CORRIGENDUM

A Correction to the Paper "Integration by Parts and Quasi-Invariance for Heat Kernel Measures on Loop Groups"

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It is asserted in Definition 4.2 in [1] that the random operators $U(t)$ defined there are unitary. As was pointed out to the author by Shizan Fang, it is clear that $U(t)$ is an isometry but it is not obvious that $U(t)$ is surjective. The purpose of this note is to fill this gap.

1. INITIAL COMMENTS

I would first like to point out that, even without verifying the surjectivity of $U(t)$ defined in Definition 4.2 in [1], all of the results and all but one proof in [1] would still be valid. Indeed, the only place where the surjectivity of $U(t)$ was used, other than for notational simplicity, was in the first proof of Theorem 4.14 in [1]. Nevertheless, Theorem 4.14 is still valid because of Theorem 6.2; see Remark 4.15 in [1]. The only notational changes that would need to be made are: (1) replace the orthogonal group $O(H_0(g))$ on $H_0(g)$ by the set $ISO(H_0(g))$ of isometries on $H_0(g)$ and (2) interpret $U(t)$ $H(t)$ as

$$U(t) \tilde{H}(t) \equiv \tilde{h}(t) + \frac{1}{2} \text{Ric} \ U(t) \ h(t).$$

In the next section we will give a more satisfying remedy to the gap in Definition 4.2 in [1], namely the fact that $U(t)$ is unitary.

2. A PROOF THAT $U(T)$ IS UNITARY

The reader is referred to [1] for the notation and definitions used in this corrigendum. Recall that $S_0 \subset H_0(g)$ is an orthonormal basis for $H_0(g)$ and

[Equation]
for any \( k_0 \in H_0(g) \) we let \( k(t) \) denote the solution to the Itô stochastic differential equation (4.2) in [1],
\[
dk(t) = -D_{g(\Omega)}k(t) + \frac{1}{2}A^{(1)}k(t) \ dt \quad \text{with} \quad k(0) = k_0. \tag{2.1}
\]
In [1], \( U(t) \) was defined as \( U(t)h := \sum_{k_0 \in S_0} (k_0, h) k(t) \) (Definition 4.2) and it was shown that \( h(t) := U(t)h \) solves Eq. (2.1) with \( h(0) = h \) (Lemma 4.3) and that \( U(t) \) is an isometry (Theorem 4.1). The surjectivity of \( U(t) \) will be an easy consequence of the next lemma.

**Lemma 2.1.** Let \( k_0, h_0 \in H_0(g) \); then
\[
E(k_0, U(t)^* h_0)^2 = E(U(t)k_0, h_0)^2 = E(k_0, U(t)h_0)^2. \tag{2.2}
\]

**Proof.** In what follows we will identify \( H_0(g) \otimes H_0(g) \) with the Hilbert–Schmidt operators \( HS(H_0(g)) \) on \( H_0(g) \) determined by identifying \( h \otimes k \in H_0(g) \otimes H_0(g) \) with the rank one operator \( (h \otimes k) u = (k, u) h \) for all \( u \in H_0(g) \). We are using \( (\cdot, \cdot) \) to denote inner product on both of the Hilbert spaces \( H_0(g) \) and \( H_0(g) \otimes H_0(g) \).

Let \( k(t) = U(t)k_0 \) and consider the random operator \( k(t) \otimes k(t) \). By Itô’s lemma,
\[
d(k(t) \otimes k(t)) = -(D_{g(\Omega)}k(t)) \otimes k(t) - k(t) \otimes D_{g(\Omega)}k(t)
+ \frac{1}{2} \left\{ A^{(1)}k(t) \otimes k(t) + k(t) \otimes A^{(1)}k(t) \right\} dt + 2 \sum_{\ell \in S_0} D_{\ell}k(t) \otimes D_{\ell}k(t) \tag{2.3}
\]
This last expression may be simplified by noticing that
\[
A^{(1)} \otimes I + I \otimes A^{(1)} + 2 \sum_{\ell \in S_0} D_{\ell} \otimes D_{\ell} = \sum_{\ell \in S_0} (D_{\ell} \otimes I + I \otimes D_{\ell})^2 : =: A^{(2)}. \tag{2.4}
\]
By Theorem 3.12 and Lemma 4.21 in Driver and Lohrenz [2], the sums in Eq. (2.4) converge strongly to a bounded self-adjoint operator \( A^{(2)} \) on \( H_0(g) \otimes H_0(g) \).

**Remark 2.2.** In [2] the operator \( D^{(2)} := (D_{\ell} \otimes I + I \otimes D_{\ell}) \) on \( H_0(g) \otimes H_0(g) \) was simply denoted by \( D_{\ell} \) and \( A^{(1)} \) on \( H_0(g) \) and \( A^{(2)} \) on \( H_0(g) \otimes H_0(g) \) were both denoted by \( A \).

With this notation, we may write Eq. (2.3) as
\[
d(k(t) \otimes k(t)) = -(D_{g(\Omega)}k(t)) \otimes k(t) - k(t) \otimes D_{g(\Omega)}k(t)
+ \frac{1}{2} A^{(2)}(k(t) \otimes k(t)) \ dt. \tag{2.5}
\]
Integrating this equation relative to $t$ and then taking expectations of the result show that

$$E(k(t) \otimes k(t)) = k_0 \otimes k_0 + \frac{1}{2} \int_0^t A^{(2)}(k(\tau) \otimes k(\tau)) \, d\tau$$

$$= k_0 \otimes k_0 + \frac{1}{2} A^{(2)} \int_0^t (k(\tau) \otimes k(\tau)) \, d\tau. \quad (2.6)$$

The solution to this last equation is

$$E(k(t) \otimes k(t)) = e^{t^{2}/2}(k_0 \otimes k_0). \quad (2.7)$$

Equation (2.6), along with the fact that $A^{(2)}$ is self-adjoint, implies

$$E(U(t) k_0, h_0)^2 = E(k(t), h_0)^2 = (e^{t^{2}/2}(k_0 \otimes k_0), h_0 \otimes h_0)$$

$$= (k_0 \otimes k_0, e^{t^{2}/2}(h_0 \otimes h_0)) = E(k_0, U(t) h_0)^2. \quad (2.8)$$

**Theorem 2.3.** The random isometry $U(t)$ defined in Definition 4.2 in [1] is unitary a.s.

**Proof.** Let $P(t) := U(t) U(t)^*$, a random projection operator. Our goal is to show that $P(t) = I$ a.s. Summing Eq. (2.2) on $k_0 \in S_0$ and using the fact that $U(t)$ is an isometry shows that

$$E \| P(t) h_0 \|^2 = E \| U(t)^* h_0 \|^2 = E \| U(t) h_0 \|^2 = \| h_0 \|^2$$

for all $h_0 \in H_0(g)$. Because $\| h_0 \|^2 \geq \| P(t) h_0 \|^2$, it follows that $\| h_0 \|^2 = \| P(t) h_0 \|^2$ a.s. or equivalently $h_0 = P(t) h_0$ a.s. Since $H_0(g)$ is separable, we may conclude that $I = P(t)$ a.s. as desired. Q.E.D

Theorem 2.3 may be strengthened as follows. Another proof of the following theorem which was discovered essentially simultaneously to the one presented here will appear in Fang [3].

**Theorem 2.4.** On a set of full measure independent of $t \geq 0$, the map $t \mapsto U(t)$ is unitary. That is, the null sets implicitly appearing in Theorem 2.3 may be chosen to be independent of $t$.

**Proof.** Let $h \in H_0(g)$. We will start by showing that there exists a null set $\Omega^*_n$ such that on $\Omega^*_n$, the map $t \mapsto \| P(t) h \|^2$ is continuous. To this end let $\{ S_n \}_{n=1}^\infty$ be a collection of finite subsets contained in $S_0$ such that $S_n$
increases to $S_0$ as $n \to \infty$. For $k_0 \in S_0$ set $k(t) = U(t) k_0$ and let $P_n(t)$ be the
finite rank projection operators

$$P_n(t) := \sum_{k \in S_n} U(t) k_0 \otimes U(t) k_0 = \sum_{k \in S_n} k(t) \otimes k(t).$$

Since $k$ is a continuous process for each $k_0 \in S_0$, there is a null set $\Omega_1$ such that

$$t \to \|P_n(t) h\|^2 = \sum_{k \in S_n} (k(t), h)^2 = \sum_{k \in S_n} (k(t) \otimes k(t), h \otimes h)$$

is continuous on $\Omega_1$ for all $h \in H_0(q)$ and $n = 1, 2, 3, \ldots$. Let $P_{m,n}(t) := P_m(t) - P_n(t)$ and suppose for concreteness that $m > n$. Then by Eq. (2.5), the skew symmetry of $D^{(2)}$, and the symmetry of $A^{(2)}$,

$$\|P_{m,n}(t) h\|^2 - \|P_{m,n}(0) h\|^2 = M_{m,n}^m + A_{m,n}^m,$$

where

$$M_{m,n}^m := \left[ \int_0^t (P_{m,n}(\tau), D^{(2)}_{jij}(h \otimes h)) d\tau \right]$$

and

$$A_{m,n}^m := \frac{1}{2} \left[ \int_0^t (P_{m,n}(\tau), A^{(2)}(h \otimes h)) d\tau \right].$$

Because $M_{m,n}^m$ is a square integrable martingale,

$$E( \sup_{0 \leq t \leq T} |M_{m,n}^m|^2 ) \leq CE |M_{m,n}^m|^2$$

which converges to zero as $m,n \to \infty$ by the dominated convergence theorem along with the facts: (1) $\|P_{m,n}(t) D_j h\|^2 \leq \|D_j h\|^2$; (2) $\sum_{j \in S_n} \|D_j h\|^2 = (-\Delta h, h) \leq \|A\|_{op} \|h\|^2$; and (3) $\lim_{m,n \to \infty} \|P_{m,n}(t) D_j h\|^2 = 0$. Similarly,
\[
\sup_{0 \leq t \leq T} |A_m^{m+} - A_{m+}^{m+}| = \sup_{0 \leq t \leq T} \left| \int_0^t \sum_{k_0 \in S_k \cap S_{k_0}} (k(\tau), Ah)(k(\tau), h) \, d\tau \right. \\
+ \left. \int_0^t \sum_{k_0 \in S_k \cap S_{k_0}} \left( \sum_{\ell \in S_k} (Df, k(\tau))^2 \right) d\tau \right. \\
\leq \int_0^T \left[ |(Ah, P_{m,n}(\tau) h)| + \sum_{\ell \in S_k} \|P_{m,n}(\tau) Df, h\|^2 \right] d\tau
\]

which converges to zero boundedly as \(m, n \to \infty\). Combining the above estimates shows that
\[
E \sup_{0 \leq t \leq T} \|P_{m,R}(t)\|^2 - \|P_{m,R}(t) h\|^2 = E \sup_{0 \leq t \leq T} \|P_{m,n}(t) h\|^4 \to 0 \quad m, n \to \infty.
\]

Therefore there exists a null set \(\Omega_h\) such that on \(\Omega'_h\), \(t \in [0, T] \to \|P(t) h\|^2\) is the uniform limit of the continuous functions and hence is continuous.

Since \(H_0(g)\) is separable, we may choose a null set \(\Omega_2\) independent of \(h \in H_0(g)\) and \(T > 0\) such that \(t \in [0, T] \to \|P(t) h\|^2\) is continuous on \(\Omega'_2\).

By Theorem 2.3, given a countable dense subset \(D \subseteq [0, T]\), there exists a null set \(\Omega_D\) such that \(P(t) = I\) on \(\Omega'_D\); i.e., \(\|P(t) h\| = \|h\|\) for all \(h \in H_0(g)\) and \(t \in D\). Let \(\Omega_0\) be the null set, \(\Omega_0 = \Omega_2 \cup \Omega_D\). Then on \(\Omega_0\), \(\|P(t) h\| = \|h\|\) for \(t \in [0, T]\) and \(h \in H_0(g)\) or equivalently \(P(t) = I\) for all \(t \in [0, T]\).

Q.E.D

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REFERENCES

3. Shizan Fang, Girsanov theorem and quasi-invariance on the path space over loop groups, Univ. de Paris VI, preprint 1997.