

Since

$$e^{\|N\|^2 G(x,\|N\|)} \leq e^{\alpha\|N\|^2}$$

and

$$G_x\left(x, \|N\|\right) \|N\|^2 e^{\|N\|^2 G(x,\|N\|)} \leq C\left(1+\|N\|\right)^{\beta} \|N\|^2 e^{\alpha\|N\|^2}$$

with the right hand members of these inequalities being integrable functions, it follows that $f(x)$ is a C^1 - function for x near 0 with

$$f'(x) = \mathbb{E}\left[G_x\left(x, \|N\|\right) \|N\|^2 e^{\|N\|^2 G(x,\|N\|)}\right].$$

In particular we have $f(0) = \mathbb{E}\left[e^{\|N\|^2 G(0,\|N\|)}\right] = 1$ and

$$f'(0) = \mathbb{E}\left[G_x\left(0, \|N\|\right) \|N\|^2\right]$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(\mathbb{E}\left[e^{\|N\|^2 G\left(\frac{1}{n},\|N\|\right)}\right]\right)^n &= \lim_{n\to\infty}\left(f\left(n^{-1}\right)\right)^n = \lim_{n\to\infty} e^{n\ln f\left(n^{-1}\right)} = \lim_{x\to 0} e^{\frac{1}{x}\ln f(x)} \\ &= \lim_{x\to 0} e^{f'(x)/f(x)} = e^{f'(0)} = e^{\mathbb{E}\left[G_x(0,\|N\|)\|N\|^2\right]}.\end{aligned}$$

□

Theorem 3.16. *Suppose that* (M, g) *is a Riemannian manifold with non-negative sectional curvatures which are bounded above by* $K = \frac{1}{2d}$*. Then for all* p *sufficiently close to* 1,

$$\sup_n \mathbb{E}\left[\det{}^{p/2}(n\mathcal{Q}^n \circ b_n)\right] < \infty.$$

Proof. For $\omega \in H_{\mathcal{P}}(\mathbb{R}^d)$, from Theorem 3.12,

$$\begin{aligned}A_m(\omega) &= n\int_0^{1/n}\left\|S'_m\left(\omega, s\right)^T S'_m\left(\omega, s\right) - I\right\| ds \\ &\leq \psi\left(\sqrt{K}\left\|\Delta_m\omega\right\|\right) K\left\|\Delta_m\omega\right\|^2 + \frac{1}{3}\psi^2\left(\sqrt{K}\left\|\Delta_m\omega\right\|\right) K^2\left\|\Delta_m\omega\right\|^4,\end{aligned}$$

and if we write $\tau_j(\omega) = \sqrt{K}\,\|\Delta_j \omega\|$,

$$\begin{aligned}
B_m(\omega) &= \sum_{j=m+1}^{n} \|S_m(\omega, \Delta_m s)\|^2 \prod_{k=m+1}^{j-1} \|C_k(\omega, \Delta_k s)\|^2 \, n \int_0^{1/n} \left\|C_j'(\omega, s)\right\|^2 ds \\
&\le \sum_{j=m+1}^{n} (\Delta_m s)^2 \, \psi^2(\tau_m(\omega)) \left[\prod_{k=m+1}^{j-1} \psi^2\left(\tau_k(\omega)\right)\right] K^2 \left(\Delta_j s\right)^2 \left\|\omega_j'\right\|^4 \psi^2(\tau_j(\omega)) \\
&= K^2 \sum_{j=m+1}^{n} \left[\prod_{k=m}^{j} \psi^2\left(\tau_k(\omega)\right)\right] \|\Delta_j \omega\|^4 \\
&= K^2 \sum_{j=m+1}^{n} \left[\prod_{k=m}^{j} \psi^2\left(\sqrt{K}\,\|\Delta_k \omega\|\right)\right] \|\Delta_j \omega\|^4 .
\end{aligned}$$

Hence if we let

$$\alpha(\omega) := \prod_{k=1}^{n} \varphi^2\left(\sqrt{K}\,\|\Delta_k \omega\|\right) = \prod_{k=1}^{n} e^{2K\|\Delta_k \omega\|^2 g\left(K\|\Delta_k\omega\|^2\right)},$$

then

$$\begin{aligned}
A_m(\omega) &\le \alpha(\omega) K \|\Delta_m \omega\|^2 + \frac{1}{3}\alpha(\omega) u\left(K\|\Delta_m \omega\|^2\right) K^2 \|\Delta_m \omega\|^4 \\
&\le \alpha(\omega)\left[K\|\Delta_m \omega\|^2 + \frac{1}{3} u\left(K\|\Delta_m \omega\|^2\right) K^2 \|\Delta_m \omega\|^4\right]
\end{aligned}$$

and

$$\begin{aligned}
B_m(\omega) &\le \alpha(\omega) K^2 \sum_{j=m+1}^{n} \left[\prod_{k=m}^{j} u\left(\sqrt{K}\,\|\Delta_k \omega\|\right)\right] \|\Delta_j \omega\|^4 \\
&\le \alpha(\omega) K^2 \sum_{j=1}^{n} u\left(\sqrt{K}\,\|\Delta_j \omega\|\right) \|\Delta_j \omega\|^4 .
\end{aligned}$$

For $\omega \in (W(\mathbb{R}^d), \mu)$,

$$\Delta_i b_n(\omega) = \Delta_i b(\omega) = b(s_i)(\omega) - b(s_{i-1})(\omega) = \omega(s_i) - \omega(s_{i-1}) = \Delta_i \omega.$$

Therefore, on $W(\mathbb{R}^d)$,

$$\begin{aligned}
&\sum_{m=1}^{n} (A_m + B_m) \circ b_n \\
&\le \alpha \circ b_n \cdot \left[\begin{array}{c} \sum_{m=1}^{n} \left[K\|\Delta_m b\|^2 + \frac{1}{3} u\left(K\|\Delta_m b\|^2\right) K^2 \|\Delta_m b\|^4\right] \\ + K^2 n \sum_{m=1}^{n} u\left(K\|\Delta_m b\|^2\right) \|\Delta_m b\|^4 \end{array}\right] \\
&= \alpha \circ b_n \cdot \sum_{m=1}^{n} \left[K\|\Delta_m b\|^2 + \left(\frac{1}{3} + n\right) K^2 u\left(\sqrt{K}\,\|\Delta_m b\|\right) \|\Delta_m b\|^4\right].
\end{aligned}$$

Now let $x = n^{-1}$ and $N_m := \sqrt{n}\Delta_m b$, so that $\{N_m\}_{m=1}^n$ is a collection of $\mathbb{R}^d$ – valued independent standard normal random variables. With this notation we have

$$\begin{aligned}
&\sum_{m=1}^n (A_m + B_m) \circ b_n \\
&\leq \alpha \circ b_n \cdot \sum_{m=1}^n \left[Kx \|N_m\|^2 + \left(\frac{1}{3} + n\right) K^2 u\left(Kx\|N_m\|^2\right) x^2 \|N_m\|^4 \right] \\
&= \alpha \circ b_n \cdot \sum_{m=1}^n \left[Kx \|N_m\|^2 + K\left(\frac{1}{3}x + 1\right) u\left(Kx\|N_m\|^2\right) Kx \|N_m\|^4 \right] \\
&= \alpha \circ b_n \cdot \sum_{m=1}^n \left[Kx + K\left(\frac{1}{3}x + 1\right) u\left(Kx\|N_m\|^2\right) Kx \|N_m\|^2 \right] \|N_m\|^2
\end{aligned}$$

and therefore, using Equation (3.3),

$$\begin{aligned}
&\det(n\mathcal{Q}^n \circ b_n) = \rho_n \circ \phi \circ b_n \\
&\leq \left(\alpha^{nd} \exp\left(\alpha^{-1} d \sum_{m=1}^n \|(n\mathcal{Q}^n_{mm} - I)\|\right)\right) \circ b_n \\
&\leq \left(\alpha^{nd} \exp\left(\alpha^{-1} d \sum_{m=1}^n (A_m + B_m)\right)\right) \circ b_n \\
&\leq (\alpha \circ b_n)^{nd} \exp\left(d \sum_{m=1}^n \left[Kx + K\left(\frac{1}{3}x + 1\right) u\left(Kx\|N_m\|^2\right) Kx\|N_m\|^2 \right] \|N_m\|^2 \right)
\end{aligned}$$

where

$$\begin{aligned}
(\alpha \circ b_n)^{nd} &= \prod_{m=1}^n \varphi^{2nd}\left(\sqrt{K}\|\Delta_m b\|\right) = \prod_{m=1}^n \varphi^{2nd}\left(\sqrt{Kx}\|N_m\|\right) \\
&= \prod_{m=1}^n e^{2ndK\|\Delta_m b\|^2 g\left(K\|\Delta_m b\|^2\right)} = \prod_{m=1}^n e^{2dK\|N_m\|^2 g\left(Kx\|N_m\|^2\right)}.
\end{aligned}$$

Let $\theta\left(x, \|N\|^2\right) = x + \left(\frac{1}{3}x + 1\right) u\left(Kx\|N\|^2\right) Kx\|N\|^2$. Then,

$$\begin{aligned}
\det(n\mathcal{Q}^n \circ b_n) &\leq \prod_{m=1}^n e^{2dK\|N_m\|^2 g\left(Kx\|N_m\|^2\right)} \exp\left(dK\theta\left(x, \|N_m\|^2\right)\|N_m\|^2\right) \\
&= \prod_{m=1}^n \exp\left(dK\left[2g\left(Kx\|N_m\|^2\right) + \theta\left(x, \|N_m\|^2\right)\right]\|N_m\|^2\right).
\end{aligned}$$

Hence

$$\mathbb{E}\left[\det{}^{p/2}(n\mathcal{Q}^n \circ b_n)\right] = \left[\mathbb{E}\exp\left(dpK\left[g\left(Kx\left\|N\right\|^2\right) + \frac{1}{2}\theta\left(x, \left\|N_m\right\|^2\right)\right]\left\|N\right\|^2\right)\right]^n$$
$$= \left[\mathbb{E}\exp\left(dpK \cdot G\left(x, \left\|N\right\|^2\right)\left\|N\right\|^2\right)\right]^{1/x}$$

where

$$G\left(x, \left\|N\right\|^2\right) = g\left(Kx\left\|N\right\|^2\right) + \frac{x}{2} + \left(\frac{1}{6}x + \frac{1}{2}\right)u\left(Kx\left\|N\right\|^2\right)Kx\left\|N\right\|^2.$$

By our choice of g and hence u, we know

$$g\left(Kx\left\|N\right\|^2\right) + \frac{1}{2}u\left(Kx\left\|N\right\|^2\right)Kx\left\|N\right\|^2 \leq .6 + \frac{1}{2}.63 = .915 < 1.$$

Therefore, for small x, $G\left(x, \left\|N\right\|^2\right) \leq .92 < 1$ for small x. Hence, if p is sufficiently close to 1, we will have

$$dpK \cdot G\left(x, \left\|N\right\|^2\right) = \frac{1}{2}pG\left(x, \left\|N\right\|^2\right) \leq \frac{1}{2}p \cdot 0.92 < \frac{1}{2}.$$

Therefore we may apply Proposition 3.15 to conclude that

$$\limsup_{n\to\infty}\mathbb{E}\left[\det{}^{p/2}(n\mathcal{Q}^n \circ b_n)\right] \leq \lim_{x\to 0}\left[\mathbb{E}\exp\left(dpK \cdot G\left(x, \left\|N\right\|^2\right)\left\|N\right\|^2\right)\right]^{1/x}$$
$$= \exp\left(\mathbb{E}\left[dpK \cdot G_x\left(0, \left\|N\right\|^2\right)\left\|N\right\|^2\right]\right)$$
$$= \exp\left(\mathbb{E}\left[dpK\left(\frac{2K}{16}\left\|N\right\|^2 + \frac{1}{2} + \frac{1}{2}K\left\|N\right\|^2\right)\left\|N\right\|^2\right]\right) < \infty.$$

□

4

Second Formula for ρ_n

Definition 4.1. *For any $\epsilon > 0$ and any partition $\mathcal{P}$ of $[0,1]$, let*

$$\begin{aligned} H_{\mathcal{P}}^{\epsilon}(\mathbb{R}^d) &= \left\{ \omega \in H_{\mathcal{P}}(\mathbb{R}^d) \;\middle|\; \int_{s_{i-1}}^{s_i} \| \omega'(s) \| \, ds < \epsilon \ for \ i = 1, \ldots, n \right\} \\ &= \{\omega \in H_{\mathcal{P}}(\mathbb{R}^d) \mid \| \Delta_i \omega \| < \epsilon \ \forall i\}, \end{aligned}$$

where $\Delta_i \omega = \omega(s_i) - \omega(s_{i-1})$. The second equality holds since $\omega_i'(s)$ is a constant in J_i for each i.

Remark 4.2. *We will now consider all the $\omega \in H_{\mathcal{P}}^{\epsilon}(\mathbb{R}^d)$, with ϵ sufficiently small, specified in the next lemma.*

Lemma 4.3. *There exists an ϵ with $\bigvee_{i=1,\ldots,n} \| \Delta_i \omega \| < \epsilon$ such that for $i = 1, 2, \ldots n$, $S_i(\omega, s)$ is invertible for $0 < s \leq \Delta_i s$.*

Proof. From Equation (3.15) with $\kappa = \frac{K}{(\Delta_i s)^2} \| \Delta_i \omega \|^2$, we see that

$$\| S_i(\omega, s) \| \leq s \left(\frac{\sinh(\sqrt{K} \| \Delta_i \omega \|)}{\sqrt{K} \| \Delta_i \omega \|} \right) \leq s \cosh(\sqrt{K} \| \Delta_i \omega \|), \tag{4.1}$$

where we have used the inequality $\frac{\sinh x}{x} \leq \cosh x$. By Taylor's Theorem with integral remainder,

$$S_i(\omega, s) = sI + \int_0^s (s-u) S_i^{(2)}(\omega, u) \, du = sI + \int_0^s (s-u) A_i(\omega, u) S_i(\omega, u) \, du.$$

Now using Theorem 3.12,

$$\begin{aligned}\left\|\int_0^s (s-u)A_i(\omega,u)S_i(\omega,u)\ du\right\| &\leq \int_0^s (s-u)\frac{K \parallel \Delta_i\omega \parallel^2}{(\Delta_i s)^2} u\cosh(\sqrt{K} \parallel \Delta_i\omega \parallel)\ du \\ &= \left(\int_0^s (s-u)u\ du\right)\frac{K \parallel \Delta_i\omega \parallel^2}{(\Delta_i s)^2}\cosh(\sqrt{K} \parallel \Delta_i\omega \parallel) \\ &= \frac{s^3}{6}\frac{K}{(\Delta_i s)^2} \parallel \Delta_i\omega \parallel^2 \cosh(\sqrt{K} \parallel \Delta_i\omega \parallel) \\ &\leq sK \parallel \Delta_i\omega \parallel^2 \cosh(\sqrt{K} \parallel \Delta_i\omega \parallel).\end{aligned}$$

Hence, if we choose an ϵ such that for $0 < x \leq \epsilon$,

$$Kx^2 \cosh(\sqrt{K}x) < 1,$$

then

$$S_i(\omega,s) = s\left(I + \frac{1}{s}\int_0^s (s-u)A_i(\omega,u)S_i(\omega,u)\ du\right)$$

is invertible for $0 < s \leq \Delta_i s$. □

In order to compute $\lim_{n\to 0}\rho_n$, we will first derive another formula for ρ_n. Define a set of tangent vectors $\{f_{i,a}(\omega,s)\}_{\substack{i=1,2,\ldots,n\\ a=1,2,\ldots,d}}$ on $T_\omega H^\epsilon_{\mathcal{P}}(\mathbb{R}^d)$ such that $f_{i,a}(\omega,s)$ is the solution to Equation (3.1) with the given initial conditions

$$f_{i,a}(\omega,0) = 0,$$

for $j = 1,\ldots,n$,

$$f'_{i,a}(\omega,s_{j-1}) = \begin{cases} e_a, & j = i \\ -F_i(\omega)\ e_a, & j = i+1 \\ 0, & \text{otherwise.}\end{cases}$$

where

$$F_i(\omega) := (S_{i+1}(\omega,\Delta_{i+1}s))^{-1}\, C_{i+1}(\omega,\Delta_{i+1}s)\ S_i(\omega,\Delta_i s), \tag{4.2}$$

where S_i and C_i are as in Definition 3.6. By Lemma 4.3, we can choose an ϵ such that $F_i(\omega)$ is defined on $H^\epsilon_{\mathcal{P}}(\mathbb{R}^d)$. Therefore,

$$f_{i,a}(\omega,s) = \begin{cases} S_i(\omega,s-s_{i-1})\ e_a, & s\in J_i \\ C_{i+1}(\omega,s-s_i)S_i(\omega,\Delta_i s)\ e_a - S_{i+1}(\omega,s-s_i)F_i(\omega)\ e_a, & s\in J_{i+1} \\ 0, & \text{otherwise,}\end{cases}$$

and hence

$$f'_{i,a}(\omega,s)=\begin{cases}F_{ii}(\omega,s)\ e_a, & s\in J_i\\ F_{i+1,i}(\omega,s)\ e_a, & s\in J_{i+1}\\ 0, & \text{otherwise,}\end{cases}$$

where for $i=1,\dots,n$,

$$\begin{aligned}F_{ii}(\omega,s)&=S'_i(\omega,s-s_{i-1}),\\ F_{i+1,i}(\omega,s)&=C'_{i+1}(\omega,s-s_i)S_i(\omega,\Delta_i s)\ -S'_{i+1}(\omega,s-s_i)F_i(\omega).\end{aligned}$$

Proposition 4.4. *For all $\sigma\in H^{\epsilon}_{\mathcal{P}}(M)$, the vectors, $\{X^{f_{i,a}}(\sigma)\}_{\substack{i=1,\dots,n\\ a=1,\dots,d}}$ form a basis for $T_\sigma H^{\epsilon}_{\mathcal{P}}(M)$, where $\sigma=\phi(\omega)$.*

Proof. From Equation (2.2),

$$X^{f_{i,a}}=//_s(\sigma)f_{i,a}(\omega,s),$$

where we have used the identification $\mathbb{R}^d$ with $T_o(M)$. Since $//_s$ is an isometry, it suffices to check that $\{f_{i,a}(\omega,s)\}_{\substack{i=1,\dots,n\\ a=1,\dots,d}}$ is a basis in $H^{\epsilon}_{\mathcal{P}}(\mathbb{R}^d)$. Suppose

$$\sum_{a=1}^{d}\sum_{i=1}^{n}c_{i,a}f_{i,a}(\omega,s)=0.$$

If we let $s=s_j$, $j=1,\dots,n$, we have

$$\sum_{a=1}^{d}c_{j,a}S_j(\omega,\Delta_j s)e_a=0.$$

But from Lemma 4.3, $S_i(\omega,\Delta_j s)$ is invertible and hence we get

$$\sum_{a=1}^{d}c_{j,a}e_a=0.$$

By linear independence of $\{e_a, a=1,\ \dots d\}$, $c_{j,a}=0$. ∎

At this point, we will now assume $\mathcal{P}_n=\left\{s_i=\frac{i}{n}\right\}_{i=0}^{n}$ and $\Delta=1/n$ throughout this section. We may now write

$$f'_{i,a}(\omega,s)=\left[1_{J_i}(s)\,S'_i(\omega,s-s_{i-1})+1_{J_{i+1}}(s)\,V'_{i+1}(\omega,s-s_i)\right]e_a \tag{4.3}$$

where $V_1 \equiv 0 \equiv V_{n+1}$ and for $2 \leq i \leq n$,

$$V_i\left(\omega, s\right) := C_i(\omega, s) S_{i-1}(\omega, \Delta) - S_i(\omega, s) F_{i-1}(\omega)$$

and

$$F_i(\omega) := S_{i+1}(\omega, \Delta)^{-1} C_{i+1}(\omega, \Delta) S_i(\omega, \Delta). \tag{4.4}$$

Observe that

$$V_i\left(\omega, \Delta\right) = C_i\left(\omega, \Delta\right) S_{i-1}\left(\omega, \Delta\right) - S_i\left(\omega, \Delta\right) S_i\left(\omega, \Delta\right)^{-1} C_i\left(\omega, \Delta\right) S_{i-1}\left(\omega, \Delta\right) = 0 \tag{4.5}$$

and that

$$V_i\left(\omega, 0\right) = S_{i-1}\left(\omega, \Delta\right). \tag{4.6}$$

Lemma 4.5. *Continuing the notation above, we have*

$$\begin{aligned}
&\int_0^1 \left\langle f'_{i,a}(\omega, s), f'_{j,c}(\omega, s)\right\rangle \, ds \\
&= \begin{cases}
\int_0^\Delta \langle S'_i(\omega, s) e_a, V'_i\left(\omega, s\right) e_c\rangle \, ds & \text{if } j = i-1 \\
\int_0^\Delta \left[\langle S'_i(\omega, s) e_a, S'_i\left(\omega, s\right) e_c\rangle + \langle V'_{i+1}\left(\omega, s\right) e_a, V'_{i+1}\left(\omega, s\right) e_c\rangle\right] \, ds & \text{if } j = i \\
\int_0^\Delta \langle V'_{i+1}(\omega, s) e_a, S'_{i+1}\left(\omega, s\right) e_c\rangle \, ds & \text{if } j = i+1 \\
0 & \text{otherwise}
\end{cases} \\
&= \begin{cases}
\int_0^\Delta \langle e_a, S'_i(\omega, s)^T V'_i\left(\omega, s\right) e_c\rangle \, ds & \text{if } j = i-1 \\
\int_0^\Delta \left[\langle e_a, S'_i(\omega, s)^T S'_i\left(\omega, s\right) e_c\rangle + \langle e_a, V'_{i+1}\left(\omega, s\right)^T V'_{i+1}\left(\omega, s\right) e_c\rangle\right] \, ds & \text{if } j = i \\
\int_0^\Delta \langle e_a, V'_{i+1}(\omega, s)^T S'_{i+1}\left(\omega, s\right) e_c\rangle \, ds & \text{if } j = i+1 \\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

Let us define the block matrix function of $s \in [0, \Delta]$ by,

$$\mathcal{F}^n_{ij}\left(\omega, s\right) = \delta_{ij} S'_i\left(\omega, s\right) + \delta_{i,j+1} V'_i\left(\omega, s\right)$$

or equivalently as

$$\mathcal{F}^n\left(\omega, s\right) := \begin{bmatrix}
S'_1\left(\omega, s\right) & 0 & \dots & 0 & 0 \\
V'_2\left(\omega, s\right) & S'_2\left(\omega, s\right) & 0 & & 0 \\
0 & V'_3\left(\omega, s\right) & S'_3\left(\omega, s\right) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & 0 & V'_n\left(\omega, s\right) & S'_n\left(\omega, s\right)
\end{bmatrix}. \tag{4.7}$$

where

$$V_i'(\omega, s) := C_i'(\omega, s) S_{i-1}(\omega, \Delta) - S_i'(\omega, s) F_{i-1}(\omega).$$

Remark 4.6. *When* $\omega = 0$ *we have* $S_i(0,s) = sI$, $C_i(0,s) = I$, $F_i(0) := \Delta^{-1} I \Delta = I$, $V_i(0,s) := \Delta I - sI$, *and* $\mathcal{F}^n(0,s) = \mathcal{T}^n$ *where* $\mathcal{T}^n_{ij} = (\delta_{ij} - \delta_{i,j+1}) I$, *i.e.*

$$\mathcal{F}^n(0,s) = \mathcal{T}^n := \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ -I & I & 0 & \dots & 0 \\ 0 & -I & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -I & I \end{bmatrix}. \tag{4.8}$$

It is also worth observing that

$$\mathcal{F}^n(\omega, 0) := \begin{bmatrix} I & 0 & \dots & 0 & 0 \\ -F_1(\omega) & I & 0 & \dots & 0 \\ 0 & -F_2(\omega) & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -F_{n-1}(\omega) & I \end{bmatrix}, \tag{4.9}$$

or equivalently that

$$\mathcal{F}^n_{ki}(\omega, 0) = \delta_{ki} I - \delta_{i,k-1} F_i(\omega). \tag{4.10}$$

Theorem 4.7. *Let* $\mathcal{F}^n(s)_{ij} = \delta_{ij} S_i'(s) + \delta_{i,j+1} V_i'(s)$ (See Equation (4.7).) *Then*

$$\det\left(\left\{\int_0^1 \left\langle f_{i,a}'(\omega,s), f_{j,c}'(\omega,s)\right\rangle \, ds\right\}\right) = \det\left(\int_0^\Delta \left(\mathcal{F}^n(\omega,s)\right)^T \mathcal{F}^n(\omega,s)\, ds\right), \tag{4.11}$$

$$\det\left(\left\{\sum_{k=1}^n \left\langle f_{i,a}'(\omega, s_{k-1}+), f_{j,c}'(\omega, s_{k-1}+)\right\rangle \Delta\right\}\right) = \Delta^{nd}, \tag{4.12}$$

and hence

$$(\rho_n \circ \phi)^2(\omega) = \det\left(\frac{1}{\Delta}\int_0^\Delta (\mathcal{F}^n(\omega,s))^T \mathcal{F}^n(\omega,s)\, ds\right). \tag{4.13}$$

Proof. Since

$$\left(\mathcal{F}^n_{ij}(\omega,s)\right)^T = \delta_{ij} S_i'(\omega,s)^T + \delta_{j,i+1} V_j'(\omega,s)^T,$$

$$\begin{aligned}
&\left[(\mathcal{F}^n(\omega,s))^T\mathcal{F}^n(\omega,s)\right]_{ij}\\
&=\sum_k\left[\delta_{ik}S_i'(\omega,s)^T+\delta_{k,i+1}V_k'(\omega,s)^T\right]\left[\delta_{kj}S_k'(\omega,s)+\delta_{k,j+1}V_k'(\omega,s)\right]\\
&=\delta_{ij}S_i'(\omega,s)^T S_i'(\omega,s)+\delta_{i,j+1}S_i'(\omega,s)^T V_k'(\omega,s)\\
&+\delta_{i+1,j}V_{i+1}'(\omega,s)^T S_{i+1}'(\omega,s)+\delta_{ij}V_{i+1}'(\omega,s)^T V_{i+1}'(\omega,s)\\
&=\delta_{ij}\left[S_i'(\omega,s)^T S_i'(\omega,s)+V_{i+1}'(\omega,s)^T V_{i+1}'(\omega,s)\right]\\
&+\delta_{i,j+1}S_i'(\omega,s)^T V_i'(\omega,s)+\delta_{i+1,j}V_{i+1}'(\omega,s)^T S_{i+1}'(\omega,s).
\end{aligned}$$

So comparing with the results from Lemma 4.5 it follows that

$$\begin{aligned}
&G^1\left(X^{f_{i,a}},X^{f_{j,c}}\right)(\omega)=\int_0^1 g\left(\frac{\nabla X^{f_{i,a}}(\omega,s)}{ds},\frac{\nabla X^{f_{j,c}}(\omega,s)}{ds}\right)\,ds\\
&=\int_0^1\left\langle f_{i,a}'(\omega,s),f_{j,c}'(\omega,s)\right\rangle\,ds=\int_0^1\left\langle e_a,\left[(\mathcal{F}^n(\omega,s))^T\mathcal{F}^n(\omega,s)\right]_{ij}e_c\right\rangle\,ds,
\end{aligned}$$

from which Equation (4.11) follows.

In order to prove Equation (4.12) we begin by observing that

$$\begin{aligned}
f_{i,a}'(\omega,s_{k-1}+)&=\left[1_{J_i}(s_{k-1}+)\,S_i'(\omega,s_{k-1}-s_{i-1})+1_{J_{i+1}}(s_{k-1}+)\,V_{i+1}'(\omega,s_{k-1}-s_i)\right]e_a\\
&=\left[\delta_{ik}S_i'(\omega,0)+\delta_{i,k-1}V_{i+1}'(\omega,0)\right]e_a\\
&=\left[\delta_{ik}I-\delta_{i,k-1}F_i(\omega)\right]e_a=\mathcal{F}_{ik}^n(\omega,0)\,e_a
\end{aligned}$$

where the last equality follows from Equation (4.10). Hence it follows that

$$\begin{aligned}
&G_{\mathcal{P}}^1\left(X^{f_{i,a}},X^{f_{j,c}}\right)(\omega)=\sum_{k=1}^n g\left(\frac{\nabla X^{f_{i,a}}(\omega,s_{k-1}+)}{ds},\frac{\nabla X^{f_{j,c}}(\omega,s_{k-1}+)}{ds}\right)\\
&=\sum_{k=1}^n\langle f_{i,a}'(\omega,s_{k-1}+),f_{j,c}'(\omega,s_{k-1}+)\rangle\Delta=\Delta\sum_{k=1}^n\left\langle\mathcal{F}_{ki}^n(\omega,0)\,e_a,\mathcal{F}_{kj}^n(\omega,0)\,e_c\right\rangle\\
&=\Delta\sum_{k=1}^n\left\langle e_a,(\mathcal{F}_{ik}^n(\omega,0))^T\mathcal{F}_{kj}^n(\omega,0)\,e_c\right\rangle=\Delta\left\langle e_a,\left[(\mathcal{F}^n(\omega,0))^T\mathcal{F}^n(\omega,0)\right]_{ij}e_c\right\rangle
\end{aligned}$$

and therefore

$$\det\left(\left\{\sum_{k=1}^n f_{i,a}'(\omega,s_{k-1}+)\cdot f_{j,c}'(\omega,s_{k-1}+)\Delta\right\}\right)=\det\left(\Delta(\mathcal{F}^n(\omega,0))^T\mathcal{F}^n(\omega,0)\right)=\Delta^{nd}.$$

Equation (4.13) now follows from Equation (2.5). $\square$

4.1 Some Identities

Definition 4.8. *For real square matrix functions,* $A(s)$ *and* $B(s)$, *of* $s \in [0, \Delta]$, *let*

$$\langle A \rangle := \frac{1}{\Delta} \int_0^\Delta A(s)\, ds$$

and

$$\begin{aligned} \mathrm{Cov}(A, B) &= \frac{1}{\Delta} \int_0^\Delta A(s)^T B(s)\, ds - \left(\frac{1}{\Delta} \int_0^\Delta A(s)\, ds \right)^T \left(\frac{1}{\Delta} \int_0^\Delta B(s)\, ds \right) \\ &= \left\langle A^T B \right\rangle - \langle A \rangle^T \langle B \rangle . \end{aligned}$$

Notice that $\langle A \rangle$ *and* $\mathrm{Cov}(A, B)$ *is again a square matrix.*

The following proposition summarizes some basic and easily proved properties of $\mathrm{Cov}(A, B)$.

Proposition 4.9. *The* ***covariance functional,*** Cov, *has the following properties:*

1. $\mathrm{Cov}(A, B)$ *is bilinear in* A *and* B.
2. $\mathrm{Cov}(A, B)$ *may be computed as*

$$\begin{aligned} \mathrm{Cov}(A, B) &= \frac{1}{\Delta} \int_0^\Delta [A(s) - \langle A \rangle]^T [B(s) - \langle B \rangle]\, ds \\ &= \left\langle [A(\cdot) - \langle A \rangle]^T [B(\cdot) - \langle B \rangle] \right\rangle . \end{aligned}$$

3. $\mathrm{Cov}(A, B) = 0$ *if either* $A(s)$ *or* $B(s)$ *is a constant function.*
4. $\mathrm{Cov}(A, A)$ *is always a symmetric non-negative matrix.*

Note : To simplify notation, for the rest of this section we will typically be omitting the argument, ω, from the expressions to follow.

Definition 4.10. *Define* $\mathcal{G}^n(s) := \mathcal{F}^n(s) - \mathcal{T}^n = \mathcal{F}^n(s) - \mathcal{F}^n(0)$, *i.e.*

$$\begin{aligned} \mathcal{G}^n_{ij}(s) &= \delta_{ij} \left[S_i'(s) - I \right] + \delta_{i,j+1} \left[V_i'(s) + I \right] \\ &= \delta_{ij} \left[S_i'(s) - I \right] + \delta_{i,j+1} \left[C_i'(s) S_{i-1}(\Delta) - S_i'(s) F_{i-1} + I \right] . \end{aligned} \tag{4.14}$$

Also let

$$\mathcal{Y}^n := \langle \mathcal{G}^n \rangle = \frac{1}{\Delta} \int_0^\Delta \mathcal{G}^n(s)\, ds. \tag{4.15}$$

Lemma 4.11. *Let* $\mathcal{V}^n_{ij} = \delta_{ij}\frac{1}{\Delta}S_i(\Delta)$, *i.e.*

$$\mathcal{V}^n := \frac{1}{\Delta}\begin{bmatrix} S_1(\Delta) & 0 & \dots & 0 \\ 0 & S_2(\Delta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & S_n(\Delta) \end{bmatrix} \tag{4.16}$$

and $\mathcal{D}^n_{ij} := \delta_{ij}\left[\frac{1}{\Delta}S_i(\Delta) - I\right]$, *i.e.* $\mathcal{D}^n = \mathcal{V}^n - \mathcal{I}^n$. *Then* $\mathcal{Y}^n = \mathcal{T}^n\mathcal{D}^n$ *and*

$$\begin{aligned}\left\langle(\mathcal{F}^n)^T\mathcal{F}^n\right\rangle &= (\mathcal{T}^n + \mathcal{Y}^n)^T(\mathcal{T}^n + \mathcal{Y}^n) + \operatorname{Cov}(\mathcal{G}^n, \mathcal{G}^n) \\ &= (\mathcal{V}^n)^T(\mathcal{T}^n)^T\mathcal{T}^n\mathcal{V}^n + \operatorname{Cov}(\mathcal{G}^n, \mathcal{G}^n).\end{aligned} \tag{4.17}$$

Proof. By the fundamental theorem of calculus along with Equations (4.5) and (4.6) we have

$$\begin{aligned}\mathcal{Y}^n_{ij} &= \frac{1}{\Delta}\int_0^\Delta \left(\delta_{ij}\left[S_i'(s) - I\right] + \delta_{i,j+1}\left[V_i'(s) + I\right]\right) ds \\ &= \frac{1}{\Delta}\left(\delta_{ij}\left[S_i(\Delta) - I\Delta\right] + \delta_{i,j+1}\left[V_i(\Delta) - V_i(0) + \Delta I\right]\right) \\ &= \delta_{ij}\left[\frac{S_i(\Delta)}{\Delta} - I\right] - \delta_{i,j+1}\left[\frac{S_{i-1}(\Delta)}{\Delta} - I\right].\end{aligned}$$

On the other hand

$$\begin{aligned}(\mathcal{T}^n\mathcal{D}^n)_{ij} &= \sum_k \left[\delta_{ik} - \delta_{i,k+1}\right] I\delta_{kj}\left[\frac{1}{\Delta}S_k(\Delta) - I\right] \\ &= \delta_{ij}\left[\frac{1}{\Delta}S_i(\Delta) - I\right] - \delta_{i-1,j}\left[\frac{1}{\Delta}S_{i-1}(\Delta) - I\right] = \mathcal{Y}^n_{ij}.\end{aligned}$$

The second assertion is a consequence of the following simple manipulations,

$$\begin{aligned}\left\langle(\mathcal{F}^n)^T\mathcal{F}^n\right\rangle &= \left\langle(\mathcal{T}^n + \mathcal{G}^n)^T(\mathcal{T}^n + \mathcal{G}^n)\right\rangle \\ &= (\mathcal{T}^n)^T\mathcal{T}^n + (\mathcal{T}^n)^T\mathcal{Y}^n + (\mathcal{Y}^n)^T\mathcal{T}^n + \left\langle(\mathcal{G}^n)^T\mathcal{G}^n\right\rangle \\ &= (\mathcal{T}^n + \mathcal{Y}^n)^T(\mathcal{T}^n + \mathcal{Y}^n) + \left\langle(\mathcal{G}^n)^T\mathcal{G}^n\right\rangle - (\mathcal{Y}^n)^T\mathcal{Y}^n \\ &= (\mathcal{T}^n + \mathcal{Y}^n)^T(\mathcal{T}^n + \mathcal{Y}^n) + \operatorname{Cov}(\mathcal{G}^n, \mathcal{G}^n).\end{aligned}$$

This completes the proof since

$$\mathcal{T}^n + \mathcal{Y}^n = \mathcal{T}^n(\mathcal{I}^n + \mathcal{D}^n) = \mathcal{T}^n\mathcal{V}^n.$$

$\square$

Corollary 4.12. *Letting* $\mathcal{M}^n = \mathrm{Cov}\left(\mathcal{G}^n, \mathcal{G}^n\right)$,

$$\mathcal{S}^n := (\mathcal{T}^n)^{-1} = \begin{bmatrix} I & 0 & \dots & 0 \\ I & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ I & \dots & I & I \end{bmatrix}, \tag{4.18}$$

and $\mathcal{V}^n$ *be as in Equation (4.16), dropping the superscript* n*, we have*

$$\left\langle \mathcal{F}^T \mathcal{F} \right\rangle = \mathcal{V}^T \mathcal{T}^T \left(\mathcal{I} + \mathcal{S}^T \left(\mathcal{V}^T\right)^{-1} \mathcal{M} \mathcal{V}^{-1} \mathcal{S} \right) \mathcal{T} \mathcal{V}$$

and

$$\det\left(\left\langle \mathcal{F}^T \mathcal{F} \right\rangle\right) = \left[\det\left(\mathcal{V}\right)\right]^2 \cdot \det\left(\mathcal{I} + \mathcal{S}^T \left(\mathcal{V}^T\right)^{-1} \mathcal{M} \mathcal{V}^{-1} \mathcal{S} \right). \tag{4.19}$$

4.2 The Key Determinant Formula

Our next goal is to expand out $\mathcal{V}^n$ and $\left((\mathcal{V}^n)^T\right)^{-1} \mathcal{M}^n (\mathcal{V}^n)^{-1}$ in powers of ω. It turns out that we need the expansion of $\mathcal{V}^n$ to order $\| \omega \|^3$ and $\left((\mathcal{V}^n)^T\right)^{-1} \mathcal{M}^n (\mathcal{V}^n)^{-1}$ to order $\| \omega \|^5$.

Notation 4.13. *Recall the definition of* $A_i(\omega, s)$ *in Equation (3.4). We will write* $A_i(0) = A_i(\omega, 0)$ *and also* $A_i(t) = A_i(\omega, t)$ *to simplify the notation. And when we write* $y = O(x)$*, we mean there exists some constant* $C > 0$ *independent of* i*,* n *and* ω *such that* $\| y \| \leq C \| x \|$.

Proposition 4.14. *There exists* $C < \infty$ *such that the following estimates hold for* $0 \leq s \leq \Delta$;

$$\left\| S_i'(s) - \left(I + \frac{1}{2} A_i(0) s^2 \right) \right\| \leq C \| \Delta_i \omega \|^3, \tag{4.20}$$

$$\left\| \frac{S_i(s)}{s} - \left(I + \frac{1}{6} A_i(0) s^2 \right) \right\| \leq C \left\| \Delta_i \omega \right\|^3, \tag{4.21}$$

$$\left\| C_i(s) - \left(I + \frac{1}{2} A_i(0) s^2 \right) \right\| \leq C \left\| \Delta_i \omega \right\|^3 \text{ and} \tag{4.22}$$

$$\left\| C_i'(s) - A_i(0) s \right\| \leq C s^{-1} \left\| \Delta_i \omega \right\|^3. \tag{4.23}$$

In the sequel we will abbreviate these type of estimates by writing

$$S_i'(s) = I + \frac{1}{2}A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right), \tag{4.24}$$

$$\frac{S_i(s)}{s} = I + \frac{1}{6}A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right), \tag{4.25}$$

$$C_i(s) = I + \frac{1}{2}A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right) \text{ and} \tag{4.26}$$

$$C_i'(s) = s^{-1}\left[A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right)\right]. \tag{4.27}$$

Proof. Let $(DA_i)(\omega'(s),\cdot,\cdot) := \frac{d}{ds}\Omega_{u(s_{i-1}+s)}$. Then

$$\begin{aligned}\|\,(DA_i)(\omega'(s),\omega'(s),\cdot)\omega'(s)\,\| &\leq \frac{1}{\Delta^3}\left(\|\,(DA_i)(\Delta_i\omega,\Delta_i\omega,\cdot)\Delta_i\omega\,\|\right)\\ &= O\left(\frac{1}{\Delta^3}\,\|\,\Delta_i\omega\,\|^3\right).\end{aligned}$$

Thus

$$\int_0^s rA_i(r)\,dr = \int_0^s r\left[A_i(0) + \int_0^r A_i'(t)\,dt\right]dr = \frac{1}{2}A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right)$$

and similarly that

$$\begin{aligned}\frac{1}{s}\int_0^s (s-r)\,rA_i(r)\,dr &= \frac{1}{s}\int_0^s (s-r)\,rA_i(0)\,dr + O\left(\|\Delta_i\omega\|^3\right)\\ &= \frac{1}{6}A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right)\end{aligned}$$

and finally

$$\int_0^s (s-u)A_i(u)du = \int_0^s (s-u)A_i(0)du + O\left(\|\Delta_i\omega\|^3\right) = \frac{1}{2}A_i(0)\,s^2 + O\left(\|\Delta_i\omega\|^3\right).$$

Combining these results with the three estimates in Proposition 3.13 with $\kappa := K\,\|\omega_i'\|^2$ proves (4.20) – (4.22). For Equation (4.23), we have

$$\begin{aligned}C_i'(s) &= \int_0^s A_i(r)\,C_i(r)\,dr = \int_0^s \left[A_i(0) + \int_0^r A_i'(t)\,dt\right]\left(I + O\left(\|\Delta_i\omega\|^2\right)\right)dr\\ &= A_i(0)\,s + s^{-1}O\left(\|\Delta_i\omega\|^3\right)\end{aligned}$$

as desired. □

Corollary 4.15. *With $\mathcal{V}^n = \mathcal{I}^n + \mathcal{D}^n$ as in Equation (4.16) we have*

$$\mathcal{V}_{ij}^n = \delta_{ij} \left(I + \frac{1}{6} A_i\,(0)\,\Delta^2 + \eta_i(\Delta) \right) \tag{4.28}$$

where

$$\eta_i(s) := \frac{S_i\,(s)}{s} - \left(I + \frac{1}{6} A_i\,(0)\,s^2 \right) \tag{4.29}$$

and $\| \eta_i(\Delta) \| = O\left(\|\Delta_i \omega\|^3 \right)$.

Lemma 4.16. *The function, F_i in Equation (4.4) satisfies,*

$$F_i = I + \frac{1}{6} A_i(0)\Delta^2 + \frac{1}{3} A_{i+1}\,(0)\,\Delta^2 + O\left(\|\Delta_i \omega\|^3 \vee \|\Delta_{i+1}\omega\|^3 \right). \tag{4.30}$$

Proof. In order to simplify notation, let $a_i := A_i\,(0)\,\Delta^2$ and $\beta_i = \|\Delta_i \omega\|^3$. Then

$$\begin{aligned} F_i &= S_{i+1}\,(\Delta)^{-1} C_{i+1}\,(\Delta)\ \ S_i\,(\Delta) = \left(\frac{S_{i+1}\,(\Delta)}{\Delta} \right)^{-1} C_{i+1}\,(\Delta)\ \ \frac{S_i\,(\Delta)}{\Delta} \\ &= \left(I + \frac{1}{6} a_{i+1} + O\,(\beta_{i+1}) \right)^{-1} \left(I + \frac{1}{2} a_{i+1} + O\,(\beta_{i+1}) \right) \left(I + \frac{1}{6} a_i + O\,(\beta_i) \right) \\ &= \left(I - \frac{1}{6} a_{i+1} + O\,(\beta_{i+1}) \right) \left(I + \frac{1}{2} a_{i+1} + O\,(\beta_{i+1}) \right) \left(I + \frac{1}{6} a_i + O\,(\beta_i) \right) \\ &= I + \left(\frac{1}{2} - \frac{1}{6} \right) a_{i+1} + \frac{1}{6} a_i + O\left(\|\Delta_i \omega\|^3 \right) + O\left(\|\Delta_{i+1} \omega\|^3 \right) \end{aligned}$$

which is equivalent to Equation (4.30). □

Theorem 4.17. *Let*

$$\mathcal{H}_{ij}^n\,(s) := \delta_{ij} A_i\,(0)\,\frac{s^2}{2} + \delta_{i,j+1} \left[A_i\,(0) \left(s\Delta - \frac{s^2}{2} - \frac{\Delta^2}{3} \right) - A_{i-1}\,(0)\,\frac{\Delta^2}{6} \right]$$

and $\Upsilon_{ij}^n\,(s) := \mathcal{G}_{ij}^n\,(s) - \mathcal{H}_{ij}^n\,(s)$. Then $\left\| \Upsilon_{ij}^n\,(s) \right\| = O\left(\|\Delta_{i-1}\omega\|^3 \vee \|\Delta_i \omega\|^3 \right)$ and since $\Upsilon_{ij}^n = 0$ unless $i \in \{j, j+1\}$, it follows that

$$\mathcal{G}^n\,(s) = \mathcal{H}^n\,(s) + \Upsilon^n\,(s) \ \ \text{and} \ \ \|\Upsilon^n\| = O\left(\bigvee_{i=1,\ldots,n} \|\Delta_i \omega\|^3 \right). \tag{4.31}$$

Proof. Let $\gamma_i = \|\Delta_{i-1}\omega\|^3 \vee \|\Delta_i \omega\|^3$. By Proposition 4.14 and Lemma 4.16,

$$\begin{aligned} C_i'(s) S_{i-1}(\Delta) &= \left[A_i\,(0)\,s\Delta + O\left(\|\Delta_i \omega\|^3 \right) \right] \Delta^{-1} S_{i-1}(\Delta) \\ &= \left[A_i\,(0)\,s\Delta + O\left(\|\Delta_i \omega\|^3 \right) \right] \left[I + \frac{1}{6} A_{i-1}\,(0)\,\Delta^2 + O\left(\|\Delta_{i-1}\omega\|^3 \right) \right] \\ &= A_i\,(0)\,s\Delta + O\,(\gamma_i) \end{aligned} \tag{4.32}$$

and

$$
\begin{aligned}
&S_i'(s)F_{i-1}\\
&=\left(I+\frac{1}{2}A_i\left(0\right)s^2+O\left(\|\Delta_i\omega\|^3\right)\right)\left(I+\frac{1}{6}A_{i-1}(0)\Delta^2+\frac{1}{3}A_i\left(0\right)\Delta^2+O\left(\gamma_i\right)\right)\\
&=I+A_{i-1}\left(0\right)\frac{\Delta^2}{6}+A_i\left(0\right)\left(\frac{s^2}{2}+\frac{\Delta^2}{3}\right)+O\left(\gamma_i\right). \qquad (4.33)
\end{aligned}
$$

Combining the last two equations shows,

$$
\begin{aligned}
V_i'\left(s\right)&=C_i'(s)S_{i-1}(\Delta)+I-S_i'(s)F_{i-1}\\
&=A_i\left(0\right)\left(s\Delta-\frac{s^2}{2}-\frac{\Delta^2}{3}\right)-A_{i-1}\left(0\right)\frac{\Delta^2}{6}+O\left(\|\Delta_{i-1}\omega\|^3\vee\|\Delta_i\omega\|^3\right). \qquad (4.34)
\end{aligned}
$$

This equation along with Equations (4.14) and (4.24) shows

$$
\begin{aligned}
\mathcal{G}_{ij}^n\left(s\right)&=\delta_{ij}\left[S_i^{'}\left(s\right)-I\right]+\delta_{i,j+1}\left[V_i'\left(s\right)+I\right]\\
&=\delta_{ij}\left[A_i\left(0\right)\frac{s^2}{2}+O\left(\|\Delta_i\omega\|^3\right)\right]\\
&+\delta_{i,j+1}\left[A_i\left(0\right)\left(s\Delta-\frac{s^2}{2}-\frac{\Delta^2}{3}\right)-A_{i-1}\left(0\right)\frac{\Delta^2}{6}+O\left(\|\Delta_{i-1}\omega\|^3\vee\|\Delta_i\omega\|^3\right)\right]\\
&=\mathcal{H}_{ij}^n\left(s\right)+O\left(\|\Delta_{i-1}\omega\|^3\vee\|\Delta_i\omega\|^3\right).
\end{aligned}
$$

□

Theorem 4.18. *The matrix $\mathcal{M}^n$ of Corollary 4.12 satisfies, $\mathcal{M}^n=\mathcal{C}^n+\tilde{\mathcal{E}}^n$ where $\tilde{\mathcal{E}}^n$ is a tri-block-diagonal matrix such that $\left\|\tilde{\mathcal{E}}^n\right\|=O\left(\bigvee_{i=1,\ldots,n}\|\Delta_i\omega\|^5\right)$ and $\mathcal{C}^n$ is the non-negative tri-block-diagonal matrix given by*

$$
\mathcal{C}_{ij}^n:=\delta_{ij}\frac{\Delta^4}{45}\left[A_i^2\left(0\right)+A_{i+1}^2\left(0\right)\right]+\frac{7}{360}\left[\delta_{i,j+1}A_i^2\left(0\right)+\delta_{i,j-1}A_j^2\left(0\right)\right]\Delta^4, \qquad (4.35)
$$

where $A_{n+1}^2(0):=0$. Equivalently,

$$
\mathcal{C}^n:=\begin{pmatrix}
\frac{1}{45}(a_1^2+a_2^2) & \frac{7}{360}a_2^2 & 0 & \cdots & 0\\
\frac{7}{360}a_2^2 & \frac{1}{45}(a_2^2+a_3^2) & \frac{7}{360}a_3^2 & \ddots & \vdots\\
0 & \frac{7}{360}a_3^2 & \ddots & \ddots & 0\\
\vdots & \ddots & \ddots & \frac{1}{45}(a_{n-1}^2+a_n^2) & \frac{7}{360}a_n^2\\
0 & \cdots & 0 & \frac{7}{360}a_n^2 & \frac{1}{45}a_n^2
\end{pmatrix}
$$

where as before, $a_i:=A_i\left(0\right)\Delta^2$.

Proof. Since $\mathcal{G}^n = \mathcal{H}^n + \Upsilon^n$ with $\|\mathcal{H}^n\| = O\left(\bigvee_{i=1,\ldots,n} \|\Delta_i\omega\|^2\right)$ and $\|\Upsilon^n\| = O\left(\bigvee_{i=1,\ldots,n} \|\Delta_i\omega\|^3\right)$, it follows that

$$\begin{aligned}\mathcal{M}^n &= \mathrm{Cov}\left(\mathcal{G}^n, \mathcal{G}^n\right) = \mathrm{Cov}\left(\mathcal{H}^n + \Upsilon^n, \mathcal{H}^n + \Upsilon^n\right)\\ &= \mathrm{Cov}\left(\mathcal{H}^n, \mathcal{H}^n\right) + \mathrm{Cov}\left(\mathcal{H}^n, \Upsilon^n\right) + \mathrm{Cov}\left(\Upsilon^n, \mathcal{H}^n\right) + \mathrm{Cov}\left(\Upsilon^n, \Upsilon^n\right)\\ &= \mathrm{Cov}\left(\mathcal{H}^n, \mathcal{H}^n\right) + \tilde{\mathcal{E}}^n = \mathcal{C}^n + \tilde{\mathcal{E}}^n\end{aligned}$$

where $\mathcal{C}^n := \mathrm{Cov}\left(\mathcal{H}^n, \mathcal{H}^n\right)$ and

$$\tilde{\mathcal{E}}^n = \mathrm{Cov}\left(\mathcal{H}^n, \Upsilon^n\right) + \mathrm{Cov}\left(\Upsilon^n, \mathcal{H}^n\right) + \mathrm{Cov}\left(\Upsilon^n, \Upsilon^n\right) = O\left(\bigvee_{i=1,\ldots,n} \|\Delta_i\omega\|^5\right).$$

Since $\mathcal{H}^n = \mathcal{K}^n + \mathcal{L}^n$ where

$$\mathcal{K}^n_{ij}\left(s\right) := \delta_{ij} A_i\left(0\right)\frac{s^2}{2} + \delta_{i,j+1} A_i(0)\left[s\Delta - \frac{s^2}{2}\right]$$

and $\mathcal{L}^n$ is the constant (in s) matrix,

$$\mathcal{L}^n_{ij} := -\delta_{i,j+1}\left[A_i\left(0\right)\frac{\Delta^2}{3} + A_{i-1}\left(0\right)\frac{\Delta^2}{6}\right],$$

it follows, using Proposition 4.9, that

$$\mathcal{C}^n = \mathrm{Cov}\left(\mathcal{H}^n, \mathcal{H}^n\right) = \mathrm{Cov}\left(\mathcal{K}^n, \mathcal{K}^n\right).$$

Since

$$\left\langle \mathcal{K}^n_{ij}\right\rangle := \left[\frac{1}{6}\delta_{ij} + \frac{1}{3}\delta_{i,j+1}\right] A_i\left(0\right)\Delta^2,$$

we have

$$\begin{aligned}\hat{\mathcal{K}}^n_{ij}\left(s\right) &:= \mathcal{K}^n_{ij}\left(s\right) - \left\langle \mathcal{K}^n_{ij}\right\rangle\\ &= \delta_{ij} A_i\left(0\right)\left(\frac{s^2}{2} - \frac{\Delta^2}{6}\right) + \delta_{i,j+1} A_i\left(0\right)\left(s\Delta - \frac{s^2}{2} - \frac{1}{3}\Delta^2\right).\end{aligned}$$

Let

$$\alpha = \frac{s^2}{2} - \frac{\Delta^2}{6},\ \beta = s\Delta - \frac{s^2}{2} - \frac{1}{3}\Delta^2.$$

By direct integration, we get

$$\langle \alpha^2 \rangle = \frac{1}{\Delta} \int_0^\Delta \alpha^2 \, ds = \frac{1}{\Delta} \int_0^\Delta \left(\frac{s^2}{2} - \frac{\Delta^2}{6} \right)^2 ds = \frac{\Delta^4}{45},$$
$$\langle \beta^2 \rangle = \frac{1}{\Delta} \int_0^\Delta \beta^2 \, ds = \frac{1}{\Delta} \int_0^\Delta \left(s\Delta - \frac{s^2}{2} - \frac{1}{3}\Delta^2 \right)^2 ds = \frac{\Delta^4}{45} \text{ and}$$
$$\langle \alpha\beta \rangle = \frac{1}{\Delta} \int_0^\Delta \alpha\beta \, ds = \frac{1}{\Delta} \int_0^\Delta \left(\frac{s^2}{2} - \frac{\Delta^2}{6} \right) \left(s\Delta - \frac{s^2}{2} - \frac{1}{3}\Delta^2 \right) ds = \frac{7\Delta^4}{360}.$$

Then we may conclude that

$$\begin{aligned}
\mathcal{C}^n_{ij} &= \operatorname{Cov}(\mathcal{K}^n, \mathcal{K}^n)_{ij} = \sum_k \left\langle \left((\hat{\mathcal{K}}^n)^T \right)_{ik} \hat{\mathcal{K}}^n_{kj} \right\rangle = \sum_k \left\langle \left(\hat{\mathcal{K}}^n_{ki} \right)^T \hat{\mathcal{K}}^n_{kj} \right\rangle \\
&= \sum_k \left\langle \left[\alpha\delta_{ki} + \beta\delta_{k,i+1} \right] A_k(0) \left[\alpha\delta_{kj} + \beta\delta_{k,j+1} \right] A_k(0) \right\rangle \\
&= \frac{\Delta^4}{45} \sum_k \left[\delta_{ki}\delta_{kj} + \delta_{k,i+1}\delta_{k,j+1} \right] A_k^2(0) + \Delta^4 \frac{7}{360} \sum_k \left[\delta_{ki}\delta_{k,j+1} + \delta_{k,i+1}\delta_{kj} \right] A_k^2(0) \\
&= \frac{\Delta^4}{45} \delta_{ij} \left(A_i^2(0) + A_{i+1}^2(0) \right) + \Delta^4 \frac{7}{360} \left[\delta_{i,j+1} + \delta_{i+1,j} \right] A_i^2(0)
\end{aligned}$$

which is equivalent to Equation (4.35). □

Putting together the previous estimates leads to the following key determinant formula.

Theorem 4.19. *As above, let* $\mathcal{S}^n = (\mathcal{T}^n)^{-1}$,

$$\mathcal{V}^n_{ij} = \delta_{ij} + \mathcal{D}^n_{ij} = \delta_{ij} \frac{1}{\Delta} S_i(\Delta) = \delta_{ij} \left(I + \frac{1}{6} A_i(0) \Delta^2 + \eta_i(\Delta) \right),$$
$$\mathcal{C}^n_{ij} = \frac{\Delta^4}{45} \delta_{ij} \left[A_i^2(0) + A_{i+1}^2(0) \right] + \frac{7}{360} \left[\delta_{i,j+1} A_i^2(0) + \delta_{i,j-1} A_j^2(0) \right] \Delta^4$$

where $\| \eta_i(\Delta) \| \le C \| \Delta_i \omega \|^3$, *$C$ is independent of i, n, ω and define*

$$\mathcal{U}^n := (\mathcal{S}^n)^T \mathcal{C}^n \mathcal{S}^n. \tag{4.36}$$

Then there exists a tri-block-diagonal matrix $\mathcal{E}^n = \mathcal{E}^n(\omega)$ *such that*

$$\|\mathcal{E}^n\| = O\left(\bigvee_{i=1,\dots,n} \|\Delta_i \omega\|^5 \right)$$

and

$$\det\left(\left\langle (\mathcal{F}^n)^T \mathcal{F}\right\rangle\right) = \left[\det\left(\mathcal{V}^n\right)\right]^2 \cdot \det\left(\mathcal{I}^n + \mathcal{U}^n\right) \cdot \det\ \left(\mathcal{I}^n + \mathcal{X}^n\right), \tag{4.37}$$

where $\mathcal{X}^n := \left(\mathcal{I}^n + \mathcal{U}^n\right)^{-1} (\mathcal{S}^n)^T \mathcal{E}^n \mathcal{S}^n$.

Proof. To ease the notation, we will drop the superscript n in this proof. From Equation (4.19) of Corollary 4.12, Corollary 4.15, and Theorem 4.18,

$$\det\left(\left\langle \mathcal{F}^T \mathcal{F}\right\rangle\right) = \left[\det\left(\mathcal{V}\right)\right]^2 \cdot \det\left(\mathcal{I} + \mathcal{S}^T \left(\mathcal{V}^T\right)^{-1} \left(\mathcal{C} + \tilde{\mathcal{E}}\right) \mathcal{V}^{-1}\mathcal{S}\right). \tag{4.38}$$

Now write $\mathcal{V}^{-1} = \mathcal{I} + \Psi$, where

$$\Psi := \sum_{n=1}^{\infty} (-1)^n \mathcal{D}^n$$

is a block-diagonal matrix satisfying, $\|\Psi\| = O\left(\bigvee_{i=1,\ldots,n} \|\Delta_i\omega\|^2\right)$. Hence for $\epsilon > 0$ sufficiently small, if $\bigvee_{i=1,\ldots,n} \|\Delta_i\omega\|^2 \leq \epsilon$, then $\|\mathcal{V}^{-1}\| \leq 2$. Furthermore,

$$\begin{aligned}\left(\mathcal{V}^T\right)^{-1} \left(\mathcal{C} + \tilde{\mathcal{E}}\right) \mathcal{V}^{-1} &= \left(\mathcal{V}^T\right)^{-1} \mathcal{C}\mathcal{V}^{-1} + \left(\mathcal{V}^T\right)^{-1} \tilde{\mathcal{E}}\mathcal{V}^{-1} \\ &= \left(\mathcal{I} + \Psi^T\right) \mathcal{C}\left(\mathcal{I} + \Psi\right) + \left(\mathcal{V}^T\right)^{-1} \tilde{\mathcal{E}}\mathcal{V}^{-1} = \mathcal{C} + \mathcal{E}\end{aligned}$$

where $\mathcal{E}$ is the tri-block-diagonal matrix defined by

$$\mathcal{E} = \mathcal{C}\Psi + \Psi^T\mathcal{C} + \Psi^T\mathcal{C}\Psi + \left(\mathcal{V}^T\right)^{-1} \tilde{\mathcal{E}}\mathcal{V}^{-1}$$

and $\mathcal{E}$ satisfies the norm estimate, $\|\mathcal{E}\| = O\left(\bigvee_{i=1,\ldots,n} \| \Delta_i\omega \|^5\right)$. Putting these results back into Equation (4.38) shows

$$\begin{aligned}\det\left(\left\langle \mathcal{F}^T \mathcal{F}\right\rangle\right) &= \left[\det\left(\mathcal{V}\right)\right]^2 \cdot \det\left(\mathcal{I} + \mathcal{S}^T \left[\mathcal{C} + \mathcal{E}\right] \mathcal{S}\right) \\ &= \left[\det\left(\mathcal{V}\right)\right]^2 \cdot \det\left(\mathcal{I} + \mathcal{U} + \mathcal{S}^T\mathcal{E}\mathcal{S}\right) \\ &= \left[\det\left(\mathcal{V}\right)\right]^2 \cdot \det\left(\left(\mathcal{I} + \mathcal{U}\right)\left(\mathcal{I} + \left(\mathcal{I} + \mathcal{U}\right)^{-1} \mathcal{S}^T\mathcal{E}\mathcal{S}\right)\right)\end{aligned}$$

from which the desired result follows. □

5

Convergence of $\{\rho_n\}_{n=1}^{\infty}$ in μ-measure

Recall that $H^{\epsilon}_{\mathcal{P}}(\mathbb{R}^d)$ was defined as

$$H^{\epsilon}_{\mathcal{P}}(\mathbb{R}^d) = \left\{\omega \in H_{\mathcal{P}}(\mathbb{R}^d) : \parallel \Delta_i \omega \parallel < \epsilon \ \forall i\right\}$$

where $\Delta_i \omega = \omega(s_i) - \omega(s_{i-1})$. Note that $\mathcal{V}^n$ and $\mathcal{X}^n$ are only defined on $H^{\epsilon}_{\mathcal{P}}(\mathbb{R}^d)$ for some ϵ satisfying Lemma 4.3.

Notation 5.1. *By abuse of notation, we will now write $A_i(\omega, 0) = A_i(\omega)$ from now on. This should not be confused with the notation $A_i(t)$ as described in Notation 4.13, where $t \in [0, \frac{1}{n})$ and we suppressed the argument ω.*

Unless stated otherwise, we will only consider equally spaced partitions $\mathcal{P}_n = \{0 < \frac{1}{n} < \cdots < \frac{n}{n} = 1\}$. By Theorem 4.7 and 4.19, ρ_n has been written as a product of 3 terms, namely $[\det(\mathcal{V}^n)]^2$, $\det(\mathcal{I}^n + \mathcal{U}^n)$ and $\det(\mathcal{I}^n + \mathcal{X}^n)$ on $H^{\epsilon}_{\mathcal{P}_n}(\mathbb{R}^d)$. We will now show that the determinant given in Theorem 4.19 has a limit as $|\mathcal{P}_n| \to 0$. The limit for each term will be computed in this order.

The following theorem is the Wong-Zakai type approximation theorem for solutions to Stratonovich stochastic differential equations. This theorem is a special case of Theorem 5.7.3 and Example 5.7.4 in [33]. Theorems of this type have a long history starting with Wong and Zakai [46, 47]. The following version maybe found in [18].

Theorem 5.2. *Let $f : \mathbb{R}^d \times \mathbb{R}^n \to End(\mathbb{R}^d, \mathbb{R}^n)$ and $f_0 : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be twice differentiable with bounded continuous derivatives. Let $\xi_0 \in \mathbb{R}^n$ and $\mathcal{P}$ be any partition of $[0,1]$. Further, let b and $b_{\mathcal{P}}$ be as in Definition 2.8 and $\xi_{\mathcal{P}}(s)$ denote the solution to the ordinary differential equation*

$$\xi'_{\mathcal{P}}(s) = f(\xi_{\mathcal{P}}(s))b'_{\mathcal{P}}(s) + f_0(\xi_{\mathcal{P}}(s)),\ \xi_{\mathcal{P}}(0) = \xi_0$$

and ξ denote the solution to the Stratonovich stochastic differential equation,

$$d\xi(s) = f(\xi(s))\delta b(s) + f_0(\xi(s))ds,\ \xi(0) = \xi_0.$$

Then for any $\alpha \in (0, \frac{1}{2}),\ p \in [1,\infty)$, there is a constant $C(p,\alpha) < \infty$ depending only on f, f_0 and M so that

$$\mathbb{E}[\sup_{s\leq 1} \parallel \xi_{\mathcal{P}}(s) - \xi(s) \parallel^p] \leq C(p,\alpha)|\mathcal{P}|^{\alpha p}.$$

Definition 5.3. *1. Let u be the solution to the Stratonovich stochastic differential equation*

$$\delta u = \mathcal{H}_u u \delta b,\ u(0) = u_0. \tag{5.1}$$

Hence u maybe viewed as μ-a.s. defined function from $W(\mathbb{R}^d) \to W(O(M))$.

2. Let $\tilde{\phi} := \pi \circ u : W(\mathbb{R}^d) \to W(M)$. This map will be called the stochastic development map.

The following facts will be stated without any proof. See [17].

Remark 5.4. *1. Using Theorem 5.2, one may show that $\tilde{\phi}$ is a "stochastic extension" of ϕ, i.e., $\tilde{\phi} = \lim_{|\mathcal{P}|\to 0} \phi \circ b_{\mathcal{P}}$. Moreover, the law of $\tilde{\phi}$ (i.e., $\mu\tilde{\phi}^{-1}$) is the Wiener measure ν on $W(M)$.*

2. One can prove that $u_s = \widetilde{//}_s(\tilde{\phi})$ where $\widetilde{//}$ is stochastic parallel transport defined in Definition 1.12.

3. The law of u under μ on $W(\mathbb{R}^d)$ and the law of $\widetilde{//}$ under ν are equal.

Definition 5.5. *Let $0 < \alpha < \frac{1}{2}$ and define for $\mathcal{P}_n = \{0 = \frac{1}{n} < \cdots < \frac{n}{n} = 1\}$,*

$$W_\alpha(\mathbb{R}^d) = \left\{ \omega \in W(\mathbb{R}^d) \middle| \; \exists \; n_0(\omega) \; s.t. \; \forall n \geq n_0, \bigvee_{i=1,\ldots,n} \| \Delta_i \omega \| \leq n^{-\alpha} \right\}.$$

Note that

$$\left(W_\alpha(\mathbb{R}^d)\right)^c = \left\{ \omega \in W(\mathbb{R}^d) \middle| \bigvee_{i=1,\ldots,n} \| \Delta_i \omega \| > n^{-\alpha} \; i.o. \right\}.$$

Lemma 5.6. *Let $0 < \alpha < \frac{1}{2}$, then*

$$\mu\left(W_\alpha(\mathbb{R}^d)\right) = 1.$$

Proof. Now

$$\begin{aligned} &\mu\left(\left\{ \omega \in W(\mathbb{R}^d) \middle| \bigvee_{i=1,\ldots,n} \| \Delta_i \omega \| > n^{-\alpha} \right\}\right) \\ &\leq \frac{1}{n^{-\alpha p}} \mathbb{E}\left[\bigvee_{i=1,\ldots,n} \| \Delta_i b \|^p \right] \\ &\leq \frac{1}{n^{-\alpha p}} \mathbb{E}\left[\sum_{i=1}^n \| \Delta_i b \|^p \right] = \frac{C_p}{n^{-\alpha p}} \left(\frac{1}{n}\right)^{\frac{p}{2}} n, \end{aligned}$$

where $C_p > 0$ is some constant. Therefore,

$$\begin{aligned} &\sum_{n=1}^\infty \mu\left(\left\{ \omega \in W(\mathbb{R}^d) \middle| \bigvee_{i=1,\ldots,n} \| \Delta_i \omega \| > n^{-\alpha} \right\}\right) \\ &\leq C_p \sum_{n=1}^\infty \left(\frac{1}{n}\right)^{\frac{p}{2} - 1 - \alpha p} = C_p \sum_{n=1}^\infty \left(\frac{1}{n}\right)^{(\frac{1}{2} - \alpha)p - 1} < \infty, \end{aligned}$$

if p is sufficiently large. Hence by Borel Cantelli Lemma,

$$\mu\left(\left\{ \omega \in W(\mathbb{R}^d) \middle| \bigvee_{i=1,\ldots,n} \| \Delta_i \omega \| > n^{-\alpha} \; i.o. \right\}\right) = 0,$$

and hence the proof. □

Notation 5.7. *Throughout the next few sections, let $b(s) : W(\mathbb{R}^d) \to \mathbb{R}^d$ be the projection map, $b(s)(\omega) = \omega(s)$ for all $0 \leq s \leq 1$ and $\omega \in W(\mathbb{R}^d)$. Note that when $W(\mathbb{R}^d)$ is equipped with Wiener measure, μ, $\{b(s) : 0 \leq s \leq 1\}$ is a Brownian motion. We further let $\phi_n = \phi \circ b_n$ and $u_n = //(\phi_n)$.*

5.1 Convergence of $\det(\mathcal{V}^n)$

Lemma 5.8.

$$\sum_{i=1}^{n}\left(\text{tr } A_i(b_n)\Delta_i s^2 - \left(-\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}\right)\right) \longrightarrow 0$$

$\mu - a.s.$ as $n \to \infty$.

Proof. Note that

$$\Delta_i b_n = b_n(s_i) - b_n(s_{i-1}) = b(s_i) - b(s_{i-1}) = \Delta_i b$$

and

$$A_i(b_n)\Delta_i s^2 = \Omega_{u_n(s_{i-1})}\left((b_n)'_i, \cdot\right)(b_n)'_i \Delta_i s^2 = \Omega_{u_n(s_{i-1})}(\Delta_i b_n, \cdot)\Delta_i b_n.$$

Let $Ric_{u_n(s)} := \sum_{i=1}^{d} \Omega_{u_n(s)}\left(\cdot, e_i\right) e_i$. Using the symmetry of Ric,

$$\begin{aligned}\sum_{i=1}^{n} \text{tr } A_i(b_n)\Delta_i s^2 &= \sum_{i=1}^{n} \text{tr } \Omega_{u_n(s_{i-1})}(\Delta_i b_n, \cdot)\Delta_i b_n = -\sum_{i=1}^{n}\left\langle Ric_{u_n(s_{i-1})}\Delta_i b_n, \Delta_i b_n\right\rangle \\ &= -\sum_{i=1}^{n}\left\langle Ric_{u_n(s_{i-1})}\Delta_i b, \Delta_i b\right\rangle.\end{aligned}$$

By Ito's formula,

$$\begin{aligned}&\left\langle Ric_{u_n(s_{i-1})}\Delta_i b, \Delta_i b\right\rangle \\ &= \left\langle Ric_{u_n(s_{i-1})}(b(s_i) - b(s_{i-1})), b(s_i) - b(s_{i-1})\right\rangle \\ &= 2\int_{s_{i-1}}^{s_i}\left\langle Ric_{u_n(s_{i-1})}((b(s) - b(s_{i-1})), db(s)\right\rangle + \int_{s_{i-1}}^{s_i} \text{tr } Ric_{u_n(s_{i-1})}\, ds \\ &= 2\int_{s_{i-1}}^{s_i}\left\langle Ric_{u_n(s_{i-1})}(b(s) - b(s_{i-1})), db(s)\right\rangle + Scal(\phi_n(s_{i-1}))\Delta_i s.\end{aligned}$$

Thus

$$\sum_{i=1}^{n}\left(\text{tr } A_i(b_n)\Delta_i s^2 + Scal(\phi_n(s_{i-1}))\frac{1}{n}\right) = -2\int_{s_{i-1}}^{s_i}\left\langle Ric_{u_n(s_{i-1})}(b(s) - b(s_{i-1})), db(s)\right\rangle.$$

Define

$$\xi_n = 2\sum_{i=1}^{n}\int_{s_{i-1}}^{s_i}\left\langle Ric_{u_n(s_{i-1})}(b(s)-b(s_{i-1}),db(s))\right\rangle$$
$$= 2\sum_{i=1}^{n}\left(\int_0^1\left\langle 1_{J_i}(s)Ric_{u_n(s_{i-1})}(b(s)-b(s_{i-1})),db(s)\right\rangle\right)$$
$$= 2\int_0^1\left\langle M_n(s),\ db(s)\right\rangle,$$

where $M_n(s)=\sum_{i=1}^{n}1_{J_i}(s)Ric_{u_n(s_{i-1})}(b(s)-b(s_{i-1}))$. To complete the proof, it suffices to show that ξ_n converges to 0 μ-a.s.. We will make use of Burkholder's Inequality,

$$\mathbb{E}\Big[\sup_{0\le t\le T}|M_t|^p\Big]\le C\mathbb{E}\Big[\langle M\rangle_T^{\frac{p}{2}}\Big],$$

where C is some constant. Thus applying this with $T=1$ and $p=4$, we have

$$\mathbb{E}\ |\xi_n|^4 \le \mathbb{E}\left[\left(C\int_0^1\ \|\ M_n(s)\ \|^2\ \ ds\right)^2\right]$$
$$= \mathbb{E}\left[C^2\int_0^1\ \|\ M_n(s)\ \|^4\ \ ds\right]\ (By\ Jensen's\ Inequality)$$
$$= \mathbb{E}\left[C^2\int_0^1\sum_{i=1}^{n}1_{J_i}(s)\ \|\ Ric_{u_n(s_{i-1})}(b(s)-b(s_{i-1}))\ \|^4\ \ ds\right]$$
$$= O\left(\int_0^1\sum_{i=1}^{n}1_{J_i}(s)\mathbb{E}\ \|\ (b(s)-b(s_{i-1}))\ \|^4\ \ ds\right)$$
$$= O\left(\int_0^1\sum_{i=1}^{n}1_{J_i}(s)(s-s_{i-1})^2\ ds\right) = O\left(\frac{1}{n^2}\right).$$

Hence $\sum_{n=1}^{\infty}\mathbb{E}|\xi_n|^4<\infty$ and thus

$$\xi_n \longrightarrow 0\ \mu-a.s..$$

□

Proposition 5.9.

$$\sum_{i=1}^{n}Scal(\phi_n(s_{i-1}))\frac{1}{n}\longrightarrow\int_0^1 Scal(\tilde{\phi}(s))\ ds$$

μ-a.s. as $n\to\infty$.

Proof. Note that we can write

$$\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n} \quad = \quad \int_0^1 \sum_{i=1}^{n} 1_{J_i}(s) Scal(\phi_n(s_{i-1}))\ ds.$$

Since $\phi_n = \phi \circ b_n \to \tilde{\phi}$ in the sup norm μ-a.s. as $n \to \infty$ and $Scal$ is a continuous function, thus

$$\begin{aligned}
&\sum_{i=1}^{n} 1_{J_i}(s) Scal(\phi_n(s_{i-1})) \\
&= \sum_{i=1}^{n} 1_{J_i}(s) \left[Scal(\phi_n(s_{i-1})) - Scal(\tilde{\phi}(s_{i-1})) \right] + \sum_{i=1}^{n} 1_{J_i}(s) Scal(\tilde{\phi}(s_{i-1})) \\
&\longrightarrow Scal(\tilde{\phi}(s)).
\end{aligned}$$

Hence we can apply the dominated convergence theorem to obtain

$$\lim_{n\to\infty} \int_0^1 \sum_{i=1}^{n} 1_{J_i}(s) Scal(\phi_n(s_{i-1}))\ ds = \int_0^1 Scal(\tilde{\phi}(s))\ ds$$

$\mu - a.s.$. ∎

Lemma 5.10.

$$\det(\mathcal{V}^n \circ b_n) = \prod_{i=1}^{n} \det \left[I + \frac{1}{6} A_i(b_n)\Delta_i s^2 + \eta_i(b_n) \right] \longrightarrow e^{-\frac{1}{6}\int_0^1 Scal(\tilde{\phi}(s))\ ds}$$

$\mu - a.s.$ as $n \to \infty$.

Proof. It suffices to consider on $W_\alpha(\mathbb{R}^d)$ with $\frac{1}{3} < \alpha < \frac{1}{2}$, since $\mu\left(W_\alpha(\mathbb{R}^d)\right) = 1$. Let $\omega \in W_\alpha(\mathbb{R}^d)$. By Definition 5.5, for $n \geq n_0(\omega)$,

$$\bigvee_{i=1,\dots,n} \| \Delta_i \omega \| < n^{-\alpha}.$$

However, we can always choose $n_0(\omega) > (\frac{1}{\epsilon})^{1/\alpha}$. Therefore $b_n(\omega) \in H^\epsilon_{\mathcal{P}_n}(\mathbb{R}^d)$ for all $n \geq n_0(\omega)$. Define

$$\zeta_i^n = \begin{cases} \frac{1}{6} A_i(b_n)\Delta_i s^2 + \eta_i(b_n), & n \geq n_0(\cdot); \\ 0, & n < n_0(\cdot), \end{cases}$$

where $\eta_i(b_n)$ was defined in Equation (4.29) and $\| \eta_i(b_n) \| = O\left(\bigvee_{i=1,\ldots,n} \| \Delta_i b \|^3\right)$. Hence we can extend the definition of $\det(\mathcal{V}^n \circ b_n)$ to be on $W(\mathbb{R}^d)$ μ-a.s. by letting

$$\det(\mathcal{V}^n \circ b_n) = \prod_{i=1}^{n} \det(I + \zeta_i^n).$$

Now using the perturbation formula in Equation (B.2) of the Appendix with $r = 2$,

$$\begin{aligned} \det\left[I + \zeta_i^n\right] &= \exp\left[\text{tr } \zeta_i^n + R_2(\zeta_i^n)\right] \\ &= \exp\left[\frac{1}{6}\text{tr } A_i(b_n)\Delta_i s^2 + \psi_i^n\right] \end{aligned}$$

where

$$\begin{aligned} \psi_i^n &:= \text{tr } \eta_i(b_n) + R_2(\zeta_i^n) \\ &= \text{tr } \eta_i(b_n) + \sum_{k=2}^{\infty}(-1)^{k+1}\text{tr } \left(\frac{1}{6}A_i(b_n)\Delta_i s^2 + \eta_i(b_n)\right)^k. \end{aligned}$$

Using Equation (B.3),

$$|R_2(\zeta_i^n)| \leq \frac{d \| \zeta_i^n \|^2}{1 - \| \zeta_i^n \|} = O\left(\bigvee_{i=1,\ldots,n} \| \Delta_i b \|^4\right)$$

and hence

$$|\psi_i^n| \leq |\text{tr } \eta_i(b_n)| + |R_2(\zeta_i^n)| = O\left(\bigvee_{i=1,\ldots,n} \| \Delta_i b \|^3\right).$$

Since we choose $\alpha > \frac{1}{3}$, on $W_\alpha(\mathbb{R}^d)$,

$$\begin{aligned} \left\| \sum_{i=1}^{n} \psi_i^n \right\| &= O\left(n \bigvee_{i=1,\ldots,n} \| \Delta_i b \|^3\right) \\ &= O\left(n \cdot n^{-3\alpha}\right) = O\left(n^{1-3\alpha}\right) \\ &\longrightarrow 0 \end{aligned}$$

as $n \to \infty$. Together with Lemma 5.8,

$$\begin{aligned} &\prod_{i=1}^{n} \det\left[I + \frac{1}{6}A_i(b_n)\Delta_i s^2 + \eta_i(b_n)\right] - e^{-\frac{1}{6}\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}} \\ &= e^{\frac{1}{6}\sum_{i=1}^{n}(\text{tr } A_i(b_n)\Delta_i s^2 + \psi_i^n)} - e^{-\frac{1}{6}\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}} \\ &= e^{-\frac{1}{6}\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}}\left[e^{\frac{1}{6}\sum_{i=1}^{n} \text{ tr } A_i(b_n)\Delta_i s^2 - \left(-\frac{1}{6}\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}\right) + \sum_{i=1}^{n}\psi_i^n} - 1\right] \\ &\longrightarrow 0 \end{aligned}$$

μ-a.s. as $n \to \infty$. Finally,

$$\begin{aligned}
&\prod_{i=1}^{n} \det\left[I + \frac{1}{6}A_i(b_n)\Delta_i s^2 + \eta_i(b_n)\right] - e^{-\frac{1}{6}\int_0^1 Scal(\tilde{\phi}(s))\ ds} \\
&= \prod_{i=1}^{n} \det\left[I + \frac{1}{6}A_i(b_n)\Delta_i s^2 + \eta_i(b_n)\right] - e^{-\frac{1}{6}\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}} \\
&+ e^{-\frac{1}{6}\sum_{i=1}^{n} Scal(\phi_n(s_{i-1}))\frac{1}{n}} - e^{-\frac{1}{6}\int_0^1 Scal(\tilde{\phi}(s))\ ds} \\
&\longrightarrow 0
\end{aligned}$$

μ-a.s. as $n \to \infty$. □

5.2 Convergence of $\det(\mathcal{I}^n + \mathcal{U}^n)$

Recall, from Equation (4.36), that $\mathcal{U}^n := (\mathcal{S}^n)^T \mathcal{C}^n \mathcal{S}^n$ where

$$\mathcal{C}^n_{ij} \circ b_n = \delta_{ij}\left(\frac{1}{45}\left(A_i^2(b_n)\Delta_i s^4 + A_{i+1}^2(b_n)\Delta_{i+1}s^4\right)\right) + 1_{\{|j-i|=1\}}\frac{7}{360}A^2_{i\vee j}(b_n)\Delta_{i\vee j}s^4$$

with $A^2_{n+1}(b_n)\Delta_{n+1}s^4 := 0$ as in Equation (4.35). In order to compute the limit of $\det(\mathcal{I}^n + \mathcal{U}^n)$ as $n \to \infty$, it will be necessary to compute

$$\mathrm{Tr}\ \left([\mathcal{U}^n]^k\right) = \mathrm{Tr}\ \left([(\mathcal{S}^n)^T \mathcal{C}^n \mathcal{S}^n]^k\right) = \mathrm{Tr}\ [(\mathcal{S}^n)^T \mathcal{C}^n (\mathcal{B}^n \mathcal{C}^n)^{k-1} \mathcal{S}^n] = \mathrm{Tr}\ \left([\mathcal{B}^n \mathcal{C}^n]^k\right),$$

where $\mathcal{B}^n : = \mathcal{S}^n(\mathcal{S}^n)^T$.

Lemma 5.11. *The matrix, $\mathcal{B}^n : = \mathcal{S}^n(\mathcal{S}^n)^T$, is given by*

$$\mathcal{B}^n_{lm} = (l \wedge m)I \text{ for } l, m = 1, 2, \ldots, n. \tag{5.2}$$

Moreover, $\mathcal{B}^n$ and $\mathcal{S}^n$ satisfy the norm estimates,

$$\| \mathcal{S}^n \| = O(n) \text{ and } \| \mathcal{B}^n \| = O(n^2).$$

Proof. By definition,

$$\begin{aligned}
\mathcal{B}^n_{lm} &= \sum_{k=1}^{n} \mathcal{S}^n_{lk}(\mathcal{S}^n_{mk})^T = \sum_{k=1}^{n} 1_{l\geq k}1_{m\geq k}I \\
&= \sum_{k=1}^{n} 1_{\{l\wedge m \geq k\}}I = (l \wedge m)I.
\end{aligned}$$

Let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{nd}$ be the eigenvalues of $\mathcal{B}^n$. Since it is a positive definite matrix, we have that $\| \mathcal{B}^n \| = \lambda_1$. Therefore, we have

$$\| \mathcal{B}^n \| = \lambda_1 \leq \sum_{i=1}^{nd} \lambda_i = \text{Tr } \mathcal{B}^n = \sum_{l=1}^{n} l \text{ tr } I = d\frac{n(n+1)}{2} = O(n^2).$$

Since

$$\begin{aligned} \| \mathcal{B}^n \| &= \sup_{\|v\|=1} \langle \mathcal{B}^n v, v \rangle = \sup_{\|v\|=1} \langle (\mathcal{S}^n)^T v, (\mathcal{S}^n)^T v \rangle \\ &= \left\| (\mathcal{S}^n)^T \right\|^2 = \|\mathcal{S}^n\|^2, \end{aligned}$$

it follows that $\| \mathcal{S}^n \| = O(n)$. □

The following definition will be useful in describing the limiting behavior of $A_m^2(b_n)\Delta_m s^4$ as $n \to \infty$.

Definition 5.12. *Define* $\Gamma : O(M) \longrightarrow \mathbb{R}^{d\times d}$ *(the* $d \times d$ *matrices) by*

$$\Gamma(v) = \sum_{i,j=1}^{d} \Big(\Omega_v(e_i, \Omega_v(e_i, \cdot)e_j)e_j + \Omega_v(e_i, \Omega_v(e_j, \cdot)e_i)e_j + \Omega_v(e_i, \Omega_v(e_j, \cdot)e_j)e_i \Big)$$

where $\{e_i\}_{i=1,2,\ldots,d}$ *is any orthonormal basis for* T_oM*.*

Notation 5.13. *For* $a_1,\ a_2,\ a_3,\ a_4 \in \mathbb{R}^d$ *and* $1 \leq m \leq n$*, let* $\widetilde{T}_m^n\left(a_1 \otimes a_2 \otimes a_3 \otimes a_4\right) : W\left(\mathbb{R}^d\right) \to \mathbb{R}^{d\times d}$ *be defined by*

$$\widetilde{T}_m^n\left(a_1 \otimes a_2 \otimes a_3 \otimes a_4\right) := \Omega_{u_n(s_{m-1})}(a_1, \Omega_{u_n(s_{m-1})}(a_2, \cdot)a_3)a_4 \in \mathbb{R}^{d\times d}.$$

If τ *is a permutation of* $\{1,2,3,4\}$*, let*

$$(\tau\widetilde{T}_m^n)(a_1 \otimes a_2 \otimes a_3 \otimes a_4) = \widetilde{T}_m^n(a_{\tau(1)} \otimes a_{\tau(2)} \otimes a_{\tau(3)} \otimes a_{\tau(4)})$$

and

$$(\tau\widetilde{T}_{m,i}^n)(a_1, a_2) = (\tau\widetilde{T}_m^n)(e_i \otimes e_i \otimes a_1 \otimes a_2)$$

where $\{e_i\}_{i=1}^d$ *is the standard orthonormal basis for* $\mathbb{R}^d$*.*

Let $\underline{0} = 0$ and for $t \in (s_{m-1}, s_m]$, let $\underline{t} = s_{m-1}$, $\Delta := s_m - s_{m-1} = 1/n$

$$\begin{aligned}\Delta b(t) &= b(t) - b(\underline{t}),\\ \Delta_m b &= \Delta b\,(s_m) = b(s_m) - b(s_{m-1}),\\ \Delta_m b^i &= \Delta_m b \cdot e_i, \text{ and}\\ \Delta b(t)^{\otimes^3} \otimes db(t) &= \Delta b(t) \otimes \Delta b(t) \otimes \Delta b(t) \otimes db(t).\end{aligned}$$

Lemma 5.14. *Let $\{e_i\}_{i=1,2,\ldots,d}$ be an orthonormal basis for T_oM, $\Delta = \frac{1}{n}$ and $m \in \{1, 2, \ldots, n\}$. Then*

$$A^2_m(b_n)\Delta^4 - \Delta^2\Gamma(u_n(s_{m-1})) = \epsilon^n_m \tag{5.3}$$

where

$$\epsilon^n_m := \frac{1}{6}\int_{s_{m-1}}^{s_m} \sum_\tau (\tau\widetilde{T}^n_m)(\Delta b(s)^{\otimes^3} \otimes db(s)) + \frac{1}{2}\int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau\widetilde{T}^n_{m,i})(\Delta b(s) \otimes db(s)).$$

The above sums range over permutations, τ, of $\{1, 2, 3, 4\}$ and $i = 1, 2, \ldots, d$.

Proof. By definition, we have

$$\begin{aligned}A^2_m(b_n)\Delta^4 &= \Omega_{u_n(s_{m-1})}(\Delta_m b, \Omega_{u_n(s_{m-1})}(\Delta_m b, \cdot)\Delta_m b)\Delta_m b\\ &= \widetilde{T}^n_m\,(\Delta_m b \otimes \Delta_m b \otimes \Delta_m b \otimes \Delta_m b)\\ &= \sum_{i,j,k,l} \widetilde{T}^n_m(e_i \otimes e_j \otimes e_k \otimes e_l)\Delta_m b^i \Delta_m b^j \Delta_m b^k \Delta_m b^l.\end{aligned}$$

By Ito's formula,

$$\begin{aligned}&\widetilde{T}^n_m(\Delta_m b \otimes \Delta_m b \otimes \Delta_m b \otimes \Delta_m b)\\ &= \int_{s_{m-1}}^{s_m} \sum_{j=1}^4 \widetilde{T}^n_m(\Delta b(s) \otimes \ldots \underbrace{db(s)}_{j^{\text{th}}\text{ - spot}} \ldots \otimes \Delta b(s))\\ &\quad + \frac{1}{4}\int_{s_{m-1}}^{s_m} \sum_{i=1}^d \sum_\tau (\tau\widetilde{T}^n_m)(e_i \otimes e_i \otimes \Delta b(s) \otimes \Delta b(s))ds\\ &= \frac{1}{6}\int_{s_{m-1}}^{s_m} \sum_\tau (\tau\widetilde{T}^n_m)(\Delta b(s)^{\otimes^3} \otimes db(s)) + \frac{1}{4}\int_{s_{m-1}}^{s_m} \sum_{i,\tau} (\tau\widetilde{T}^n_{m,i})(\Delta b(s) \otimes \Delta b(s))\; ds.\end{aligned} \tag{5.4}$$

Now we claim that

$$\int_{s_{m-1}}^{s_m} \Delta b(s) \otimes \Delta b(s)\ ds = \int_{s_{m-1}}^{s_m} (s_m - t)\Delta b(t) \vee db(t) + \frac{\Delta^2}{2} \sum_{i=1}^{d} e_i \otimes e_i, \tag{5.5}$$

where $a_1 \vee a_2 = a_1 \otimes a_2 + a_2 \otimes a_1$. Let

$$\begin{aligned} W_s &= \int_{s_{m-1}}^{s} (s-t)\Delta b(t) \vee db(t) + \frac{(s-s_{m-1})^2}{2} \sum_{i=1}^{d} e_i \otimes e_i \qquad (5.6) \\ &= s\int_{s_{m-1}}^{s} \Delta b(t) \vee db(t) - \int_{s_{m-1}}^{s} t\Delta b(t) \vee db(t) + \frac{(s-s_{m-1})^2}{2} \sum_{i=1}^{d} e_i \otimes e_i \end{aligned}$$

and observe that W_{s_m} is equal to the right side of Equation (5.5). Since,

$$\begin{aligned} dW_s &= \left(\int_{s_{m-1}}^{s} \Delta b(t) \vee db(t)\right) ds + s\Delta b(s) \vee db(s) - s\Delta b(s) \vee db(s) \\ &\quad + \left((s-s_{m-1}) \sum_{i=1} e_i \otimes e_i\right) ds \\ &= \left(\int_{s_{m-1}}^{s} \Delta b(t) \vee db(t) + (s-s_{m-1}) \sum_{i=1} e_i \otimes e_i\right)\ ds \\ &= (\Delta b(s) \otimes \Delta b(s))\ ds, \end{aligned}$$

it follows that

$$W_{s_m} = \int_{s_{m-1}}^{s_m} \Delta b(s) \otimes \Delta b(s)\ ds$$

which verifies Equation (5.5).

From Equation (5.5), we have

$$\begin{aligned} &\int_{s_{m-1}}^{s_m} \sum_{i,\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \otimes \Delta b(s))\ ds \\ &= \int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \vee db(s)) + \frac{\Delta^2}{2} \sum_{i,\tau} \sum_{j=1}^{d} (\tau \widetilde{T}^n_{m,i})(e_j \otimes e_j) \\ &= \int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \vee db(s)) + \frac{\Delta^2}{2} \sum_{\tau} \sum_{i,j=1}^{d} (\tau \widetilde{T}^n_{m})(e_i \otimes e_i \otimes e_j \otimes e_j) \\ &= 2\int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \otimes db(s)) + \frac{\Delta^2}{2} 8\Gamma(u(s_{m-1})) \\ &= 2\int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \otimes db(s)) + 4\Delta^2\Gamma(u_n(s_{m-1})). \end{aligned}$$

When we sum over all permutations of $\{1, 2, 3, 4\}$, we will have $4! = 24$ terms. However, Γ is a sum of 3 terms, hence we will end up with 8 copies of Γ, which explains the factor 8 in the second last equality. Combining this equation with Equation (5.4) proves Equation (5.3). $\square$

Lemma 5.15. *There exists* $C < \infty$ *such that for any* $\lambda > 0$, $n \in \mathbb{N}$, *and* $\{c_m\}_{m=1}^n \subset [0, \lambda]$,

$$\mathbb{E}\left\|\sum_{m=1}^n c_m \epsilon_m^n\right\|^2 \le C\frac{\lambda^2}{n^3},$$

which as usual we abbreviate as

$$\mathbb{E}\left\|\sum_{m=1}^n c_m \epsilon_m^n\right\|^2 = O\left(\frac{\lambda^2}{n^3}\right).$$

Proof. We may write

$$\sum_{m=1}^n c_m \epsilon_m^n = \xi_n + \chi_n$$

where

$$\xi_n = \frac{1}{6}\sum_{m=1}^n c_m \int_{s_{m-1}}^{s_m} \sum_\tau (\tau \widetilde{T}_m^n)(\Delta b(s)^{\otimes^3} \otimes db(s))$$

and

$$\chi_n = \frac{1}{2}\sum_{m=1}^n c_m \int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau \widetilde{T}_{m,i}^n)(\Delta b(s) \otimes db(s)).$$

Using the isometry property of the Ito integral, we have

$$\begin{aligned}
\mathbb{E}\,(6\xi_n)^2 &= \mathbb{E}\left\|\sum_{m=1}^n c_m \int_{s_{m-1}}^{s_m} \sum_\tau (\tau \widetilde{T}_m^n)(\Delta b(s)^{\otimes^3} \otimes db(s))\right\|^2 \\
&= \mathbb{E}\left\|\int_0^1 \sum_{m=1}^n c_m 1_{J_m}(s) \sum_\tau (\tau \widetilde{T}_m^n)(\Delta b(s)^{\otimes^3} \otimes db(s))\right\|^2 \\
&= O\left(\mathbb{E}\int_0^1 \sum_{m=1}^n c_m^2 1_{J_m}(s) \parallel \Delta b(s) \parallel^6 \; ds\right) \\
&= O\left(\int_0^1 \sum_{m=1}^n c_m^2 1_{J_m}(s)(s - s_{m-1})^3 \; ds\right) \\
&= O\left(\sum_{m=1}^n c_m^2 \frac{1}{n^4}\right) = O\left(\frac{\lambda^2}{n^3}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}\left(2\chi_n\right)^2 =&\mathbb{E}\left\|\sum_{m=1}^{n} c_m \int_{s_{m-1}}^{s_m} (s_m - s) \sum_{i,\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \otimes db(s))\Big)\right\|^2 \\ &= \mathbb{E}\left\|\int_0^1 \sum_{m=1}^{n} c_m 1_{J_m(s)}(s_m - s) \sum_{i=1}^{d}\sum_{\tau} (\tau \widetilde{T}^n_{m,i})(\Delta b(s) \otimes db(s))\right\|^2 \\ &= O\left(\mathbb{E}\int_0^1 \sum_{m=1}^{n} c_m^2 1_{J_m}(s)(s_m - s)^2 \parallel \Delta b(s) \parallel^2 \ ds\right) \\ &= O\left(\int_0^1 \sum_{m=1}^{n} c_m^2 1_{J_m}(s)(s_m - s)^2 (s - s_{m-1}) \ ds\right) \\ &= O\left(\sum_{m=1}^{n} c_m^2 \frac{1}{n^4}\right) = O\left(\frac{\lambda^2}{n^3}\right).\end{aligned}$$

Thus

$$\left(\mathbb{E}\left\|\sum_{m=1}^{n} c_m \epsilon^n_m\right\|^2\right)^{\frac{1}{2}} \leq \left(\mathbb{E}\left\|\xi_n\right\|^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\left\|\chi_n\right\|^2\right)^{\frac{1}{2}} = O\left(\sqrt{\frac{\lambda^2}{n^3}}\right)$$

and hence

$$\mathbb{E}\left\|\sum_{m=1}^{n} c_m \epsilon^n_m\right\|^2 = O\left(\frac{\lambda^2}{n^3}\right).$$

□

Definition 5.16. *For $l, m = 1, 2, \ldots n$, define Λ^n_{lm} and Φ^n_{lm} by*

$$\begin{aligned}\Lambda^n_{lm} = &\Big(l \wedge (m-1)\Big)\frac{7}{360}\Gamma(u_n(s_{m-1}))\frac{1}{n^2} + \Big(l \wedge m\Big)\frac{1}{45}\Gamma(u_n(s_m))\frac{1}{n^2}1_{\{m<n\}} \\ &+ \Big(l \wedge m\Big)\frac{1}{45}\Gamma(u_n(s_{m-1}))\frac{1}{n^2} + \Big(l \wedge (m+1)\Big)\frac{7}{360}\Gamma(u_n(s_m))\frac{1}{n^2}1_{\{m<n\}}\end{aligned}$$

and

$$\begin{aligned}\Phi^n_{lm} = &\Big(l \wedge (m-1)\Big)\frac{7}{360}\epsilon^n_m + \Big(l \wedge m\Big)\frac{1}{45}\epsilon^n_{m+1}1_{\{m<n\}} \\ &+ \Big(l \wedge m\Big)\frac{1}{45}\epsilon^n_m + \Big(l \wedge (m+1)\Big)\frac{7}{360}\epsilon^n_{m+1}1_{\{m<n\}}.\end{aligned}$$

With this notation along with Equation (5.2) we have

$$\begin{aligned}(\mathcal{B}^n\mathcal{C}^n)_{lm} \circ b_n = \left(\sum_{k=1}^{n} \mathcal{B}^n_{lk}\mathcal{C}^n_{km}\right) \circ b_n = \left(\sum_{k=1}^{n} (l \wedge k)\mathcal{C}^n_{km}\right) \circ b_n \\ = \Lambda^n_{lm} + \Phi^n_{lm}. \qquad (5.7)\end{aligned}$$

Notation 5.17. *For any $k \in \mathbb{N}$ and $d \times d$ matrices $\{M_k\}_{l=1}^{k}$, let*

$$\Big[\prod_{l=1}^{k}\Big] M_k := M_1 M_2 \cdots M_k.$$

Theorem 5.18. *For each $k = 1, 2, \ldots,$ define $\gamma_k^n : W\left(\mathbb{R}^d\right) \to \mathbb{R}$ by*

$$\gamma_k^n := \sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} \operatorname{tr}\ \Big[\prod_{l=1}^{k}\Big] \Lambda_{r_i, r_{i+1}}^n,$$

where $r_{k+1} = r_1$. Then

$$\lim_{n\to\infty} \mathbb{E}\left|\operatorname{Tr}\ ([\mathcal{U}^n \circ b_n]^k) - \gamma_k^n\right| = 0$$

and in particular

$$\operatorname{Tr}\ \left([\mathcal{U}^n \circ b_n]^k\right) - \gamma_k^n \longrightarrow 0 \text{ (in } \mu - \text{measure) as } n \to \infty.$$

Proof. For fixed $k \in \mathbb{N}$,

$$\begin{aligned}
\operatorname{Tr}\ \left([\mathcal{B}^n\mathcal{C}^n \circ b_n]^k\right) &= \left(\operatorname{tr}\ \sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} (\mathcal{B}^n\mathcal{C}^n)_{r_1,r_2}(\mathcal{B}^n\mathcal{C}^n)_{r_2,r_3}\cdots(\mathcal{B}^n\mathcal{C}^n)_{r_k,r_1}\right) \circ b_n \\
&= \operatorname{tr}\ \sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} \Lambda_{r_1,r_2}^n \Lambda_{r_2,r_3}^n \cdots \Lambda_{r_k,r_1}^n + \Xi^n \\
&= \gamma_k^n + \Xi^n,
\end{aligned}$$

where Ξ^n consists of a finite sum of terms of the form

$$\operatorname{tr}\ \sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} c_{r_1,r_2} c_{r_2,r_3} \ldots c_{r_k,r_1} \theta_{r_1} \theta_{r_2} \ldots \theta_{r_k},$$

with

$$\theta_{r_i} \in \left\{\frac{1}{n^2}\Gamma(u_n(s_{r_i-1})), \frac{1}{n^2}\Gamma(u_n(s_{r_i})), \epsilon_{r_i}^n, \epsilon_{r_i+1}^n\right\},$$

$1 \le c_{r_i,r_{i+1}} \le n$ for $i \le n$ and for at least one r_i, $\theta_{r_i} = \epsilon_{r_i}^n$. Since trace is invariant under cyclic permutation, we can assume that $\theta_{r_k} = \epsilon_{r_k}^n$. To finish the proof it suffices to show, $\lim_{n\to\infty} \mathbb{E} \parallel \Xi^n \parallel = 0$ and for this it suffices to show,

$$\lim_{n\to\infty} \mathbb{E}\left\|\sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} c_{r_1,r_2} c_{r_2,r_3} \ldots c_{r_k,r_1} \theta_{r_1} \theta_{r_2} \ldots \theta_{r_k}\right\| = 0.$$

Let $r = (r_1, \dots r_k)$ and write

$$c(r) = \prod_{i=1}^{k} c_{r_i, r_{i+1}},$$

where by convention, $r_{k+1} := r_1$. Then

$$\begin{aligned} T_n &:= \left(\mathbb{E} \left\| \sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} c(r)\theta_{r_1}\theta_{r_2}\dots\theta_{r_{k-1}}\epsilon_{r_k} \right\| \right)^2 \\ &= \left(\mathbb{E} \left\| \sum_{r_{k-1}=1}^{n} \cdots \sum_{r_1=1}^{n} \theta_{r_1}\theta_{r_2}\dots\theta_{r_{k-1}} \sum_{r_k=1}^{n} c(r)\epsilon_{r_k} \right\| \right)^2 \\ &\le \left(\mathbb{E} \sum_{r_{k-1}=1}^{n} \cdots \sum_{r_1=1}^{n} \left(\|\theta_{r_1}\theta_{r_2}\dots\theta_{r_{k-1}}\| \left\| \sum_{r_k=1}^{n} c(r)\epsilon_{r_k} \right\| \right) \right)^2 \\ &\le \mathbb{E} \left(\sum_{r_1,\dots,r_{k-1}} \prod_{i=1}^{k-1} \|\theta_{r_i}\|^2 \right) \mathbb{E} \left(\sum_{r_1,\dots,r_{k-1}} \left\| \sum_{r_k=1}^{n} c(r)\epsilon_{r_k} \right\|^2 \right), \end{aligned} \tag{5.8}$$

where the last inequality is a consequence of the Cauchy Schwartz inequality. Let $C_1 := \sup_{v \in O(M)} \|\Gamma(v)\| < \infty$, then

$$\left\| \frac{1}{n^2}\Gamma(u(s_{i-1})) \right\|^{2(k-1)} \le C_1^{2(k-1)} \left(\frac{1}{n}\right)^{4(k-1)} = O\left(\frac{1}{n^{4(k-1)}}\right).$$

Using Equation (5.3), we observe that there exists a constant C_2 such that

$$\mathbb{E} \ \| \epsilon_m^n \|^{2(k-1)} \le \mathbb{E} \ \left[C_2 \left(\| \Delta_m b \|^4 + \frac{1}{n^2} \right) \right]^{2(k-1)} = O\left(\frac{1}{n^{4(k-1)}}\right).$$

Thus we can find a constant $C(k)$ such that

$$\mathbb{E} \ \| \theta_{r_i} \|^{2(k-1)} \le C \left(\frac{1}{n}\right)^{4(k-1)}.$$

By Holder's Inequality,

$$\begin{aligned} \mathbb{E} \left(\sum_{r_{k-1}=1}^{n} \cdots \sum_{r_1=1}^{n} \prod_{i=1}^{k-1} \|\theta_{r_i}\|^2 \right) &\le \sum_{r_{k-1}=1}^{n} \cdots \sum_{r_1=1}^{n} \prod_{i=1}^{k-1} \left(\mathbb{E} \ \| \theta_{r_i} \|^{2(k-1)} \right)^{\frac{1}{k-1}} \\ = \sum_{r_{k-1}=1}^{n} \cdots \sum_{r_1=1}^{n} C \left(\frac{1}{n}\right)^{4(k-1)} &= C \left(\frac{1}{n}\right)^{3(k-1)}. \end{aligned} \tag{5.9}$$

Note that $\sup_r |c(r)| \leq n^k$. Using Lemma 5.15, with $\lambda = n^k$, we have

$$\mathbb{E}\left(\sum_{r_{k-1}=1}^{n}\cdots\sum_{r_1=1}^{n}\left\|\sum_{r_k=1}^{n}c(r)\epsilon_{r_k}\right\|^2\right) = \sum_{r_{k-1}=1}^{n}\cdots\sum_{r_1=1}^{n}\mathbb{E}\left\|\sum_{r_k=1}^{n}c(r)\epsilon_{r_k}\right\|^2$$
$$= \sum_{r_{k-1}=1}^{n}\cdots\sum_{r_1=1}^{n}O\left(\frac{n^{2k}}{n^3}\right) = n^{k-1}O\left(\frac{n^{2k}}{n^3}\right) = n^{3(k-1)}O\left(\frac{1}{n}\right). \tag{5.10}$$

Hence using Equations (5.9) and (5.10), we have from Equation (5.8),

$$T_n \leq C\frac{1}{n^{3(k-1)}}n^{3(k-1)}\frac{1}{n} = O\left(\frac{1}{n}\right).$$

Thus putting all together, we have

$$\mathbb{E} \parallel \Xi^n \parallel \leq d \cdot \sqrt{T_n} = O\left(\frac{1}{\sqrt{n}}\right).$$

□

Definition 5.19. *Let*

$$h(s,t) = (s \wedge t)\Gamma(u(t))$$

and for $k \in \mathbb{N}$, *let*

$$\gamma_k = \left(\frac{1}{12}\right)^k \int_{s_k=0}^{1}\cdots\int_{s_1=0}^{1}\operatorname{tr}\left[\prod_{l=1}^{k}\right]h(s_l, s_{l+1})\ ds_1\ldots ds_k,$$

where again by convention, $s_{k+1} := s_1$.

Proposition 5.20. *Continuing the notation in the above definition,*

$$\gamma_k^n \longrightarrow \gamma_k$$

μ*-a.s. as* $n \to \infty$.

Proof. To begin with, let us write

$$\begin{aligned} s &= (s_1, s_2, \ldots s_k) \in [0,1)^k, \\ r &= (r_1, r_2, \ldots, r_k) \in W_n^k := \{1, 2, \ldots n\}^k, \\ ds &= ds_1 \ldots ds_k \end{aligned}$$

and

$$H(s) = \Big[\prod_{l=1}^{k}\Big] h(s_l, s_{l+1})$$

where $s_{k+1} = s_1$. Denote $r_l^- := r_l - 1$ and

$$V_n(r) = V_n(r_1, r_2, \ldots, r_k) := J_{r_1} \times J_{r_2} \cdots J_{r_k} = \left(\frac{r_1^-}{n}, \frac{r_1}{n}\right] \times \left(\frac{r_2^-}{n}, \frac{r_2}{n}\right] \cdots \left(\frac{r_k^-}{n}, \frac{r_k}{n}\right].$$

With this new notation and $r_{k+1} = r_1$,

$$\begin{aligned}
\gamma_k^n - \gamma_k &= \sum_{r \in W_n^k} \mathrm{tr}\ \Big[\prod_{l=1}^{k}\Big] \Lambda_{r_i, r_{i+1}}^n - \left(\frac{1}{12}\right)^k \int_{s \in [0,1)^k} \mathrm{tr}\ H(s)\ ds \\
&= \sum_{r \in W_n^k} \left(\mathrm{tr}\ \Big[\prod_{l=1}^{k}\Big] \Lambda_{r_i, r_{i+1}}^n - \left(\frac{1}{12}\right)^k \int_{s \in V_n(r)} \mathrm{tr}\ H(s)\ ds \right) \\
&= \sum_{r \in W_n^k} \int_{s \in V_n(r)} \left(n^k \mathrm{tr}\ \Big[\prod_{l=1}^{k}\Big] \Lambda_{r_i, r_{i+1}}^n - \left(\frac{1}{12}\right)^k \mathrm{tr}\ H(s) \right)\ ds \\
&= \int_{s \in [0,1)^k} \sum_{r \in W_n^k} 1_{V_n(r)}(s) \left(\mathrm{tr}\ \Big[\prod_{l=1}^{k}\Big] n\Lambda_{r_i, r_{i+1}}^n - \left(\frac{1}{12}\right)^k \mathrm{tr}\ H(s) \right)\ ds.
\end{aligned}$$

Now $u_n \to u$ in the sup norm μ-a.s. and $\Gamma(\cdot)$ is continuous, thus

$$\sum_{l,m=1}^{n} 1_{J_l \times J_m}(\tau, t) \left(\frac{l}{n} \wedge \frac{m-1}{n}\right) \Gamma\left(u_n\left(\frac{m-1}{n}\right)\right) - h(\tau, t) \longrightarrow 0 \text{ and}$$

$$\sum_{l,m=1}^{n} 1_{J_l \times J_m}(\tau, t) \left(\frac{l}{n} \wedge \frac{m}{n}\right) \Gamma\left(u_n\left(\frac{m-1}{n}\right)\right) - h(\tau, t) \longrightarrow 0$$

μ-a.s. as $n \to \infty$. For $(\tau, t) \in [0,1)^2$, by using Definition 5.16,

$$\begin{aligned}
&\sum_{l,m=1}^{n} 1_{J_l \times J_m}(\tau, t) n\Lambda_{lm}^n - \frac{1}{12} h(\tau, t) \\
&= \sum_{l,m=1}^{n} 1_{J_l \times J_m}(\tau, t) \left\{ \begin{array}{l} \frac{7}{360}\left(\frac{l}{n} \wedge \frac{m-1}{n}\right) \Gamma\left(u_n\left(\frac{m-1}{n}\right)\right) + \frac{1}{45}\left(\frac{l}{n} \wedge \frac{m}{n}\right) \Gamma\left(u_n\left(\frac{m}{n}\right)\right) 1_{\{m<n\}} \\ + \frac{1}{45}\left(\frac{l}{n} \wedge \frac{m}{n}\right) \Gamma\left(u_n\left(\frac{m-1}{n}\right)\right) + \frac{7}{360}\left(\frac{l}{n} \wedge \frac{m+1}{n}\right) \Gamma\left(u_n\left(\frac{m}{n}\right)\right) 1_{\{m<n\}} \end{array} \right\} \\
&\qquad - \frac{1}{12} h(\tau, t) \\
&\longrightarrow 0
\end{aligned}$$

μ-a.s. as $n \to \infty$. And since taking trace and products are continuous operations, for $s \in [0,1)^k$, we hence have

$$g_n(s) := \sum_{r \in W_n^k} 1_{V_n(r)}(s) \left(\text{tr } \left[\prod_{l=1}^{k} \right] n\Lambda^n_{r_i, r_{i+1}} - \left(\frac{1}{12}\right)^k \text{tr } H(s) \right)$$
$$= \sum_{r \in W_n^k} 1_{V_n(r)}(s) \left(\text{tr } \left[\prod_{l=1}^{k} \right] n\Lambda^n_{r_i, r_{i+1}} - \text{tr } \left[\prod_{l=1}^{k} \right] \frac{1}{12} h(s_l, s_{l+1}) \right) \longrightarrow 0$$

μ-a.s. as $n \to \infty$. Thus we can now apply dominated convergence theorem and this gives us

$$\gamma_k^n - \gamma_k = \int_{s \in [0,1)^k} g_n(s)\ ds \longrightarrow 0$$

as $n \to \infty$. □

Remark 5.21. *If S_v is the sectional curvature of the manifold and $\sup_{v \in O(M)} \parallel S_v \parallel < \frac{3}{17d}$, then for any orthonormal frame $\{e_i\}_{i=1}^d \subseteq T_o M$,*

$$\sup_{v \in O(M)} \parallel \Omega_v(e_i, \cdot) e_j \parallel < 2/d,$$

and hence

$$\sup_{v \in O(M)} \parallel \Gamma(v) \parallel < 3d^2 \cdot 4/d^2 = 12.$$

See Definition D.1 and Proposition D.2. Let $\kappa = \sup_{v \in O(M)} \parallel \Gamma(v) \parallel /12$. Then $\kappa < 1$ and hence

$$|\gamma_k^n| \le \sum_{r_k=1}^{n} \cdots \sum_{r_1=1}^{n} d \left(\frac{1}{12n}\right)^k \sup_{v \in O(M)} \parallel \Gamma(v) \parallel^k = d\kappa^k.$$

A standard result in probability states that $x_n \to x$ in μ-measure iff for every subsequence of $\{x_n,\ n \ge 1\}$ has itself a subsequence converging μ-a.s. to x. Hence for any continuous function f, $f(x_n)$ converges to $f(x)$ in μ-measure. In the next proof, we will be using this result without any comment.

Theorem 5.22. *If $\sup_{v \in O(M)} \parallel S_v \parallel < \frac{3}{17d}$, then*

$$\det[(\mathcal{I}^n + \mathcal{U}^n) \circ b_n] \longrightarrow e^{\gamma}$$

in μ-measure as $n \to \infty$, where γ is defined as

$$\gamma := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \gamma_k.$$

Proof. To ease the notation, in this proof only, we will write $\mathcal{U}^n = \mathcal{U}^n \circ b_n$ and $\mathcal{I}^n = \mathcal{I}^n \circ b_n$. By Remark 5.21, $|\gamma_k^n| \leq d\kappa^k$, $\kappa = \sup_{v \in O(M)} \parallel \Gamma(v) \parallel /12 < 1$. Observe that $\mathcal{U}^n$ satisfies the hypothesis in Lemma B.1. Thus we can apply the formula in Equation (B.1). Hence

$$\det(\mathcal{I}^n + \mathcal{U}^n) = \exp(\Psi_r + R_{r+1}),$$

where $\Psi_r = \sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \mathrm{Tr}\ ([\mathcal{U}^n]^k)$ and $|R_{r+1}| \leq \frac{1}{r+1} \mathrm{Tr}\ ([\mathcal{U}^n]^{r+1})$. Therefore

$$\begin{aligned} &\left| \det(\mathcal{I}^n + \mathcal{U}^n) - \exp\left(\sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \gamma_k^n \right) \right| \\ &= \exp\left(\sum_{k=1}^{r} \frac{(-1)^{k+1}}{k} \gamma_k^n \right) \left| \exp\left(\sum_{k=1}^{r} \frac{(-1)^{k+1}}{k} \left(\mathrm{Tr}\ ([\mathcal{U}^n]^k) - \gamma_k^n \right) + R_{r+1} \right) - 1 \right| \\ &\leq \exp(d\kappa/(1-\kappa)) \left| \exp\left(\sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \left(\mathrm{Tr}\ ([\mathcal{U}^n]^k) - \gamma_k^n \right) + R_{r+1} \right) - 1 \right|. \end{aligned}$$

The last inequality follows from

$$\exp\left(\sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \gamma_k^n \right) \leq \exp\left(\sum_{k=1}^{r} \frac{1}{k} d\kappa^k \right) < \exp(d\kappa/(1-\kappa)).$$

Now by Theorem 5.18, we have

$$\mathrm{Tr}\ ([\mathcal{U}^n]^k) - \gamma_k^n \to 0$$

in μ-measure as $n \to \infty$. Together with $\parallel \gamma_k^n \parallel \leq d\kappa^k$, we will have

$$\limsup_{n \to \infty} |R_{r+1}| \leq \frac{1}{(r+1)} d\kappa^{r+1} \leq d\kappa^{r+1},$$

in μ-measure. Therefore

$$\begin{aligned} &\limsup_{n \to \infty} \left| \det[\mathcal{I}^n + \mathcal{U}^n] - \exp\left(\sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \gamma_k^n \right) \right| \\ &\leq \exp(d\kappa/(1-\kappa)) \left| \exp\left(d\kappa^{r+1} \right) - 1 \right| \end{aligned} \tag{5.11}$$

in μ-measure. From Proposition 5.20, we know that $\gamma_k^n \to \gamma_k$ and hence $\gamma_k \leq d\kappa^k$, which implies $p_r = \sum_{k=1}^r \frac{(-1)^{k+1}}{k}\gamma_k$ converges to γ and $|p_r| \leq \frac{d\kappa}{1-\kappa}$. Thus

$$\begin{aligned}
&\lim_{n\to\infty}\left|e^{\gamma} - \exp\left(\sum_{k=1}^{r}\frac{(-1)^{k+1}}{k}\gamma_k^n\right)\right| \leq |e^{\gamma} - e^{p_r}| + \lim_{n\to\infty}\left|e^{p_r} - \exp\left(\sum_{k=1}^{r}\frac{(-1)^{k+1}}{k}\gamma_k^n\right)\right| \\
&\leq e^{p_r}\left|\exp\left(\sum_{k=r+1}^{\infty}(-1)^{k+1}\frac{1}{k}\gamma_k\right) - 1\right| \leq e^{p_r}\left|\exp\left(\sum_{k=r+1}^{\infty}\frac{1}{k}d\kappa^k\right) - 1\right| \\
&\leq e^{d\kappa/(1-\kappa)}\left|\exp\left(\frac{d\kappa^{r+1}}{1-\kappa}\right) - 1\right| \qquad (5.12)
\end{aligned}$$

μ-a.s.. Therefore using Equations (5.11) and (5.12),

$$\begin{aligned}
&\limsup_{n\to\infty}|\det[\mathcal{I}^n + \mathcal{U}^n] - e^{\gamma}| \\
&\qquad\leq \limsup_{n\to\infty}\left\{\begin{array}{c}\left|e^{\gamma} - \exp\left(\sum_{k=1}^{r}\frac{(-1)^{k+1}}{k}\gamma_k^n\right)\right| \\ + \left|\det[\mathcal{I}^n + \mathcal{U}^n] - \exp\left(\sum_{k=1}^{r}\frac{(-1)^{k+1}}{k}\gamma_k^n\right)\right|\end{array}\right\} \\
&\qquad\leq \exp(d\kappa/(1-\kappa))\left|\exp\left(\frac{d\kappa^{r+1}}{1-\kappa}\right) - 1\right| + \exp(d\kappa/(1-\kappa))\left|\exp(d\kappa^{r+1}) - 1\right|
\end{aligned}$$

in μ-measure. Since the inequality holds for every r and as $r \to \infty$,

$$\left|\exp(a\kappa^{r+1}) - 1\right| \to 0,$$

for any constant a, we thus have

$$\limsup_{n\to\infty}|\ \det[(\mathcal{I}^n + \mathcal{U}^n) \circ b_n] - e^{\gamma}\ | = 0$$

in μ-measure. □

Definition 5.23. *Define an integral operator* $K_u : L^2([0,1] \to \mathbb{R}^d) \longrightarrow L^2([0,1] \to \mathbb{R}^d)$ *by*

$$\begin{aligned}
(K_u f)(s) &:= \int_0^1 h(s,t)f(t)\ dt \\
&= \int_0^1 (s \wedge t)\ \Gamma(u(t))f(t)\ dt,
\end{aligned}$$

where Γ *was defined in Definition 5.12.*

Proposition 5.24. *K_u is trace class.*

Proof. Note that $L^2([0,1] \to \mathbb{R}^d)$ is a separable Hilbert space. Let $(Af)(s) = \int_0^1 (s \wedge t) f(t)\, dt$ and $(Bf)(s) = \Gamma(u(s))f(s)$. Thus $K_u = AB$. By Proposition A.12, A is trace class. Since B is bounded, by Proposition A.14, tr $|AB| < \infty$ and hence K_u is trace class. □

Proposition 5.25.

$$e^{\gamma} = \det\left(I + \frac{1}{12}K_u\right).$$

Proof. We will use Equation (B.2) to prove the statement. Note that $\| \frac{1}{12}K_u \| < 1$ by Remark 5.21. Thus

$$\det\left(I + \frac{1}{12}K_u\right) = e^{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}\mathrm{tr}\ \left(\frac{1}{12}K_u\right)^k}.$$

Therefore it suffices to show that for $k \in \mathbb{N}$,

$$\begin{aligned} \mathrm{tr}\ (K_u^k) &= \mathrm{tr} \int_{s_k=0}^{1} \cdots \int_{s_1=0}^{1} \Big[\prod_{l=1}^{k}\Big] h(s_l, s_{l+1})\ ds_1 \ldots ds_k \\ &= \gamma_k(u), \end{aligned}$$

where by convention, $s_{k+1} = s_1$. Let

$$(K_u^k f)(s) = \int_0^1 p_k(s, s_1) f(s_1)\ ds_1,$$

where

$$p_1(s, s_1) = h(s, s_1)$$

and for $k \geq 2$,

$$p_k(s, s_1) = \int_{s_k=0}^{1} \cdots \int_{s_2=0}^{1} h(s, s_2)\Big[\prod_{l=2}^{k}\Big] h(s_l, s_{l+1})\ ds_2 \cdots ds_k.$$

Let $\Omega \subseteq L^2([0,1])$ be an orthonormal basis and hence

$$\{\varpi e_i \mid \varpi \in \Omega,\ i = 1, 2, \ldots d\}$$

is an orthonormal basis for $L^2([0,1],\mathbb{R}^d)$. Let $(,)$ denote the inner product on $L^2([0,1],\mathbb{R}^d)$ and $\langle,\rangle$ be the inner product on $\mathbb{R}^d$. Then

$$\begin{aligned}
\text{tr}\ (K_u^k) &= \sum_{i,\varpi\in\Omega}(K_u^k\varpi e_i,\ \varpi e_i) = \sum_{i,\varpi\in\Omega}\int_0^1\int_0^1 \langle p_k(s,t)\varpi(t)e_i,\ \varpi(s)e_i\rangle\ dt\ ds \\
&= \sum_{\varpi\in\Omega}\int_0^1\int_0^1 \varpi(t)\varpi(s)\text{tr}\ p_k(s,t)\ dt\ ds \\
&= \int_0^1\sum_{\varpi\in\Omega}\left(\int_0^1 \varpi(t)\text{tr}\ p_k(s,t)\ dt\right)\varpi(s)\ ds \\
&= \int_0^1 \text{tr}\ p_k(s,s)\ ds = \text{tr}\ \int_{s_k=0}^1\cdots\int_{s_1=0}^1\Big[\prod_{l=1}^k\Big]h(s_l,s_{l+1})\ ds_1\ldots ds_k.
\end{aligned}$$

Since K_u is trace class, all the interchanging between the integrals and the sums are valid. □

5.3 Convergence of $\det(\mathcal{I}^n+\mathcal{X}^n)$

Lemma 5.26.

$$\det\left[(\mathcal{I}^n+\mathcal{X}^n)\circ b_n\right]\longrightarrow 1$$

$\mu - a.s.$ as $n\to\infty$.

Proof. By Lemma 5.6, it suffices to consider $W_\alpha(\mathbb{R}^d)$ and $\frac{7}{15}<\alpha<\frac{1}{2}$. For each $\omega\in W_\alpha(\mathbb{R}^d)$, there exists $n_0(\omega)$, dependent on the path ω such that

$$\bigvee_{i=1,\ldots,n}\parallel\Delta_i\omega\parallel<\ n^{-\alpha}<\epsilon,$$

for all $n\geq n_0(\omega)>(\frac{1}{\epsilon})^{1/\alpha}$. Thus $b_n(\omega)\in H^\epsilon_{\mathcal{P}_n}(\mathbb{R}^d)$ for all $n\geq n_0(\omega)$. For $n<n_0(\omega)$, define $(\mathcal{X}^n\circ b_n)(\omega)$ to be zero. Hence $\mathcal{X}^n\circ b_n$ is a μ-a.s. defined map on $W(\mathbb{R}^d)$. Recall

$$\mathcal{X}^n=(\mathcal{I}^n+\mathcal{U}^n)^{-1}(\mathcal{S}^n)^T\mathcal{E}^n\mathcal{S}^n.$$

Now observe that $\mathcal{I}^n+\mathcal{U}^n$ is a positive definite matrix with eigenvalues greater than or equal to 1. Thus since the norm of a symmetric matrix is equal to the maximum eigenvalue and the eigenvalues of $(\mathcal{I}^n + \mathcal{U}^n)^{-1}$ is the reciprocal of the eigenvalues of $\mathcal{I}^n + \mathcal{U}^n$, we have

$$\left\|(\mathcal{I}^n + \mathcal{U}^n)^{-1}\right\| \leq 1.$$

Note that $\| \mathcal{E}^n \circ b_n \| = O\left(\bigvee_{i=1,\dots,n} \| \Delta_i b \|^5\right)$ for $n \geq n_0(\cdot)$. Thus by Lemma 5.11, on $W_\alpha(\mathbb{R}^d)$, we have

$$\| \mathcal{X}^n \circ b_n \| = O\left(n^2 \bigvee_{i=1,\dots,n} \| \Delta_i b \|^5\right).$$

Let $\| \mathcal{X}^n \circ b_n \| \leq C[n^2 \bigvee_{i=1,\dots,n} \| \Delta_i b \|^5]$, C is a fixed constant independent of ω and n. Therefore

$$\| \mathcal{X}^n \circ b_n \| \leq Cn^2 \bigvee_{i=1,\dots,n} \| \Delta_i b \|^5 < Cn^{2-5\alpha} < Cn^{-\frac{1}{3}}$$

and hence $\| \mathcal{X}^n \| < 1$. Thus using Equation (B.2), we have

$$\begin{aligned} \det\left[\mathcal{I}^n + \mathcal{X}^n\right] &= \exp\left[\sum_{k=1}^{\infty} \frac{1}{k}(-1)^{k+1} \mathrm{Tr}\ \left[(\mathcal{X}^n)^k\right]\right] \\ &= \exp\left[\mathrm{Tr}\ \mathcal{X}^n - \frac{1}{2}\mathrm{Tr}\ \left[(\mathcal{X}^n)^2\right] + R_3\left(\mathcal{X}^n\right)\right], \end{aligned}$$

where $R_3(\mathcal{X}^n) = \sum_{k=3}^{\infty} \frac{1}{k}(-1)^{k+1}\mathrm{Tr}\ \left[(\mathcal{X}^n)^k\right]$. Note that $\mathcal{X}^n$ is a $nd \times nd$ matrix and hence,

$$|R_3(\mathcal{X}^n)| \leq \frac{ndC^3 n^{3(2-5\alpha)}}{1 - Cn^{2-5\alpha}} = \frac{dC^3 n^{3(2-5\alpha)+1}}{1 - Cn^{2-5\alpha}}.$$

By the choice of α, $\tau := 3(2-5\alpha) + 1 < 0$ and hence

$$|R_3(\mathcal{X}^n)| = O\left(n^{-\tau}\right) \longrightarrow 0$$

as $n \to \infty$. Let $P := (\mathcal{I}^n + \mathcal{U}^n)^{-1}$, $S = \mathcal{S}^n$ and $E = \mathcal{E}^n$ to simplify notation. Then, using Proposition C.1 twice,

$$\begin{aligned} |\mathrm{Tr}\ \mathcal{X}^n| &= \left|\mathrm{Tr}\left(PS^T ES\right)\right| = \left|\mathrm{Tr}\left(SPS^T E\right)\right| \\ &\leq \|E\|\,\mathrm{Tr}\left(SPS^T\right) = \|E\|\,\mathrm{Tr}\left(S^T SP\right) \leq \|E\|\,\|P\|\,\mathrm{Tr}\left(S^T S\right) \end{aligned}$$

where $\mathrm{Tr}\left(S^T S\right) = O\left(n^2\right)$. This is because from Lemma 5.11, we have

$$\mathrm{Tr}\ \left(S^T S\right) = \mathrm{Tr}\ \mathcal{B}^n = \sum_{m=1}^{n} m\mathrm{tr}\ I = \frac{dn(n+1)}{2}.$$

Thus

$$\begin{aligned}|\text{Tr } \mathcal{X}^n| &\leq O(n^2) \parallel E \parallel = O\left(n^2(n^{-\alpha})^5\right) \\ &= O\left(n^2(n^{-7/15})^5\right) = O(n^{-1/3}) \\ &\longrightarrow 0\end{aligned}$$

as $n \to \infty$. Similarly, using Lemma C.1 twice and the fact that P is positive definite so that SPS^T is positive definite, we have

$$\begin{aligned}\left|\text{Tr}\left[(\mathcal{X}^n)^2\right]\right| &= \left|\text{Tr}\left(PS^TESPS^TES\right)\right| = \left|\text{Tr}\left(SPS^TESPS^TE\right)\right| \\ &\leq \text{Tr}\left(SPS^T\right)\left\|ESPS^TE\right\| = \text{Tr}\left(S^TSP\right)\left\|ESPS^TE\right\| \\ &\leq \text{Tr}\left(S^TS\right)\cdot\|P\|\left\|ESPS^TE\right\| \\ &\leq O(n^2)\left\|ESPS^TE\right\| \leq O(n^2)\|E\|^2\|S\|^2 \\ &= O(n^4)\|E\|^2 = O\left(n^4(n^{-\alpha})^{10}\right) \\ &= O\left(n^4(n^{-7/15})^{10}\right) = O\left(n^4n^{-14/3}\right) \\ &= O\left(n^{\frac{12-14}{3}}\right) = O(n^{-2/3}) \\ &\longrightarrow 0\end{aligned}$$

as $n \to \infty$. Therefore

$$\begin{aligned}\det\left[(\mathcal{I}^n + \mathcal{X}^n)\circ b_n\right] &= \exp\left[\text{Tr } \mathcal{X}^n - \frac{1}{2}\text{Tr } \left[(\mathcal{X}^n)^2\right] + R_3\left(\mathcal{X}^n\right)\right]\circ b_n \\ &\longrightarrow 1\end{aligned}$$

μ-a.s. as $n \to \infty$. □

As a random variable on $(W(\mathbb{R}^d),\ \mu)$, the next theorem shows that $\rho_n \circ \phi \circ b_n$ converges in μ-measure.

Theorem 5.27. *If* $\sup_{v\in O(M)} \parallel S_v \parallel < \frac{3}{17d}$, *then*

$$\rho_n \circ \phi \circ b_n \longrightarrow e^{-\frac{1}{6}\int_0^1 Scal(\tilde{\phi}(s))\ ds}\sqrt{\det\left(I + \frac{1}{12}K_u\right)}\ .$$

in μ-measure as $n \to \infty$.

Proof. From Theorem 4.19,

$$\det[\langle(\mathcal{F}^n)^T\mathcal{F}^n\rangle] = [\det(\mathcal{V}^n)]^2 \det(\mathcal{I}^n + \mathcal{U}^n)\det\Big[\mathcal{I}^n + \Big(\mathcal{I}^n + \mathcal{U}^n\Big)^{-1}(\mathcal{S}^n)^T\mathcal{E}^n\mathcal{S}^n\Big].$$

Using Lemma 5.10, Theorem 5.22, Proposition 5.25 and Lemma 5.26, we have

$$\begin{aligned}\det[\langle(\mathcal{F}^n)^T\mathcal{F}^n\rangle \circ b_n] &\longrightarrow e^{\frac{1}{3}\int_0^1 Scal(\tilde{\phi}(s))\ ds+\gamma} \\ &= e^{\frac{1}{3}\int_0^1 Scal(\tilde{\phi}(s))\ ds}\det\left(I + \frac{1}{12}K_u\right)\end{aligned}$$

in μ-measure as $n \to \infty$. But by Theorem 4.7,

$$(\rho_n \circ \phi)^2 = \det[\langle(\mathcal{F}^n)^T\mathcal{F}^n\rangle].$$

Thus taking the square root completes the proof. □

6

L^1 Convergence of $\{\rho_n\}_{n=1}^{\infty}$

Definition 6.1. *Let $\mu_{G^1_{\mathcal{P}}}$ be defined as in Theorem 1.4 on $H_{\mathcal{P}}(\mathbb{R}^d)$ by the density*

$$\frac{1}{Z^1_{\mathcal{P}}} e^{-\frac{1}{2}E_{\mathbb{R}^d}} Vol_{G^1_{\mathcal{P}}}$$

where $Z^1_{\mathcal{P}} := (2\pi)^{\frac{dn}{2}}$ and $E_{\mathbb{R}^d}(\omega) := \int_0^1 \| \omega'(s) \|^2 \, ds$.

The following theorem can be found in [1] and hence we will omit the proof.

Theorem 6.2. *Let $\mu_{G^1_{\mathcal{P}}}$ be defined as in Definition 6.1. Write $\phi_{\mathcal{P}} = \phi|_{H_{\mathcal{P}}(\mathbb{R}^d)}$. Then $\mu_{G^1_{\mathcal{P}}}$ is the pullback of $\nu_{G^1_{\mathcal{P}}}$ by $\phi_{\mathcal{P}}$. i.e.*

$$\mu_{G^1_{\mathcal{P}}} = (\phi_{\mathcal{P}})^* \nu_{G^1_{\mathcal{P}}}.$$

Let $\pi_{\mathcal{P}} : W(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ be given by

$$\pi_{\mathcal{P}}(\omega) := \Big(\omega(s_1),\ \omega(s_2),\ \ldots, \omega(s_n)\Big).$$

Note that $\pi_{\mathcal{P}} : H_{\mathcal{P}}(\mathbb{R}^d) \to (\mathbb{R}^d)^n$ is a linear isomorphism of finite dimensional vector spaces. Let $i_{\mathcal{P}} : (\mathbb{R}^d)^n \to H_{\mathcal{P}}(\mathbb{R}^d)$ denote the inverse of $\pi_{\mathcal{P}}|_{H_{\mathcal{P}}(\mathbb{R}^d)}$.

Lemma 6.3. *Let $(y_1, \ldots y_n)$ denote the standard cartesian coordinates on $(\mathbb{R}^d)^n$ and $y_0 := 0$. Then*

$$i^*_{\mathcal{P}} \mu_{G^1_{\mathcal{P}}} = \frac{1}{Z^1_{\mathcal{P}}} \left(\prod_{i=1}^{n} (\Delta_i s)^{-\frac{d}{2}} \exp\left(-\frac{1}{2\Delta_i s} \| y_i - y_{i-1} \|^2 \right) \right) \, dy_1 dy_2 \ldots dy_n. \tag{6.1}$$

This equation can also be written as

$$i^*_{\mathcal{P}}\mu_{G^1_{\mathcal{P}}} = \left(\prod_{i=1}^{n} p_{\Delta_i s}(y_{i-1},\ y_i)\right)\ dy_1 dy_2 \dots dy_n$$

where $p_s(x,\ y) := (\frac{1}{2\pi})^{-\frac{d}{2}} \exp(-\frac{\|x-y\|}{2s})$ *is the heat kernel on* $\mathbb{R}^d$*.*

Proof. Let $\omega \in H_{\mathcal{P}}(\mathbb{R}^d)$, then

$$E(\omega) = \int_0^1 \parallel \omega'(s) \parallel^2\ ds = \sum_{i=1}^{n} \left(\frac{\parallel \Delta_i \omega \parallel}{\Delta_i s}\right)^2 \Delta_i s = \sum_{i=1}^{n} \frac{1}{\Delta_i s} \parallel \Delta_i \omega \parallel^2 .$$

Hence if $\omega = i_{\mathcal{P}}(y)$, then

$$\int_0^1 \parallel \omega'(s) \parallel^2\ ds = \sum_{i=1}^{n} \frac{1}{\Delta_i s} \parallel y_i - y_{i-1} \parallel^2 = \sum_{i=1}^{n} \parallel \xi_i \parallel^2 \tag{6.2}$$

where $\xi_i := (\Delta_i s)^{-\frac{1}{2}}(y_i - y_{i-1})$. This last equation shows that the linear transformation

$$\omega \in H_{\mathcal{P}}(\mathbb{R}^d) \to \left\{(\Delta_i s)^{-\frac{1}{2}}\Big(\omega(s_i) - \omega(s_{i-1})\Big)\right\}_{i=1}^{n} \in (\mathbb{R}^d)^n$$

is an isometry of vector spaces and therefore

$$i^*_{\mathcal{P}} Vol_{G^1_{\mathcal{P}}} = d\xi_1 d\xi_2 \dots d\xi_n. \tag{6.3}$$

Now an easy computation shows that

$$d\xi_1 d\xi_2 \dots d\xi_n = \left(\prod_{i=1}^{n} (\Delta_i s)^{-\frac{d}{2}}\right)\ dy_1 dy_2 \dots dy_n. \tag{6.4}$$

From Equations (6.2), (6.3) and (6.4), we see that Equation (6.1) is valid. □

We are now ready to prove our main result.

Theorem 6.4. *Let* M *be a compact Riemannian manifold,* $f : W(M) \to \mathbb{R}$ *be a bounded continuous function and* $\mathcal{P} = \{0 < \frac{1}{n} < \dots < \frac{n}{n} = 1\}$ *is an equally spaced partition. Let* b *be brownian motion,* u *solves Equation (5.1),* $\tilde{\phi} = \pi \circ u$ *and* $\widetilde{//}$ *is* *stochastic parallel translation defined in Definition 1.12. Suppose* $\sup_{v\in O(M)} \parallel S_v \parallel < \frac{3}{17d}$*, where* S_v *is sectional curvature and* $\rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}$ *is uniformly integrable. Then*

$$\begin{aligned}\int_{H_{\mathcal{P}}(M)} f(\sigma)\ d\nu_{\mathcal{P}}(\sigma) &= \int_{H_{\mathcal{P}}(M)} f(\sigma)\ \rho_{\mathcal{P}}(\sigma)\ d\nu_{G^1_{\mathcal{P}}}(\sigma) \\ \underset{|\mathcal{P}|\to 0}{\longrightarrow} & \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 Scal(\sigma(s))\ ds} \sqrt{\det\left(I + \frac{1}{12} K_{\widetilde{//}(\sigma)}\right)}\ d\nu(\sigma)\end{aligned}$$

where ν *is Wiener measure on* M*.* $Scal$ *is the scalar curvature of the manifold and* $K_{\widetilde{//}} : L^2([0,1] \to \mathbb{R}^d) \longrightarrow L^2([0,1] \to \mathbb{R}^d)$ *is given by*

$$(K_{\widetilde{//}} h)(s) = \int_0^1 (s \wedge t)\ \Gamma(\widetilde{//}_t) h(t)\ dt$$

where

$$\Gamma(\widetilde{//}) = \sum_{i,j=1}^{d} \left(\Omega_{\widetilde{//}}(e_i, \Omega_{\widetilde{//}}(e_i, \cdot)e_j)e_j + \Omega_{\widetilde{//}}(e_i, \Omega_{\widetilde{//}}(e_j, \cdot)e_i)e_j + \Omega_{\widetilde{//}}(e_i, \Omega_{\widetilde{//}}(e_j, \cdot)e_j)e_i\right)$$

for any orthonormal basis $\{e_i\}_{i=1}^d \subseteq T_o M$*, and*

$$\Omega_{\widetilde{//}}(a, c) := \widetilde{//}^{-1} R\left(\widetilde{//}a, \widetilde{//}c\right) \widetilde{//}$$

for any vectors $a, c \in T_o M$*,* R *being the curvature tensor.*

Proof. By Theorem 5.2, $f \circ \phi \circ b_{\mathcal{P}} := f_{\mathcal{P}}$ converges to $f \circ \tilde{\phi}\ \mu - a.s.$ as $|\mathcal{P}| \to 0$. Thus we have

$$\begin{aligned}\int_{H_{\mathcal{P}}(M)} f\ d\nu_{\mathcal{P}} &= \int_{H_{\mathcal{P}}(M)} f\ \rho_{\mathcal{P}}\ d\nu_{G^1_{\mathcal{P}}} \\ &= \int_{H_{\mathcal{P}}(\mathbb{R}^d)} (f\rho_{\mathcal{P}}) \circ \phi\ d\mu_{G^1_{\mathcal{P}}}\ (\textit{By Theorem } 6.2) \\ &= \int_{W(\mathbb{R}^d)} (f\rho_{\mathcal{P}}) \circ \phi \circ b_{\mathcal{P}}\ d\mu\ (\textit{By Lemma } 6.3) \\ &= \int_{W(\mathbb{R}^d)} f_{\mathcal{P}} \cdot \rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}\ d\mu.\end{aligned}$$

By the assumption on $\rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}$, since f is bounded and continuous, we have

$f_{\mathcal{P}} \cdot \ \rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}$ is uniformly integrable. Therefore by Theorem 5.27,

$$\begin{aligned}\int_{H_{\mathcal{P}}(M)} f(\sigma)\ d\nu_{\mathcal{P}}(\sigma) &= \int_{W(\mathbb{R}^d)} (f_{\mathcal{P}} \cdot \rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}})(\omega)\ d\mu(\omega) \\ &\longrightarrow \int_{W(\mathbb{R}^d)} \left(f \circ \tilde{\phi} \cdot e^{-\frac{1}{6}\int_0^1 Scal(\tilde{\phi}(s))\ ds} \sqrt{\det\left(I + \frac{1}{12}K_u\right)}\right)(\omega)\ d\mu(\omega) \\ &= \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 Scal(\sigma(s))\ ds} \sqrt{\det\left(I + \frac{1}{12}K_{\widetilde{//}(\sigma)}\right)}\ d\nu(\sigma)\end{aligned}$$

as $|\mathcal{P}| \to 0$, where K_u was defined in Definition 5.23. Note that $\nu = \mu\tilde{\phi}^{-1}$ and $u = \widetilde{//}(\tilde{\phi})$ from (1) and (2) of Remark 5.4 respectively. $\square$

Corollary 6.5. *Let M be a compact Riemannian manifold, $f : W(M) \to \mathbb{R}$ be a bounded continuous function and $\mathcal{P}$ is an equally spaced partition. Suppose that $\forall v \in O(M)$, $0 \le S_v < \frac{3}{17d}$. Then*

$$\int_{H_{\mathcal{P}}(M)} f(\sigma)\ d\nu_{\mathcal{P}}(\sigma) \underset{|\mathcal{P}| \to 0}{\longrightarrow} \int_{W(M)} f(\sigma) e^{-\frac{1}{6}\int_0^1 Scal(\sigma(s))\ ds} \sqrt{\det\left(I + \frac{1}{12}K_{\widetilde{//}(\sigma)}\right)}\ d\nu(\sigma).$$

Proof. By Theorem 3.16, we know that $\rho_{\mathcal{P}} \circ \phi \circ b_{\mathcal{P}}$ is uniformly integrable under the assumptions on the sectional curvature. Hence the corollary now follows from Theorem 6.4. $\square$

6.1 Further Questions

Two questions immediately arise from Corollary 6.5.

1. Can we remove the upper bound restriction on the sectional curvature? In other words, does the result hold true for any compact manifold with non-negative sectional curvature?

2. Can the result be extended to an arbitrary compact manifold?

The restriction of the result on a non-negative manifold arises when we are trying to prove the uniform integrability of $\rho_{\mathcal{P}}$. Thus one way to improve the result is to obtain a better upper estimate for $\rho_{\mathcal{P}}$, and show that without any restrictions on the sectional curvature of the manifold, $\rho_{\mathcal{P}}$ is still uniformly integrable.

One possible research problem will be to consider a different metric on $H(M)$. For example, by considering a L^2-metric G^0 on $TH(M)$ where

$$G^0(X,\ X) := \int_0^1 g\left(X(s),\ X(s)\right)\ ds,$$

for $X \in T_\sigma H(M)$. Instead of considering Wiener space $W(M)$, one can consider the space of pinned paths, equipped with pinned Wiener measure and carry out a similar analysis.

A

Trace Class Operators

References for this section can be found in [20] and [44].

Definition A.1. *Let X be a complex Banach space and A be a bounded linear operator on X. The resolvent set $\rho(A)$ of A is the set of complex numbers λ for which $(\lambda I - A)^{-1}$ exists as a bounded operator with domain X. The spectrum $\sigma(A)$ of A is the complement of $\rho(A)$. The function $R(\lambda; A) = (\lambda I - A)^{-1}$, defined on $\rho(A)$ is called the resolvent function of A or simply the resolvent of A.*

It is a standard result that $\sigma(A)$ is a closed and nonempty set. Let D be a region in the complex plane. A function $x(\cdot)$ defined on D with values in X is said to be analytic at $z_0 \in D$ if the limit of $\frac{x(z_0+h)-x(z_0)}{h}$ exists in X as h goes to zero in $\mathbb{C}$. Starting from this point, one can develop a theory of vector-valued analytic functions which is almost exactly parallel to the usual theory, i.e. a meromorphic function has a Laurent series and definition of a pole as in the usual theory. With this notion, one can prove that the resolvent $R(\lambda; A)$ is analytic in $\rho(A)$. (See Lemma 2 in VII.3.1 of [19].)

Definition A.2. *A point λ_0 is said to be an isolated point of the spectrum of A, $\sigma(A)$ if there is a neighborhood U of λ_0 such that $\sigma(A) \cap U = \{\lambda_0\}$. An isolated point λ_0 of $\sigma(A)$ is called a pole of A, or simply a pole, if $R(\lambda; A)$ has a pole at λ_0. By the order $\nu(\lambda_0)$ of a pole λ_0 is meant the order of λ_0 as a pole of $R(\lambda; A)$.*

The next important result enables us to define a notion of algebraic multiplicity for an eigenvalue of a compact operator.

Theorem A.3. *If A is a compact operator, then its spectrum is discrete and has no accumulation point in the complex plane, except possibly at $\lambda = 0$. Every non-zero number in $\sigma(A)$ is a pole of A.*

For a proof, refer to Theorem 5 in VII.4.5 of [19].

Definition A.4. *For any compact operator A and $\lambda \in \sigma(A) \backslash \{0\}$, let $\nu(\lambda)$ be the order of the pole λ. We will now define the algebraic multiplicity of λ as $\nu(\lambda)$.*

Definition A.5. *Let $\mathcal{H}$ be a separable Hilbert space with $\langle , \rangle$ as inner product and $\{\phi_i\}_{i=1}^{\infty}$ be an orthonormal basis. For any positive compact operator A, define*

$$\text{tr } A = \sum_{i=1}^{\infty} \langle A\phi_i,\ \phi_i \rangle.$$

Given any compact operator A, A is said to be trace class if $\text{tr } |A| < \infty$.

Remark A.6. *It can be shown that the trace of a positive operator is independent of the orthonormal basis used in the definition.*

Definition A.7. *If A is trace class, then define* $\text{tr } A$ *as*

$$\text{tr } A = \sum_{i=1}^{\infty} \langle A\phi_i,\ \phi_i \rangle$$

where $\{\phi_i\}_{i=1}^{\infty}$ is any orthonormal basis. Let I be the identity. We can also define $\det(I + A)$ *by*

$$\det(I + A) = \prod_{i=1}^{n} (1 + \lambda_i(A))$$

where $\lambda_i(A)$ are the eigenvalues of A, repeated according to its algebraic multiplicity. (See [44].)

Let $\mathcal{J}_1$ be the space of trace class operators, equipped with a norm $\| \cdot \|_1$ by

$$\| A \|_1 = \text{tr } |A|.$$

This norm should not be confused with the operator norm, $\| \cdot \|$.

Given any compact operator A, denote the eigenvalues $\lambda_1(A), \lambda_2(A), \ldots \lambda_n(A), \ldots$ ordered so that $|\lambda_1| \geq |\lambda_2| \geq \ldots$ such that each eigenvalue is counted up to its algebraic multiplicity. The next theorem is Theorem 19 in Section XI.9.20 of [20].

Theorem A.8. *The function* tr $(\cdot)$ *is a bounded linear functional on* $\mathcal{J}_1$*, i.e.*

$$|\text{tr } A| \leq \| A \|_1$$

for any trace class operator A*. In fact,*

$$\text{tr } A = \sum_{i=1}^{\infty} \lambda_i(A).$$

Let $\{\mu_n(A)\}_{i=1}^{\infty}$ be the eigenvalues of $|A|$. Then we have the following theorem, taken from Theorem 1.15 in [44].

Theorem A.9. *Let* ϕ *be a non-negative monotone increasing function on* $[0, \infty)$ *so that* $t \mapsto \phi(e^t)$ *is convex. Then for any compact operator* A*,*

$$\sum_{i=1}^{\infty} \phi(|\lambda_i(A)|) \leq \sum_{i=1}^{\infty} \phi(\mu_i(A)).$$

Moreover, for any compact operators A *and* B*,*

$$\sum_{i=1}^{\infty} \phi(\mu_i(AB)) \leq \sum_{i=1}^{\infty} \phi(\mu_i(A)\mu_i(B)).$$

The following theorem shows that $\det(I + \cdot)$ is a continuous function on $\mathcal{J}_1$. See Theorem 3.4 in [44].

Theorem A.10. $A \mapsto \det(I + A)$ *is a continuous function on* $\mathcal{J}_1$*. In fact,*

$$|\det(I + A) - \det(I + B)| \leq \| A - B \|_1 \exp(\| A \|_1 + \| B \|_1 + 1).$$

In general, it is difficult to determine if an integral operator is trace class. However, the following theorem, taken from Theorem 2.12 of [44], gives a condition for which an integral operator is trace class. See also Section XI.4 of [41].

Theorem A.11. *Let μ be a Baire measure on a locally compact space X. Let K be a function on $X \times X$ which is continuous and Hermitian positive, that is*

$$\sum_{i,j=1}^{N} \overline{z_i} z_j K(x_i, x_j) \geq 0$$

for any $x_1, \ldots, x_N \in X$, $z_1, \ldots, z_N \in \mathbb{C}^N$ and for any N. Then $K(x,x) \geq 0$ for all x. Suppose that in addition,

$$\int K(x,x)\ d\mu(x) < \infty.$$

Then there exists a unique trace class integral operator A such that

$$(Af)(x) = \int K(x,y) f(y)\ d\mu(y)$$

and

$$\| A \|_1 = \int K(x,x)\ d\mu.$$

Proposition A.12. *Let $(Af)(s) = \int_0^1 (s \wedge t) f(t)\ dt$. Then A is a trace class operator.*

Proof. Let $X = [0,1]$ and μ be Lebesgue measure. Using Theorem A.11, it suffices to show that $s \wedge t$ is Hermitian positive. Let $z_1, z_2, \ldots, z_N$ be any complex numbers and let $x_1, \ldots, x_N \in [0,1]$. The proof is by induction. Clearly when $N = 1$, it is trivial. Suppose it is true for all values from $k = 1, 2, \ldots, N-1$. Without loss of generality, we can assume that $x_1 \leq x_k$, $k = 2, \ldots N$. Hence $x_1 \wedge x_k = x_1$ for any k. Thus

$$\begin{aligned}
\sum_{j=1}^{N} \overline{z_1} z_j (x_1 \wedge x_j) + \sum_{j=1}^{N} \overline{z_j} z_1 (x_j \wedge x_1) &= x_1 \sum_{j=1}^{N} \overline{z_1} z_j + x_1 \sum_{j=1}^{N} \overline{z_j} z_1 \\
&= x_1 \left(\sum_{j=1}^{N} \overline{z_j} \right) \left(\sum_{j=1}^{N} z_j \right) - x_1 \sum_{i,j=2}^{N} \overline{z_i} z_j.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i,j=1}^{N} \overline{z_i} z_j (x_i \wedge x_j) &= x_1 \left(\sum_{j=1}^{N} \overline{z_j} \right) \left(\sum_{j=1}^{N} z_j \right) - x_1 \sum_{i,j=2}^{N} \overline{z_i} z_j + \sum_{i,j=2}^{N} \overline{z_i} z_j (x_i \wedge x_j) \\
&= x_1 \left(\sum_{j=1}^{N} \overline{z_j} \right) \left(\sum_{j=1}^{N} z_j \right) + \sum_{i,j=2}^{N} \overline{z_i} z_j (c_i \wedge c_j),
\end{aligned}$$

where $c_i = x_i - x_1 \geq 0,\ i = 2, \ldots, N$. Thus by induction hypothesis,

$$\sum_{i,j=2}^{N} \overline{z_i} z_j (c_i \wedge c_j) \geq 0$$

and hence

$$\sum_{i,j=1}^{N} \overline{z_i} z_j (x_i \wedge x_j) \geq 0.$$

□

Proposition A.13. *Any bounded operator B can be written as a linear combination of four unitary operators.*

Proof. Let $\star$ denotes the adjoint. Then we can write

$$B = \frac{1}{2}(B + B^\star) - \frac{i}{2}\left[i\left(B - B^\star\right)\right]$$

as a linear combination of 2 self adjoint operators. So without loss of generality, assume that B is self adjoint and $\| B \| \leq 1$. Then $A \pm i\sqrt{I - A^2}$ are unitary and

$$B = \frac{1}{2}(A + i\sqrt{I - A^2}) + \frac{1}{2}(A - i\sqrt{I - A^2}).$$

□

Proposition A.14. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose A is trace class and B is a bounded operator. Then*

$$\text{tr } |AB| \leq \| B \| \text{ tr } |A|$$

where $\| \cdot \|$ is the operator norm.

Proof. By Proposition A.13, we can assume that B is unitary. Let $T = |AB| = \sqrt{(AB)^\star (AB)}$ where $\star$ denotes the adjoint. Since $|A|$ is a self adjoint trace class operator, by Hilbert-Schmidt theorem, there exists a complete orthonormal basis $\{\phi_i\}_{i=1}^{\infty}$, such that

$$|A|\phi_i = \lambda_i \phi_i.$$

Thus let $\{\varphi_i\}_{i=1}^{\infty}$ be another complete set of orthonormal basis such that

$$B\varphi_i = \phi_i.$$

Now by Cauchy Schwartz inequality,

$$|\langle T\varphi_i,\ \varphi_i\rangle| \leq \| T\varphi_i \| .$$

But

$$\begin{aligned}\| T\varphi_i \|^2 &= \langle T\varphi_i,\ T\varphi_i\rangle = \langle T^\star T\varphi_i,\ \varphi_i\rangle = \langle B^\star A^\star AB\varphi_i,\ \varphi_i\rangle \\ &= \langle (A^\star A)B\varphi_i,\ B\varphi_i\rangle = \langle (A^\star A)\phi_i,\ \phi_i\rangle = \lambda_i^2.\end{aligned}$$

Thus

$$\| T\phi_i \| \leq \lambda_i.$$

Hence

$$\sum_{i=1}^{\infty}\langle T\phi_i,\ \phi_i\rangle \leq \sum_{i=1}^{\infty} \| T\phi_i \| \leq \sum_{i=1}^{\infty} \lambda_i < \infty,$$

since A is trace class and thus $\sum_{i=1}^{\infty} \lambda_i < \infty$. This completes the proof. □

B

Perturbation Formulas

Lemma B.1. *Let U be a $d \times d$ matrix and for $r \in \mathbb{N}$, let*

$$\Psi_r(U) := \sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \text{tr } U^k.$$

If U is positive definite, then there exists $R_{r+1}(U)$ such that

$$\det(I + U) = \exp\left(\Psi_r(U) + R_{r+1}(U)\right), \tag{B.1}$$

where

$$|R_{r+1}(U)| \leq \frac{1}{r+1} \text{tr } U^{r+1}.$$

If U is any $d \times d$ matrix (not necessarily positive) such that $\| U \| < 1$, then

$$\det(I + U) = \exp\left(\Psi_r(U) + R_{r+1}(U)\right), \tag{B.2}$$

with

$$R_{r+1}(U) = \sum_{k=r+1}^{\infty} (-1)^{k+1} \frac{1}{k} \text{tr } U^k$$

such that

$$|R_{r+1}(U)| \leq \frac{d \| U \|^{r+1}}{1 - \| U \|}. \tag{B.3}$$

Proof. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ be the set of eigenvalues of U. Then

$$\begin{aligned} \det(I + U) &= \prod_{i=1}^{d} (1 + \lambda_i) \\ &= \exp\left(\sum_{i=1}^{d} \ln(1 + \lambda_i)\right), \end{aligned} \tag{B.4}$$

since by assumption, $\lambda_i \geq 0$ for all i. But by applying Taylor's Theorem to $\ln(1+x)$, we have

$$\ln(1+x) = \sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} x^k + R_{r+1}(x),$$

where

$$R_{r+1}(x) = x^{k+1} \frac{(-1)^r}{r+1} \frac{1}{(1+c)^{r+1}}$$

for some $c \in (0, x)$. Therefore,

$$|R_{r+1}(x)| \leq \frac{1}{r+1} x^{r+1}.$$

Since

$$\text{tr } U^k = \sum_{i=1}^{d} \lambda_i^k,$$

we have

$$\begin{aligned}
\sum_{i=1}^{d} \ln(1+\lambda_i) &= \sum_{i=1}^{d} \left(\sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \lambda_i^k + R_{r+1}(\lambda_i) \right) \\
&= \sum_{k=1}^{r} (-1)^{k+1} \frac{1}{k} \text{tr } U^k + \sum_{i=1}^{d} R_{r+1}(\lambda_i) \\
&= \Psi_r(U) + R_{r+1}(U),
\end{aligned}$$

where $R_{r+1}(U) := \sum_{i=1}^{d} R_{r+1}(\lambda_i)$ and $|R_{r+1}(U)| \leq \sum_{i=1}^{d} \lambda_i^{r+1} = \frac{1}{r+1} \text{tr } U^{r+1}$. To prove Equation (B.2), we assume that $\| U \| < 1$. Thus the eigenvalues of U, $\lambda_i < 1$ for all i. Therefore Equation (B.4) holds. But $\ln(1+\lambda_i) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \lambda_i^k$ and this sum converges absolutely. Thus

$$\begin{aligned}
\sum_{i=1}^{d} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \lambda_i^k &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \sum_{i=1}^{d} \lambda_i^k \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \text{tr } U^k.
\end{aligned}$$

Hence this proves Equation (B.2). In this case,

$$\begin{aligned}
|R_{r+1}(U)| = \left| \sum_{k=r+1}^{\infty} (-1)^{k+1} \frac{1}{k} \text{tr } U^k \right| &\leq \sum_{k=r+1}^{\infty} \frac{1}{k} \left| \text{tr } U^k \right| \\
&\leq \sum_{k=r+1}^{\infty} d \| U \|^k = \frac{d \| U \|^{r+1}}{1 - \| U \|}. \qquad \text{(B.5)}
\end{aligned}$$

Alternatively, Equation (B.2) can be proved by a rewriting of the standard formula,

$$\begin{aligned}\frac{d}{ds}\log(\det(I - sU)) &= -\mathrm{tr}((I - sU)^{-1}U) \\ &= -\mathrm{tr}\left(\sum_{k=0}^{\infty} s^k U^k U\right) \\ &= -\sum_{k=0}^{\infty} s^k \mathrm{tr}\ (U^{k+1}).\end{aligned}$$

□

Let $A \in \mathcal{J}_1$. Suppose $-z^{-1}$ lies in the resolvent set of A, i.e. $-z^{-1} \notin \sigma(A)$. Then the mapping $A \to \ln(I + zA)$ is defined and is continuous on A. (See Lemma 15 in Section XI.9.22 of [20].) Thus, one can define a function $\widetilde{\det}$ on $\mathcal{J}_1$ in the following way.

Definition B.2. *Define for* $-z^{-1} \notin \sigma(A)$,

$$\widetilde{\det}(I + zA) = \exp(\mathrm{tr}\ \ln(I + zA)).$$

This function $\widetilde{\det}(I + zA)$ is analytic in z and has only removable singularities at the points z such that $-z^{-1} \in \sigma(T)$. In fact, the definition $\det(I + \cdot)$ defined in Definition A.7 coincides with $\widetilde{\det}(I + \cdot)$. (See Lemma 16 and 22 in Section XI.9.22 of [20].) We can now show that Equation (B.2) holds even for trace class operator.

Lemma B.3. *Let A be a trace class operator on a Hilbert space $\mathcal{H}$, with $\| A \| < 1$ where $\| \cdot \|$ is the operator norm. Then Equation (B.2) holds with U replaced by A.*

Proof. Because $\alpha = \| A \| < 1$, $-1 \notin \sigma(A)$. Thus $\widetilde{\det}(I + A)$ is well defined. Since

$$\det(I + A) = \widetilde{\det}(I + A) = \exp(\mathrm{tr}\ \ln(I + A)),$$

it suffices to show that

$$\mathrm{tr}\ \ln(I + A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \mathrm{tr}\ A^k.$$

If $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\mu_i\}_{i=1}^{\infty}$ are the eigenvalues of A and $|A|$ respectively, repeated according to algebraic multiplicity and arranged in decreasing order of absolute values, then by applying Theorem A.9 with $\phi(t) = t$,

$$\sum_{i=1}^{\infty} |\lambda_i| \leq \sum_{i=1}^{\infty} \mu_i = \| A \|_1 .$$

Since $\sup_i |\lambda_i| \leq \alpha$, hence

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k}|\lambda_i|^k = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \frac{1}{k}|\lambda_i|^k &\leq \sum_{k=1}^{\infty} \alpha^{k-1} \sum_{i=1}^{\infty} |\lambda_i| \\ &\leq \sum_{k=0}^{\infty} \alpha^k \| A \|_1 = \| A \|_1 \frac{1}{1-\alpha} < \infty.\end{aligned}$$

Hence we can apply Fubini's Theorem and by Theorem A.8,

$$\begin{aligned}\text{tr } \ln(I + A) &= \sum_{i=1}^{\infty} \ln(1 + \lambda_i) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\lambda_i^k \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{i=1}^{\infty} \lambda_i^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \text{tr } A^k.\end{aligned}$$

□

C

Matrix Inequalities

Proposition C.1. *If A, B are two $N \times N$ matrices with B being positive semi definite, then*

$$|\mathrm{tr}(AB)| \leq \|A\| \ \mathrm{tr} \ B \tag{C.1}$$

and in particular by taking $B = I$,

$$|\mathrm{tr} \ A| \leq N \|A\| . \tag{C.2}$$

Proof. Let $\{e_i\}_{i=1,2,\ldots N}$ be an orthonormal basis of eigenvectors of B with corresponding eigenvalues, $\{\lambda_i \geq 0\}_{i=1,2,\ldots N}$. Then

$$\begin{aligned}|\mathrm{tr}\,(AB)| &= \left|\sum_{i=1}^{N} \langle ABe_i, e_i\rangle\right| = \left|\sum_{i=1}^{N} \lambda_i \langle Ae_i, e_i\rangle\right| \\ &\leq \sum_{i=1}^{N} \lambda_i |\langle Ae_i, e_i\rangle| \leq \sum_{i=1}^{N} \lambda_i \|A\| = \|A\| \ \mathrm{tr} \ B.\end{aligned}$$

□

Proposition C.2. *Suppose that M is a positive definite $N \times N$ matrix and $\alpha > 0$. Then*

$$\det(M) \leq \left(\frac{\mathrm{tr}\,(M)}{N}\right)^N \leq \alpha^N e^{\mathrm{tr}\left(\alpha^{-1}M - I\right)}. \tag{C.3}$$

Moreover if $\alpha \geq 1$, then

$$\det(M) \leq \alpha^N e^{\alpha^{-1}\,\mathrm{tr}(M-I)}. \tag{C.4}$$

Proof. Let $\{\lambda_i\}_{i=1}^N$ be the eigenvalues of M, then

$$\det M = \lambda_1 \cdots \lambda_N = \left(\lambda_1^{1/N} \cdots \lambda_N^{1/N}\right)^N \leq \left(\sum_{i=1}^N \frac{1}{N}\lambda_i\right)^N = \left(\frac{\operatorname{tr}(M)}{N}\right)^N.$$

Now suppose that $\alpha > 0$. Then

$$\begin{aligned}\det M &= \alpha^N \det\left(\alpha^{-1}M\right) \leq \alpha^N \left(\frac{\operatorname{tr}(\alpha^{-1}M)}{N}\right)^N \\ &\leq \alpha^N \left(1 + \frac{\operatorname{tr}(\alpha^{-1}M - I)}{N}\right)^N \leq \alpha^N e^{\operatorname{tr}\left(\alpha^{-1}M - I\right)},\end{aligned}$$

where the last inequality follows from the inequality

$$\left(1 + \frac{x}{N}\right)^N \leq e^x \text{ for all } N \in \mathbb{N} \text{ and } x \geq -N.$$

Indeed, elementary calculus shows that $f(x) := e^{-x}\left(1 + \frac{x}{N}\right)^N$ for $x \geq -N$ has a global maximum of one at $x = 0$.

Alternatively,

$$\begin{aligned}\det(M) &= \alpha^N \det\left(\alpha^{-1}M\right) = \alpha^N \det\left(\alpha^{-1}M - I + I\right) \\ &= \alpha^N \prod_{i=1}^N \left(1 + \left(\alpha^{-1}\lambda_i - 1\right)\right) \leq \alpha^N \prod_{i=1}^N e^{\left(\alpha^{-1}\lambda_i - 1\right)} = \alpha^N e^{\operatorname{tr}\left(\alpha^{-1}M - I\right)}\end{aligned}$$

wherein we have used the inequality, $1 + x \leq e^x$, which results from the convexity of the exponential function. Finally, by optimizing this inequality over $\alpha > 0$, (take $\alpha = N^{-1} \operatorname{tr} M$), the previous inequality implies $\det M \leq \left(N^{-1} \operatorname{tr} M\right)^N$. ∎

D

Bounds on Curvature

Definition D.1. *Let M be a manifold, $u \in O(M)$ and Ω_u as defined in Definition 2.1. Define for any vectors $x, y, w, z \in T_o M$,*

$$R_u(x, y, w, z) = \langle \Omega_u(x, y)w, z \rangle,$$

and for any linearly independent vectors x, y,

$$S_u(x, y) = \frac{R_u(x, y, x, y)}{\| x \|^2 \| y \|^2 - \langle x, y \rangle^2}.$$

The curvature tensor, R_u satisfies the following symmetries,

$$R_u(x, y, w, z) = -R_u(y, x, w, z), \tag{D.1}$$

$$R_u(x, y, w, z) = -R_u(x, y, z, w), \tag{D.2}$$

$$R_u(x, y, w, z) = R_u(w, z, x, y), \tag{D.3}$$

$$R_u(x, y, w, z) + R_u(w, x, y, z) + R_u(y, w, x, z) = 0. \tag{D.4}$$

The last equality is called the Bianchi Identity.

S_u is called the sectional curvature and if we define $S_u(V) := S_u(x, y)$ where V is the plane spanned by any linearly independent vectors x, y, then it can be shown that $S_u(V)$ is independent of the choice of x and y. We will write

$$\| S_u \| = \sup\{ |S_u(V)| \mid V \text{ is a plane in } T_0 M \},$$

and

$$K = \sup_{u \in O(M)} \| S_u \| .$$

Since we are considering a compact manifold M, K is finite. Immediate from this definition, we have

$$|R_u(x,y,x,y)| \leq \| S_u \| \| x \|^2 \| y \|^2 \tag{D.5}$$

for any $x, y \in T_oM$.

For any vector v, Equation (D.3) shows that $\Omega_u(v,\cdot)v$ is a symmetric matrix. It follows that

$$\| \Omega_u(v,\cdot)v \| = \sup_{\|w\|=1} \| \Omega_u(v,w)v \| = \sup_{\|w\|=1} |\langle \Omega_u(v,w)v, w\rangle| \leq \| S_u \| \| v \|^2 .$$

The last inequality follows from Equation (D.5). As a consequence, if we take the supremum over $O(M)$, then

$$\sup_{u \in O(M)} \| \Omega_u(v,\cdot)v \| \leq K \| v \|^2 .$$

Proposition D.2. *If* $\sup_{u\in O(M)} \| S_u \| < \frac{3}{17d}$, *then for any orthonormal frame* $\{e_i\}_{i=1}^d \subseteq T_oM$,

$$\sup_{u \in O(M)} \| \Omega_u(e_i,\cdot)e_j \| < \frac{2}{d},$$

where d is the dimension of the manifold.

Proof. Let $x, y, w, z \in T_oM$. Then using Equations (D.1), (D.2) and (D.3),

$$\begin{aligned}
A :=\ & R_u(x+w, y+z, x+w, y+z) - R_u(x+w, y, x+w, y) - R_u(x+w, z, x+w, z) \\
& - R_u(x, y+z, x, y+z) - R_u(w, y+z, w, y+z) + R_u(w,y,w,y) + R_u(x,z,x,z) \\
& = R_u(x+w, y, x+w, z) + R_u(x+w, z, x+w, y) \\
& - R_u(x,y,x,z) - R_u(x,z,x,y) - R_u(x,y,x,y) \\
& - R_u(w,y,w,z) - R_u(w,z,w,y) - R_u(w,z,w,z) \\
& = R_u(x,y,w,z) + R_u(w,y,x,z) + R_u(x,z,w,y) + R_u(w,z,x,y) \\
& - R_u(x,y,x,y) - R_u(w,z,w,z) \\
& = 2R_u(x,y,w,z) + 2R_u(w,y,x,z) - R_u(x,y,x,y) - R_u(w,z,w,z).
\end{aligned}$$

Similarly,

$$\begin{aligned}
B := &R_u(x+z, y+w, x+z, y+w) - R_u(x+z, y, x+z, y) - R_u(x+z, w, x+z, w)\\
&- R_u(x, y+w, x, y+w) - R_u(z, y+w, z, y+w) + R_u(x, w, x, w) + R_u(z, y, z, y)\\
&= R_u(x+z, y, x+z, w) + R_u(x+z, w, x+z, y)\\
&- R_u(x, y, x, w) - R_u(x, w, x, y) - R_u(x, y, x, y)\\
&- R_u(z, y, z, w) - R_u(z, w, z, y) - R_u(z, w, z, w)\\
&= R_u(x, y, z, w) + R_u(z, y, x, w) + R_u(x, w, z, y) + R_u(z, w, x, y)\\
&- R_u(x, y, x, y) - R_u(z, w, z, w)\\
&= 2R_u(x, y, z, w) + 2R_u(z, y, x, w) - R_u(x, y, x, y) - R_u(w, z, w, z).
\end{aligned}$$

Thus

$$\begin{aligned}
A - B &= 2R_u(x, y, w, z) + 2R_u(w, y, x, z) - 2R_u(x, y, z, w) - 2R_u(z, y, x, w)\\
&= 2R_u(x, y, w, z) + 2R_u(x, y, w, z) + 2R_u(w, y, x, z) + 2R_u(y, z, x, w)\\
&= 4R_u(x, y, w, z) + 2R_u(w, y, x, z) + 2R_u(x, w, y, z)\\
&= 4R_u(x, y, w, z) - 2R_u(y, x, w, z)\\
&= 6R_u(x, y, w, z).
\end{aligned}$$

The second last equality follows from the Bianchi Identity. Hence for any unit vectors x, y, w, z,

$$\begin{aligned}
|A| &\leq \| S_u \| \left[\| x+w \|^2 \| y+z \|^2 + 2 \| x+w \|^2 + 2 \| y+z \|^2 + 2\right]\\
&\leq \| S_u \| \left[4^2 + 2 \cdot 8 + 2\right] = 34 \| S_u \| .
\end{aligned}$$

Similarly, $|B| \leq 34 \| S_u \|$. Therefore, for any unit vectors x, y, w, z,

$$|R_u(x, y, w, z)| = \frac{1}{6}|A - B| \leq \frac{34}{3} \| S_u \| .$$

and hence

$$\sup_{\|x\|=\|y\|=\|w\|=\|z\|=1} |R_u(x, z, w, z)| \leq \frac{34}{3} \| S_u \| .$$

Now for any $u \in O(M)$ and orthonormal frame $\{e_i\}_{i=1}^d$, let $\Omega_{ij}(y) := \Omega_u(e_i, y)e_j$. Then

$$\sup_{\|y\|=1} \| \Omega_{ij}(y) \| = \sup_{\|y\|=1} \left\langle \Omega_{ij}(y), \frac{\Omega_{ij}(y)}{\| \Omega_{ij}(y) \|} \right\rangle = \sup_{\|y\|=1} \left| R_u \left(e_i, y, e_j, \frac{\Omega_{ij}(y)}{\| \Omega_{ij}(y) \|} \right) \right|$$
$$\leq \sup_{\|x\|=\|y\|=\|w\|=\|z\|=1} |R_u(x, y, w, z)| \leq \frac{34}{3} \| S_u \| .$$

So, for any $u \in O(M)$ and orthonormal frame $\{e_i\}_{i=1}^d \subseteq T_oM$, $\| \Omega_{ij} \| \leq \frac{34}{3} \| S_u \|$. If we choose $\sup_{u \in O(M)} \| S_u \| < 3/(17d)$, then

$$\sup_{u \in O(M)} \| \Omega_u(e_i, \cdot)e_j \| < \frac{34}{3} \cdot \frac{3}{17d} = \frac{2}{d}.$$

□

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