Classifications of Bundle Connection Pairs
by Parallel Translation and Lassos*

BRUCE K. DRIVER

Department of Mathematics, C-012, University of California, San Diego, La Jolla, California 92093

Let $M$ be a connected manifold and $G$ be a closed Lie subgroup of $GL(F)$—
the general linear group on a finite dimensional vector space $F$. Denote by $(L_g)$ the
space of $H^1$-loops in $M$ starting at a fixed point (w). Let $\mathcal{M}$ be the set of
$P \mapsto G(\mathcal{M}, F)$ module conjugation by an element of $G$, such that $P(\alpha) = P(\alpha)P(\beta)$ for $\alpha, \beta \in G$, or the concatenation of the loops $\alpha$ and $\beta$, $P(\alpha) = P(\beta)$ if $\alpha = \beta$ and $P(\alpha)$ is a "locally" condition. It is shown that the bundle connection pairs $(E, F)$ (up to
equivalence), with structure group $G$ and fiber $F$, are in one to one correspondence
with $\mathcal{M}$—a similar result has been announced by Kobayashi. The correspondence is
induced by the parallel translation operators of connections. Furthermore, if the
manifold $(M)$ is simply connected, then the space of bundle connection pairs can be
classified by a collection of Lie algebra valued 1-forms on the manifold $\Omega_2$, called
integrated lassos). These 1-forms are related to the differentials of elements of $\mathcal{M}$.
This last result generalizes Weil's characterization of $U(1)$—line bundle connection
pairs by the curvature 2-form. It is also a generalization of Groth's results to base
manifolds $(M)$ other than $\mathbb{R}$. © 1999 Academic Press, Inc.

1. INTRODUCTION

Let $F$ be a closed imaginary valued (iR is the Lie algebra of $U(1)$) 2-form
on a connected manifold $M$. Set $\omega = (2\pi)^{-1} F$, and denote by $[\omega]$, the De
Rham cohomology class of $\omega$. Then it is known [We] that $F$ is the curv-
ature 2-form for a $U(1)$-connection $\nabla$ on some complex line bundle $E$ over
$M$ if and only if $[\omega]$ is integral. (For a short review see Presley and Segal
[PrS] Section 4.5 and for a detailed treatment see Kostant [Ko]. That is,$\int_M \omega$ should be an integer for all integral 2-chains $(c)$. Alternatively $[\omega]$ should be in the image of the natural homomorphism $\ast: H^2(M, \mathbb{Z}) \to H^1(M, \mathbb{R})$, where $H^2(M, \mathbb{R})$ stands for any one of the equivalent
cohomologies—Cech, De Rham, or simplicial—and $H^1(M, \mathbb{Z})$ is either Cech
or simplicial. The theory of the first Chern class gives a 1–1 correspondence

*This research was supported in part by N.S.F. Grant No. DMS 84-01997, while the
author was at the Institute for Advanced Study, Princeton, NJ 08540.

0022-1236/99 $3.00
Copyright © 1999 by Academic Press, Inc.
All rights of reproduction in any form reserved.

185
Theorem 2.2.1 of [Ko]). On the other hand (Theorem 2.5.1 of [Ko]), if M is not simply connected, ρ−1(F) is not one to one correspondence with \(H^*(M)\) the homomorphisms from \(H^*(M)\) to \(U(1)\). In the physical literature, the property that \(ρ−1\) is not 1 1 is called the gauge field copy problem.

In [Ko], there is another description of the space \(E\) in terms of the parallel translation operators on loops. Let \(Ω:=\{\varepsilon \in [0,1], M: 0=\varepsilon \} \) be the space of piecewise smooth loops on \(M\). Let \(P^B\) be parallel translation on \(Ω\) with respect to the connection \(V\). Since \(P^B(\varepsilon)\) is an endomorphism of \(E(\varepsilon)\), the fiber of \(E\), \(\varepsilon\) is a 1-dimensional complex vector space, \(P^B(\varepsilon)\) can be identified with a complex number. Proposition 1.23 of [Ko] (also see Theorem 5.1 below) states that two \((E, V)\)-bundles with connectives, \((E, V)\) and \((E', V')\), are equivalent if and only if \(P^B=P'^B \) on \(Ω\).

The purpose of this paper is to prove analogous statements for more general vector bundles. We are interested in considering \(G\)-vector bundles, where \(G\) is a closed subgroup of \(GL(V)\) (see Section 2). The goal is to classify the space, \(L=\mathcal{H}(M, \nabla, G)\), of \(G\)-vector bundle connection pairs \((E, \nabla)\) modulo equivalence.

The first question is what should play the role of the curvature 2-form when the structure group \(G\) is not \(U(1)\). The obvious choice of using the curvature tensor as before is not satisfactory. The main reason being that the curvature tensor (\(\sigma^V\)) is a 2-form valued in \(\text{End}(\mathcal{H})\) – the endomorphism bundle associated to \(E\). So in order to specify an \(F\), one has to first specify the bundle. Leaving this issue aside for a moment, the natural analogue of \(DF\) is the Bianchi identity \(D\sigma^V=0\), where \(D\sigma^V\) is the curvature differential of \(\sigma^V\) in the sense of Gross [G1], when \(G\) is not commutative. The equation \(D\sigma^V=0\) corresponds to the Bianchi identity.

In Theorem 6.1 and Theorem 6.2 of this paper we extend this result to general simply connected manifolds \(M\). This requires an added condition on a 2-form which is an analogue of the integrality condition for \(F\) if \(G=U(1)\). In fact, for \((E, V)\)-bundles, the added condition reduces to the integrality condition, see Corollary 6.1.

In case where \(M\) is not simply connected, the lassos are no longer in one to one correspondence with \(\sigma\). In close analogy to the \(U(1)\) case, given an "irreducible" lasso \((L_\delta)\), the set of potential \(L_\delta\) associated to \(V\) is still just the set of all \(\delta\). As discussed above, the copy conjugation by elements of \(G\) is in one to one correspondence with the set of homomorphisms of \(H^*(M)\) to the center of \(G\), see Theorem 7.1. Therefore, lassos suffer from a gauge copy problem. For example, Wu and Yang [WY] have shown when \(G\) is not commutative that the curvature \(\omega^V\) is not sufficient to determine the connection \(V\) up to the equivalence even when \(M=R^n\). (See also [MS] and its bibliography.) Nevertheless, K. Mackenzie [Ma] has announced a criterion for the existence of principal bundle connections with prescribed curvature form. This result is not in the spirit of this paper, since the formulation requires the bundle to be prescribed at the outset.

In [G1], Gross proposes to use "lassos" as an analogue for the curvature tensor. In order to define a lasso, let \(\gamma\) be the space of \(H^\gamma\)-paths on \(M\) starting at \(m\) (see Section 2). The lasso associated to a connection \(V\) on a vector bundle \(E\) is \(L^\gamma(\gamma, V) \approx \text{End}(\mathcal{H})\) for \(\gamma \in \mathcal{Y}\) and \(\mathcal{Y} \subset T_{\gamma}M\) (see Definition 4.3). It should be noted, if \(G=U(1)\), that \(L^\gamma(\gamma, V) \approx \mathcal{U}(1)\) is essentially the curvature 2-form again. We refer the reader to the discussion in [G1] motivating the study of these objects as related to "quantized gauge fields." Such path dependent ojjects have been discussed in the physics literature, see Birula [B] and Mandelstam [Man3].

Now choose a local trivialization \(\varphi\) over a neighborhood of \(m\), and use \(\varphi\) to identify \(E_m\) with the \([k]k\)k space \(V\). Because of this identification we may consider \(L^\gamma(\gamma, V) \approx \mathfrak{g}\), the Lie Algebra of \(G\) which is a subspace of \(\text{End}(V)\). \(L^\gamma(\gamma, V) \approx \mathfrak{g}\) is now only defined up to conjugation by an element of \(G\). A lasso is an example of a more general object called a path 2-form. A path 2-form \(L_\gamma\) is a (smooth) function \(L_\gamma\) with values in \(\mathfrak{g}\), defined on \([0, u, v) \in \mathcal{Y}\); \(L_\gamma(u, v)\) is the skew symmetric in \(u, v\) and \(\mathfrak{g}\). The main theorems of [G1] give necessary and sufficient conditions for a path 2-form \(L_\gamma\) to be a lasso in case \(M=R^n\). These conditions generalize the equation \(d\sigma^V=0\) for \(G=U(1)\). The conditions formulated in [G1] are intrinsic conditions, i.e., not requiring a vector bundle or a connection for their formulation. Furthermore, it is shown in [G1] that the lasso moduli conjugation by elements of the structure group \(G\) are in one to one correspondence with elements of \(\text{End}(\mathcal{H})\) (see Section 2).
simply connected are not sufficient to guarantee that a path 2-form is a
1-lasso on \( \mathcal{P} \) (Example 7.1). However (Theorem 7.2 and Corollary 7.1), the
theorem of Definition 6.1 does imply that the path 2-form is the "pullback" of a
1-lasso on the path space of the universal cover of \( M \). The author feels the
simply connected case may be a good application to the non-abelian cohomology
theory in Decker [DI-1] and Brown [BR].

Despite the difficulty with the lassos for non-simply connected
manifolds—indeed, there is a path 2-form \( (M, V, G) \) which can always
be classified by the parallel translation operators. In Theorem 5.1 (also see
[Kob,]), we show that \( \mathcal{E}(M, V, G) \) is in one to one correspondence with a
subset \( \mathcal{M} \) of \( \mathcal{C}^\infty(W, G) \) (see Definitions 5.2-5.4).

It should be remarked that by Theorem 4.1 (see also Theorem 2.2 and
Corollary 2.1.6 of [G1]) the differential of the parallel translation operator \( F^P \)
may be expressed in terms of the lasso \( L^1 \). So Theorem 6.1 involving the
lassos may be thought of as the infinitesimal version of Theorem 5.1.

This paper is divided into seven sections. In section 2, some basic
definitions and notations are introduced. Section 3 is a review of some
basic properties about the Hilbert manifold of \( H^1 \)-paths \( \mathcal{P} \) on a manifold
\( M \). This section also contains some technical results which are needed for
determining when a function, which has \( \mathcal{P} \) as either the domain or the
range space, is smooth. In section 4, the parallel translation operator is
shown to be smooth, and its differential is computed (Theorem 4.1.6) in terms of
lassos and integrated lassos (see Definitions 4.2 and 4.3). Section 5
gives the loop characterization of \( \mathcal{E} \), see Theorem 5.1. Section 6 deals with
the lasso characterization of \( \mathcal{E} \), see Theorems 6.1 and 6.2 and Corollary 6.1.
Section 7 contains some remarks for non-simply connected manifolds.
The results of Sections 5-7 have already been discussed in this introduction.

It is a pleasure to thank Leonard Gross and Mitchell Rothstein for many
useful discussions, and the Institute for Advanced Study where much of this
work was done.

2. Notation

For the purposes of this paper all manifolds will be \( C^\infty \); however, there
will be numerous occasions for using functions of different degrees of
smoothness. The following prefixes will be used to denote the smoothness of a
particular function: \( C^k \) for \( r \)-continuous derivatives \( (C = C^0) \), \( AC \) for
absolutely continuous, \( H^k \) for absolutely continuous and the derivative in
\( L^k \), and \( PC^k \) for piecewise \( C^k \).

Throughout this paper \( (E, V, \eta, N, G) \) or \( E \) for short will denote a vector
bundle \( E \) over \( M \) with fiber \( V \) (\( V \) is a complex or real finite dimensional
vector space), \( \eta : E \to M \) is the projection map, and \( G \) is the structure
group. The group \( G \) is assumed to be a closed Lie subgroup of \( GL(V) \), the
linear automorphisms of \( V \). To say that \( E \) has structure group \( G \) means
that there is a distinguished class of "admissible" local trivializations \( (\psi, U) \)
of \( E \) covering \( M \) for which the transition functions are \( G \)-valued. (Here,
\( U \) is an open subset of \( M \) and \( (\psi, \phi) : \pi^{-1}(U) \to U \times V \) is a diffeomorphism.)

Suppose \( (\psi, U) \) defined above is admissible, then there is a \( C^\infty \)
function \( g : U \to G \) such that \( (\psi, \phi) : (x_0, \phi(x_0)) \to (x_0, g(x_0)) \) for all \( (x_0, e) \in U \times V \). In (the sequel, the phrase local trivialization will always
mean an admissible local trivialization.) To simplify such statements, it is often
covenient to write \( E \) for \( \pi^{-1}(\psi) \), and \( \psi \) if \( \psi \) is any function on \( E \).

Given a covariant derivative (or connection) \( \nabla \) on \( E \) and a local
trivialization \( (\psi, U) \), the associated connection one form \( \omega \) is defined by
\[ \omega^\mathcal{A}(\psi) = \nabla_{(m \to \psi(m) \cdot \xi)} U \to \pi^{-1}(U) \] for \( v \in TU \).

(As a general rule, an argument of a function which is enclosed by the
brackets \( \{ \cdot \} \) will indicate that the function is linear or fiber linear in
that variable.) The terminology covariant derivative and connection will be
used interchangeably in this paper. We will only consider connections on \( E \)
compatible with the structure group \( G \). This means for all \( \psi \in U \), \( A^\mathcal{A} \) is a 1-form on \( U \) taking values in the Lie algebra \( \mathfrak{g} \) of \( G \),
which may be considered to be a subspace of \( \operatorname{End}(V) \).

Remark 2.1. For later purposes we note that \( A^\mathcal{A} \) is related to \( A^\mathcal{\psi} \) by
\[ A^\mathcal{\psi} = g^{-1} A^\mathcal{A} g + g^{-1} d g \]
\[ = g^{-1} A^\mathcal{A} g - d (g^{-1}) g \quad \text{on} \quad \pi^{-1}(U \cap V), \]

(2.1)

where \( (\psi, U) \) and \( (\phi, V) \) are two local trivializations of \( E \).

The curvature of a connection \( (\psi, U) \) is \( F^E \in A(T^*M) \otimes \operatorname{End}(E) \) defined by
\[ F^E(X, Y) = \{ \nabla_X \nabla_Y \nabla^{-1}(X, Y) \} \]
\[ = \{ \nabla_{X, Y} Y^{-1} \nabla_{X, Y} \} \]

(2.2)

for \( X \) and \( Y \) vector fields on \( M \). In terms of a local trivialization \( (\psi, U) \), the
local expression for the curvature is
\[ \psi_* \xi F^E(m) \psi_* \alpha \to A^\mathcal{A} \alpha A^\mathcal{A} + A^\mathcal{A} A^\mathcal{A} \]

(2.3)

for \( m \in U \), where the following notation is being used. If \( \alpha \) is a k-form, then
\( \alpha(m) \) denotes restriction of \( \alpha \) to \( A^\mathcal{A} \). The term \( A^\mathcal{A} A^\mathcal{A} \) in (2.3)
denotes the 2-form with value in \( \mathfrak{g} \) given by
\[ A^\mathcal{\psi} \cdot A^\mathcal{\psi} \alpha \to (A^\mathcal{\psi} \alpha, A^\mathcal{\psi} \alpha) \]

(2.4)

for \( u, v \in T_m M \).
Definition 2.1. Given a 1-form $A$ on an open subset $(U)$ of $M$ taking values in $H(T^*U)$, the curvature $(F^A)$ of $A$ is

$$F^A = dA + A \wedge A.$$  

(2.5)

Of course in terms of (2.5) and (2.1) one may write $F^\phi(m) = \phi \wedge F^A(m) \phi^{-1}$ for $m \in U$.

Definition 2.2. If $A$ is a $\mathfrak{g}$-valued 1-form on an open set $U$ of $M$, and $\sigma: [0, 1] \rightarrow U$ is a $C^1$-path, then parallel translation along $\sigma$ with respect to $A$ up to time $s$ is the element $P^\sigma_s(e) \in G$, where $P^\sigma_s(e)$ satisfies

$$\frac{d}{ds} P^\sigma_s(e) + A(\sigma'(s)) P^\sigma_s(e) = 0$$

and $P^\sigma_1(e) = 1 \in G$.

Remark 2.2. If $A = \alpha^a$ is the connection 1-form of a covariant derivative $(\nabla)$ with respect to a local trivialization $(\phi, U)$, the parallel translation operator $P^\phi_s(e)$ can be expressed as

$$P^\phi_s(e) = \phi^{-1}_0(e \tau^a) P^\sigma_s(\phi)_0^a,$$  

(2.6)

for any $C^1$-path in $U$.

If $f: N \rightarrow M$ is a map, the notation $\Gamma_k(L)$ will be used to denote the $C^0$-sections $s: \pi^{-1}(L) \rightarrow f$ along $f$. As indicated above $\Gamma_k(E)$ would then denote the continuous sections along $f$, with analogous statements for different prefixes.

3. HILBERT MANIFOLDS OF BARED LOOPS

This section introduces notation and reviews some basic facts about certain submanifolds of the Hilbert manifold of $H^1$-curves on a manifold $M$. (For a thorough treatment of this material see [K11].)

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space, and $J$ be a subinterval of $I = [0, 1]$. Put $H^1(J, V) = \{ e \in AC^1(J, V) \}$ and $\Vert e \Vert_1 = \langle e(t) \rangle + \langle e'(t) \rangle$ where $\Vert e \Vert_1 = \int_e (\langle e(t) \rangle + \langle e'(t) \rangle) dt$ is an infinite dimensional Hilbert space. Let $M$ be a fixed $n$-dimensional manifold. A path $J \rightarrow M$ will be called $H^1$, if $\frac{d}{dt} X(t)$ is $H^1$ for each subinterval $K \subset J$ and coordinate chart $(x, U)$ of $(M)$ for which $(U) \subset K$. (The notion of a curve being $H^1$ is chart independent.) The set of $H^1$ curves $J \rightarrow M$ will be denoted by $H^1(J, M)$.

It is fairly standard that $H^1(J, M)$ is Hilbert manifold modeled on $H^1(I, R^n)$ (The manifold structure will be described presently.) For the basic definitions and facts about infinite dimensional manifolds, see Lang [L] and Eells [E2]. For discussions on spaces of maps between manifolds as infinite dimensional manifolds, see [EI1-2], [ES1-2], [EI], [P] and [PS] and especially [KL1-3].

In order to describe the manifold structure on $H^1(J, M)$, fix a Riemannian metric on $M$, which will be denoted by $(g)$ or $\langle \cdot, \cdot \rangle$. Let $\exp$ denote the geodesic flow associated to the Levi-Civita connection of $g$, and $\alpha^{\cdot}$ be the induced metric on $M$. It is easy to check that the space $H^1(J, M)$ may alternatively be characterized as

$$H^1(J, M) = \{ e \in AC^1(J, M) : \Vert e \Vert_1 < \infty \}$$  

(3.1)

where

$$\Vert X \Vert_1 = \int_J \langle X(t), X(t) \rangle dt.$$  

(3.2)

for any Borel measurable $X: J \rightarrow TM$. Let $D$ denote the covariant derivative on $M$ of the Levi-Civita connection for $g$. For any $\alpha \in H^1(J, V)$, let $\rho^\alpha$ be the pullback of $D$ to $AC^1(J, TM)$, the absolutely continuous sections of $TM$ along $\alpha$. To be more explicit, suppose that $\Gamma$ is the connection form for $D$ with respect to some local trivialization of $TM$, and $X: J \rightarrow R^n$ is the local expression of a section along $\alpha$, then

$$\langle \rho^\alpha (\Gamma)(t), X(t) \rangle = \langle \Gamma(t), X(t) \rangle.$$  

(3.3)

For any $\alpha \in H^1(J, M)$, let

$$T_\alpha H^1(J, M) = \{ X \in AC^1(J, TM) : \Vert X \Vert_1 < \infty \},$$  

(3.4)

where $\Vert X \Vert_1$ is defined as

$$\Vert X \Vert_1 = \int_J \langle X(t), X(t) \rangle dt + \int e^{\rho^\alpha(X)}(t) dt.$$  

As the notation suggests this will be the tangent space to $H^1(J, M)$. In view of (3.3) $T_\alpha H^1(J, M)$ may alternatively be described as the set of $H^1$ sections along $\alpha$, a notion independent of the metric $g$.

In order to describe the coordinate charts on $H^1(J, M)$, we recall a basic fact from Riemannian geometry.

Theorem 3.1. Let $(M, g)$ be a Riemannian manifold and $K \subset M$ be a compact set, then there exists an $\varepsilon > 0$ such that for each point $p \in K$ the exponential map restricted to $\{ X \in T_p K : \langle X(p), X \rangle < \varepsilon^2 \}$ is a diffeomorphism onto the neighborhood $B_p(\varepsilon) = \{ \varepsilon \in M : \varepsilon \in p \}$ which is geodesically convex. (Recall that a neighborhood $N$ is geodesically convex if for any pair
of points p, q ∈ N there exists a unique geodesic of minimal length joining p to q which lies entirely in N.) Furthermore, let \( K_1 = \{ p \in M : d(p, K) < \epsilon \} \) and \( K = \inf \{ d(p, k) : k \in K \} \) and \( U = \{ (p, q) \in \mathbb{R} \times (0, \infty) : d(p, q) < \epsilon \} \) (an open subset of \( \mathbb{R} \times M \)). Then there is a \( C^\infty \) map \( \gamma : U \rightarrow TM \) such that \( \gamma(0, p) = p \) and \( \exp_p \gamma(t, p) = q \). As detailed in Section 2, \( \exp_p \) denotes \( \exp \) restricted to its domain intersected with \( T_pM \).

Proof. This is a slight generalization of the method outlined in Spivak [5p, pg 49], problem 32] for the case where \( K \) is a point. Also see pages 32–36 of [1]. To generalize this to arbitrary \( K \) compact, it is enough to show that each point of \( M \) has a neighborhood for which the theorem holds. (This fact is theorem 1.9.10 of [KL]). But for this case the proof outlined in Spivak still goes through, if one observes by smoothness that the key estimates hold uniformly at a point. Q.E.D.

We are now in a position to describe the coordinate neighborhoods on \( H^1(I, J) \). Let \( \sigma \in C^2(J, M) \) choose \( \epsilon > 0 \) as in Theorem 3.1 for the compact set \( K = \) image of \( \sigma \). Put \( W(\sigma, \epsilon) = \{ x \in H^1(I, J) : d(\sigma(t), \sigma(t')) < \epsilon \text{ for } t, t' \} \), then for each \( \xi \in W(\sigma, \epsilon) \) and \( \tau \in \text{the point } \sigma(t), \sigma(t') \text{ is in } U_\sigma \) as above. Hence the map \( V_\xi : W(\sigma, \epsilon) \rightarrow H^1(I, J) \) by \( V_\xi(x) = (x \circ \sigma, \sigma(t), \sigma(t')) \) is \( C^\infty \) by Lemma 3.1 below and is bijective, where

\[
\|\|_\tau = \sup \|\langle x(t), x'(t)\rangle^{\tau/\epsilon} ; t \in I \|
\]

**Theorem 3.2.** (Restatement of 2.3.23 Theorem of [KL]). The collection of charts \( \{ (X_\xi, W(\sigma, \epsilon)) : \sigma \in C^2(J, M) \} \) induces \( C^\infty \) manifold structure on \( H^1(I, J) \) which is modeled on any one of the equivalent Hilbert spaces \( T_pH^1(I, J) \) for \( p \in C^1(J, M) \). In particular the charts \( (X_\xi, W(\sigma, \epsilon)) \) are \( C^\infty \) related.

For a proof of this theorem we refer the reader to [KL] or [1].

**Remark 3.1.** The sets \( W(\sigma, \epsilon) \) do not form a basis for the topology on \( H^1(I, J) \), but sets of the form \( U_\sigma(x, T_pH^1(I, J)) \triangleq \|\|_\tau \) do form a basis.

**Remark 3.2.** \( T_pH^1(I, J) \) is naturally isomorphic to \( H^1(I, T_pM) \), where \( m = \sigma(a), f = [a, b] \), and the inner product on \( T_pM \) is the metric (x) restricted to \( T_pM \). To demonstrate this let \( \phi(t) \) be a parallel translation in \( T_pM \) with respect to the Levi-Civita connection \( D \). For \( x \in H^1(I, T_pM) \) put \( X(t) = q(t)(X(t')) \), so that \( X \in T_pH^1(I, J) \). Then \( \phi(DX(t)) = \phi(t)(X(t')) \), hence \( \|\|_\tau \) for all \( x \in H^1(I, T_pM) \) because \( \phi(t) \) preserves the metric \( g(t)(x) \) on \( T_pM \). Thus \( X \rightarrow X(t = 0) = T_pH^1(I, J) \) is a Hilbert space isomorphism.

For the last purposes it is convenient to develop some techniques for showing that maps having \( H^1(I, J) \) as the range or the domain space are smooth. The reader might wish to skip to Lemma 3.2 and only refer to these results when necessary. The next proposition is useful for computing derivatives of functions on \( H^1 \).

**Proposition 3.1.** Let \( f : B \rightarrow K \) be a continuous map between two Banach spaces \( B \) and \( K \) with norms both denoted by \( \| \cdot \| \). Let \( Df \) be a dense sub-space and assume there exists a continuous function \( P : B \rightarrow \text{Hom}(B, K) \) such that for all \( d, d' \in D \), \( P(f(x))df(x) \) and \( f(x)df(x) \) exists and is equal to \( F(d)(d') \). Then \( f \) is in \( C^1 \) and the differential of \( f \) is \( F \).

**Proof.** By the fundamental theorem of calculus,

\[
(f(d + d') - f(d)) = \int_0^1 \langle \partial_x f\rangle \langle d\rangle \text{dt} = \int_0^1 \langle P(d + td') \rangle \text{dt}
\]

for all \( d, d' \in D \). Both sides of this last equation are continuous function on all of \( B \times B \), so in fact the last equation hold for all \( d, d' \in D \). For \( d, d' \),

\[
\langle f(d + d') - f(d) \rangle \text{dt} \leq \langle \int_0^1 |P(d + td')| \text{dt} \rangle \langle f(x) \rangle \text{dt} |d'
\]

This shows that \( F \) is differentiable with differential equal to \( F \). Q.E.D.

**Remark 3.3.** It is clear that Proposition 3.1 is true if the domain of \( f(\mathbb{D}(\mathbb{F})) \) is an open convex subset of \( B \) and \( D \) is a dense convex subset of \( D(\mathbb{F}) \).

**Lemma 3.1.** Suppose \( X \in H^1(I, \mathbb{R}^n) \) and \( f \) is a \( C^1 \) function defined in a neighborhood (N) of the graph of \( X \) taking values in \( \mathbb{R}^n \). (A function \( f(\mathbb{F})(x) \) is \( C^1 \) if all partial derivatives of \( f \) with respect to the \( x \) variables are \( C^1 \)). In \( (u, x) \), then for \( \varepsilon > 0 \) sufficiently small, the map \( Y = f(\mathbb{F}) \) is smooth about \( \mathbb{X} \) inside \( H^1(I, \mathbb{R}^n) \).

**Proof.** This is a special case of 1.2.5 Lemma of [KL]. A direct proof is easily given using Proposition 3.1. The fact that \( f \) is continuous on \( B(x, \varepsilon) \) is fairly straightforward by uniform continuity arguments. Let
BUNDLE-CONNECTION CLASSIFICATIONS

Proof. Let \( \sigma \in C^k(I, M) \), then the lemma is equivalent to showing that 
\[(i, X) \in \exp_{(\sigma(t))}(0, \xi(t)) \\
\text{is } C^k \text{ for near zero in } H^I(\La_{\psi}, M), \]
where \( \xi(t) \) is parallel translation along \( \sigma \) as in Remark 3.2. But this is easily seen to be the case by the same techniques of Lemma 3.1. Q.E.D.

DEFINITION 3.1. For each \( i \in I \), put \( \beta_i(e) = \sigma_i(t) \in M \) for \( \sigma \in H^I(I, M) - \)
\( \beta_i \in C^k \) by Lemma 3.2. If \( i = 1 \), write \( \beta \) for \( \beta_1 \).

DEFINITION 3.2. The base spaces at \( m \) are \( \P_m = \beta_1^{-1}(m) \).

DEFINITION 3.3. The loop space based at \( m \) is \( \Omega_m = \{ e \in \P_m : e(1) = e(0) = m \} \).

The space \( \Omega_m \) is a submanifold of \( H^I(I, M) \) and the space \( \Omega_m \) is a submanifold of \( \P_m \). This is seen by the implicit function theorem (see Corollary 2.1 of [L1]) coupled with the fact that the differential of \( \beta \) is surjective. The tangent bundle of \( \Omega_m \) may be identified with the subbundle of \( T \P_m \) given by \( \{ x \in T \P_m : X(1) = 0 \} \), and the tangent bundle of \( \Omega_m \) may be identified with the subbundle of \( T \P_m \) given by \( \{ x \in T \P_m : X(1) = 0 \} \).

4. PARALLEL TRANSLATION, LADIES, AND INTEGRATED LADIES

Our next goal is to show that parallel translation along a curve in \( \P_m \) is well defined (Corollary 4.1). In fact, it will be shown that the parallel translation operator \( P^e \) is a \( C^k \)-function on \( \P_m \). We first prove a local version of this fact. This discussion closely parallels the discussion in [G1] in the case \( M = \bb{R}^n \).

LEMMA 4.1. Let \( V \) be a finite dimensional inner product space. Then there exists a unique \( C^k \) function \( P : L^2(I, End(V)) \rightarrow H^I(I, Aut(V)) \)
\((H^I(I, Aut(V)) = \{ e \in H^I(I, End(V)) : e(t) \in Aut(V) \text{ for all } t \} \), an open subset of \( H^I(I, End(V)) \) which satisfies:

DE1. \((d/dt)P(A)(t) + A(t)P(A)(t) = 0 \text{ for almost all } t \),

DE2. \( P(A)(0) = Id \in Aut(V) \).

Furthermore, the differential of \( P \) is
\[ DP(A)(B)(t) = -P(A)(t)^{-1} B(t)P(A)(t) \text{ for almost all } t. \]

(DuhAME's Principle)
Remark 4.1. The interval \( t \) may be replaced by any other interval with obvious changes in notation.

Proof. By standard theorems on linear, ordinary differential equations, for each \( A \in C^\infty(I, \text{End}(V)) \) there is a unique continuous (in fact \( C^\infty \) in \( t \)) function \( (P, \cdot) \) satisfying DE1 and DE2. We will now show that this solution satisfies the estimates

\[
|P(A)(t) - P(B)(t)| \leq K(|A|_1 + |B|_1) \|A - B\|_1
\]

and

\[
\frac{d}{dt} |P(A)(t) - P(B)(t)| \leq K(|A|_1 + |B|_1) \|A - B\|_1,
\]

(4.2)

where \( |\cdot| \) denotes any of the equivalent norms on \( \text{End}(V), \|\cdot\|_1 \), is the \( L^2 \)-norm on \( C^\infty(I, \text{End}(V)) \) where the norm on \( \text{End}(V) \) is taken to be the Hilbert-Schmidt norm. The function \( K \) denotes a continuous positive function increasing in each of its arguments. This last estimate shows that the function \( P \) extends uniquely to all \( L^2(I, \text{End}(V)) \) from \( C^\infty(I, \text{End}(V)) \). The extended function will again be called \( P \).

In the argument below, \( K \) will denote a continuous function which is increasing in any of its arguments. We will now prove the assertion in (4.3).

Set \( f(t) = h(t) = |P(A)(t) - P(B)(t)| \), and then by DE1 we can estimate \( h(t) \) by

\[
\frac{d}{dt} h(t) = |A(t) - B(t)| |P(B)(t)| + |A(t)||f(t)|.
\]

(4.3)

Upon integration and by the fact that \( h(0) = 0 \) we find the inequality

\[
f(t) \leq \int_0^t |A(s) - B(s)| |P(B)(s)| ds + \int_0^t |A(s)||f(s)| ds.
\]

(4.4a)

Now first suppose that \( A = 0 \) in (4.4a); using Gronwall's lemma (see [D3]) we find the estimate

\[
|P(B)(t)| \leq K(|B|_1) = [1 + |B|_1 \exp(|B|_1)]
\]

Putting this last inequality into (4.4a) one finds

\[
f(t) \leq \|A - B\|_1 K(|B|_1) + \int_0^t |A(s)||f(s)| ds.
\]

(4.4b)

Applying Gronwall's lemma to (4.4b) yields (4.2). The inequality in (4.3) is easily derived from (4.2) by integration.
above to conclude the formal expression for the differential is actually continuous.

Q.E.D.

From this result follows a number of corollaries.

COROLLARY 4.1. Let $P^u$ be the parallel translation operator on $E$ with respect to the connection $V$. Then $P^u:\mathcal{P}(M) \to \text{Hom}(E,E)$ is $C^\infty$, where $\text{Hom}(E,E)$ is the vector bundle over $M$ with the fiber over $p \in M$ given by $\text{Hom}(E,E)_p$.

Proof. Let $\sigma \in \mathcal{P}_M$, and choose local trivializations $\{(\psi_i, U_i)\}_{i=1}^n$, and subintervals $I_i = [t_{i-1}, t_i]$ for $i$ from $1$ to $k$ such that $0 = t_0 < t_1 < \cdots < t_k = 1$ and $\psi(U_i) \subset U_i$. Let $A^\sigma$ be the connection form associated to the local trivialization $(\psi_i, U_i)$, see Section 2. Choose $c > 0$ such that $W(\sigma, c)$ is a coordinate neighborhood of $\mathcal{P}_M$ as in Theorem 3.2, and such that for $x \in W(\sigma, c)$, $\tilde{M} \subset U_i$. Then by using a proof similar to that for Lemma 3.1, the map $x \in W(\sigma, c) \to A^\sigma(x, t_i) \in \tilde{L}(\tilde{J}(x, \text{End}(V)))$ is $C^\infty$. Hence by Lemma 4.1, the functions $P(x) \equiv P(x(t))$ are $C^\infty$ where $P(x)$ is the solution to the differential equation:

\[ \frac{d}{dt} P(x(t)) + A^\sigma(x(t)) \cdot P(x(t)) = 0 \quad \text{with} \quad P(x(t_0)) = \text{Id}. \]  

(4.9)

From this it follows that $P^u$ is $C^\infty$, since $P^u$ may be written (see (2.6)) as

\[ (x^u) = \psi_{-1} \psi(x, t_i) \psi_{-1} \psi(x, t_r) \]  

(4.10)

where $\psi(x, t_i), \psi(x, t_r)$, and $\psi(x, t)$ are for $x \in U_i \cap U_r$. Q.E.D.

COROLLARY 4.2. $T_\mathcal{P}$ is trivial. More precisely let $Q(\sigma)$ denote the parallel translation operator on $TM$ with respect to the Levi-Civita connection $D$ along a curve $u$ in $\sigma$. Then $F : T_\mathcal{P} \to \mathcal{P} \times H^J(l, TM)$ given by $F(\tau) = (Q(\tau)^{-1} \tau, \tau(t))$ is a vector bundle isomorphism where $(Q(\tau)^{-1} \tau)^{-1} \tau(t) = Q(\tau)^{-1} \tau(t)$, and $H^J(l, TM) = \{ X \in H^J(l, TM) : X(0) = 0 \}$.

Proof. It is clear that $F$ is bijective and fiber linear, so the only issue is the smoothness of $F$ and $F^{-1}$. Let $t \in C\mathcal{P}_M$, and $\upsilon \in C^\infty$ such that $W(\upsilon, \epsilon)$ is a coordinate neighborhood of $\mathcal{P}_M$. To make notation manageable, let $\Sigma(\upsilon, x) = \exp(\upsilon(\tilde{H}(\tilde{J}(\tilde{X}(\upsilon,x)))) \tilde{X}(\upsilon,x))$ for $\tilde{X} \in H^J(l, TM)$ with $|\tilde{X}|_\upsilon < \epsilon$ (Note that $\tilde{L}$ is $C^\infty$, since it is the inverse of a coordinate chart.) Also let

$q(X(t)) = Q_{\mathcal{P}}(l_{\mathcal{P}})(X(t))$. The local expression $(\vec{F})$ for $F$ with respect to the coordinate patch $W(\upsilon, \epsilon)$ is

\[ (\vec{F}) = (X, q(X)^{-1} \Sigma(\upsilon, x)) \in H^J(l, TM) \times H^J(l, TM) \]

for $X \in H^J(l, TM)$ such that $|X|_\epsilon < \epsilon$. In this last expression $\Sigma(\upsilon, x) = (d/dt)_{t=0} \tilde{X} + \tau(t) \in T_{\mathcal{P}(\upsilon)}$, and $(X^{-1} \Sigma(\upsilon, x)) = q^{-1}(X^{-1} \Sigma(\upsilon, x))(t)$. Due to Proposition 3.2, it is enough to prove that

\[ G(\upsilon, x) = (q^{-1} \Sigma(\upsilon, x) \upsilon)^{\ast} \tilde{M} \]

is $C^\infty$ if the $\tilde{J}$ are as in Corollary 4.1. Let $\psi$ be a local trivialization of $TM$ over the open set $U$, where now $\psi \in \mathcal{P}$. Again using the notation of Corollary 4.1 (applied to the vector bundle $TM$) we may write

\[ G(\upsilon, x) = q^{-1}(\Sigma(\upsilon, x) \upsilon)^{\ast} \tilde{M} = P(\upsilon, x)^{-1} \psi^\ast \Sigma(\upsilon, x). \]

By Lemma 3.1, the map $(\upsilon, x) \to q^{-1}(\Sigma(\upsilon, x) \upsilon)^{\ast} \tilde{M} = H^J(l, R) \times H^J(l, R)^2 = H^J(l, V)$ is $C^\infty$ and by the proof of Corollary 4.1, $P^u$ is $C^\infty$. Hence by Corollary 4.1 applied to $(\psi \circ P(\upsilon, x) \upsilon)^{\ast} \tilde{M}$ and properties $S1$ and $S2$ stated in the proof of Lemma 4.1, we conclude that $G(\upsilon, x)$ is $C^\infty$. Similarly one can show that $F^{-1}$ is also $C^\infty$.

Q.E.D.

COROLLARY 4.3. Let $E$ be any vector bundle over $M$ with covariant derivative $\nabla$. Put $\beta_\mathcal{P} \to E$ equal to $\beta(\sigma) = \nabla(\sigma)$. Then the pullback bundle $\beta^\ast E$ over $\mathcal{P}$ is trivial. The map $\beta : \beta^\ast E \to E$, given by $H(\tau, \zeta) = P(\tau)^{-1} \zeta$ is a fiber linear isomorphism. Recall that $\beta^\ast E$ is the bundle over $\mathcal{P}_M$ with fibers $\beta^\ast E_x = (\sigma(x), \zeta \in E_{\mathcal{P}(x)} = E_{\mathcal{P}(x)})^\ast$.

Proof. Similar to Corollary 4.2 but easier.

Q.E.D.

Now that we know $P^u$ is $C^\infty$, it is natural to compute its differential. This computation of the differential is shown in Theorem 2.2 of [G1] for the case where the base manifold is $R^4$, and in Corollary 2.16 for general $M$. To be complete I will rederive this result using a slightly different derivation. In order to state the results it is necessary to introduce the main objects of study for this paper.

DEFINITION 4.1. Let $W$ be a finite dimensional vector space. The elements of $T_{\mathcal{P}_M} \mathcal{P}(TM) \otimes W$ will be $W$-valued path two forms. More explicitly, a path two form is a smooth function $(L)$ such that for each $\sigma \in \mathcal{P}_M$, $L(\sigma)(\cdot, \cdot)$ is an alternating bi-linear function of $T_{\mathcal{P}(x)}M$ with values in $W$. Particular cases of interest will be for $W = \text{End}(E_{\mathcal{P}(x)})$ and $W = \xi$, the Lie algebra of the structure group $\mathcal{G}$.
DEFINITION 4.2. Given a connection $\nabla$ on $E$ as in section 2, then the Lasso associated to $\nabla$ is the $\text{End}(E_\omega)$-valued path two form $L^\omega$ defined by
\[ L^\omega(u, v)(x) = P^\omega(x)^{-1} F^\omega(x)(u, v) P^\omega(x) \] (4.11) for $u, v \in T_xM$. (It is clear that $L^\omega$ is $C^\omega$ since $P^\omega$ and $F^\omega$ are $C^\omega$.) In order to define the "integrated lasso" we need the following result.

PROPOSITION 4.1. For each $\alpha \in \mathcal{P}_\omega$, and $r \in I$, put $\alpha'(t) = \alpha(t)$, then the map $(\alpha, \alpha') : I \times T_\omega \to \mathcal{P}_\omega$ is $C^\omega$.

Proof. Fix $r \in I$, and $\alpha \in \mathcal{P}_\omega$. Choose $\xi \in C^\omega_{\text{reg}}$, and $c > 0$ such that $W(\xi, c)$ is a neighborhood of $\alpha$. Then $W(\alpha', c)$ is a neighborhood of $\alpha'$. Let $g(t)$ be parallel translation along the curve $x \in TM$ with respect to the connection $D$. Then locally the operation $(\alpha, \alpha') \to \alpha'$ may be expressed as $x \to (x \to f(t, x))$ from a neighborhood of 0 in $H[I, T_xM]$ to $H[I, T_xM]$, where
\[ f(t, x, u) = \exp_{\alpha(x)}^\omega t \exp_{\alpha'_x} \] for $t \in I$ and $u$ in a neighborhood of 0 in $T_xM$. By Lemma 2.7 of [G1], the map $(\alpha, \alpha') : I \times H[I, T_xM] \to H[I, T_xM]$ is jointly continuous. Since for fixed $x$, the map $x \to \xi' = \xi$ from $H[I, T_xM]$ to $H[I, T_xM]$ is linear, it follows that $(\alpha, \alpha') : I \times H[I, T_xM]$ is $C^\omega$. Now $f(t, x, u)$ is $C^\omega$ on its domain, so by taking derivatives as in the proof of Lemma 3.1, the map $(\alpha, \alpha') : I \times H[I, T_xM]$ is $C^\omega$. Therefore, the composite map $x \to (f(t, x), \alpha'(t))$ is $C^\omega$. Integrating by parts on the second term of this last equation yields
\[ X^\omega = -P^\omega \left[ \int_0^1 \frac{d}{dt} P^\omega(\alpha)(t)^{-1} \left( \frac{d}{dt} P^\omega(\alpha)(t)^{\omega} \right) \right] P^\omega(\alpha)(t) dt \]
\[ \left[ \frac{d}{dt} A'(x(\tau)) \right] P^\omega(\alpha)(t) dt, \]
where (4.1) (Duhamel's principle) was used to derive the first equality and the definition of the exterior derivative (4.1) was used for the second. Integrating by parts on the second term of this last equation yields
\[ X^\omega = P^\omega \left[ \int_0^1 \frac{d}{dt} P^\omega(\alpha)(t)^{-1} \left( \frac{d}{dt} P^\omega(\alpha)(t)^{\omega} \right) \right] P^\omega(\alpha)(t) dt \]
\[ -[\alpha'(t), x(\tau))] P^\omega(\alpha)(t) dt \]
\[ A'(x(\tau)) P^\omega(\alpha)(t) \right] \]
\[ \int_0^1 \left[ \frac{d}{dt} A'(x(\tau)) \right] P^\omega(\alpha)(t) dt, \]
where
\[ X^\omega = P^\omega \left[ \int_0^1 \frac{d}{dt} P^\omega(\alpha)(t)^{-1} \left( \frac{d}{dt} P^\omega(\alpha)(t)^{\omega} \right) \right] P^\omega(\alpha)(t) dt \]
and
\[ X^\omega = \int_0^1 F^\omega(u, v)(x(t)) \frac{d}{dt} A'(x(t)) \]
\[ + \int_0^1 [F^\omega(u, v), A'(x(t))] P^\omega(\alpha)(t) dt \]
\[ \frac{d}{dt} A'(x(t)) \]
\[ \frac{d}{dt} A'(x(t)) \]
for $u$ and $v \in T_u U_1$. Now recall (see equation (4.10)) that
\[
 f(u) = \phi P^u (u) = P^u (u) h_{\omega \sigma} (\tau(t_{11})) P^u (u) \ldots h_{\omega \sigma} (\tau(t_{1i})) P^u (u),
\]
(4.17)
where $h_{\omega \sigma} (x) = (\psi_{\omega \sigma} (x), \psi_{\omega \sigma} (x))$ for $x \in U_{1i} \cap U_{1j}$. To ease notation, let
\[
 R_i = h_{\omega \sigma} (\tau(t_{1i})) P^u (u) \ldots h_{\omega \sigma} (\tau(t_{1i})) P^u (u) \psi_{\omega \sigma} (x).
\]
and
\[
 K_i (u) = h_{\omega \sigma} (\tau(t_{1i})) P^u (u).
\]
Then by the product rule
\[
 Y_f = (X P^u) K_i - 1 + P^u (u) (X K_i - 1) R_i - 2 + P^u (u) K_i - 1 (X K_i - 1) R_i - 2 + \ldots + P^u (u) K_i - 1 K_i - 2 \ldots K_i (K_i).
\]
(4.18)
So by (4.14)
\[
 X K_i = h_{\omega \sigma} (\tau(t_{1i})) (\psi_{\omega \sigma} (x), \psi_{\omega \sigma} (x)) R_i - 1 + P^u (u) A' (X (\tau(t_{1i})) P^u (u) + \psi_{\omega \sigma} (x), \psi_{\omega \sigma} (x)) + d_{\omega \sigma} (X (\tau(t_{1i}))) P^u (u)
\]
which may be written by (2.1) as
\[
 X K_i = K_i, \psi_{\omega \sigma} (x), \psi_{\omega \sigma} (x) K_i + K_i A' (X (\tau(t_{1i}))).
\]
(4.19)
Plugging (4.19) into (4.18) one finds that the “boundary terms” involving $A' (X (\tau(t_{1i})))$ all cancel except for the term $A' (X (\tau(t_{1i})))$. Noting that
\[
P^{u} (u) K_i - 1 K_i - 2 \ldots K_i - 1 \psi_{\omega \sigma} (x), \psi_{\omega \sigma} (x) = f(u) \int_{\alpha} P^{u} (u) (x) - 1 F^u (\alpha (t)), X (t) > - P^{u} (u) (x) dt,
\]
(4.20)
the expression in (4.18) sums to (4.13). Q.E.D.

As an immediate corollary we have:

**Corollary 4.4.** If $X \in T_q \mathcal{P}$ such that $X (0) = X (1) = 0$, then
\[
 X P^{u} (u) = P^{u} (u) \sigma' (X).
\]
(4.21)

**Bundle-Connection Classifications**

**Corollary 4.5.** Under the assumptions above,
\[
 (\beta' D)_{\eta} \mathcal{L}^{\eta} = \{ L^{\eta} (u), \eta' (X) \} + P^{u} (u) (u) (\eta \otimes D_{\xi, \eta} F^{u} (\eta (1)) P^{u} (u),
\]
(4.22)
where $\beta' : \mathcal{P} \to M$ is defined by $\beta' (z) = \sigma' (z)$, and $\beta' D$ is the pullback of the Levi-Civita connection $D$ to sections along $\beta$. (Both sides of equation (4.21) are $\in T_{\mathcal{L}^{\eta}} (\mathcal{P})$ where $X \in T_{\mathcal{L}^{\eta}} \mathcal{P}$.)

Before proving this corollary we shall use some notation which will be useful in the proof.

**Definition 4.4.** Let $P$ be a function on $\mathcal{P}$ taking values in some set, then $P (z) = P (z)$ for all $z \in \mathcal{P}$.

**Remark 4.3.** If $P^{u} (u)$ is the parallel translation operator of a section $\mathcal{V}$, then $P^{u} (u)$ is parallel translation along $\sigma$ up to time $s$. We also write $P^{u} (u)$ for parallel translation along a curve $\sigma$ up to time $s$ for a curve which is defined on some interval $(-\epsilon, \epsilon)$ about the origin. This should not cause confusion, since when there is any ambiguity the two definitions agree.

**Definition 4.5.** Let $\sigma$ and $t$ be in $\mathcal{P}$ such that $\sigma (t) = \sigma (0)$. Then $\sigma (t)$ is defined to be the element in $\mathcal{P}$ given by,
\[
 \sigma (t) = \{ (t_{2i}), 0 \leq t_{2i} \leq 1/2 \}
\]
(4.23a)
Similarly if $X \in T_{\sigma} \mathcal{P}$ and $Y \in T_{\sigma} \mathcal{P}$ with $X (0) = Y (1)$, then $XY \in T_{\sigma} \mathcal{P}$ is defined by
\[
 XY (t) = \{ t_{2i}, 0 \leq t_{2i} \leq 1/2 \}
\]
(4.23b)

**Proposition 4.2.** Suppose that $\gamma$ and $\eta$ are $C^{0}$ maps of $(-\epsilon, \epsilon)$ to $\mathcal{P}$ such that $\gamma (s) (1) = \eta (s) (0)$ for all $s \in (-\epsilon, \epsilon)$, then the map
\[
 (s \rightarrow \gamma (s) (1) (s \rightarrow \eta (s) (1)) : (-\epsilon, \epsilon) \rightarrow \mathcal{P}
\]
(4.24)
is also of class $C^{0}$ and furthermore
\[
 \frac{d}{ds} \eta (s) (1) = \eta (s)' (1).
\]

**Proof.** An easy application of Proposition 3.2. Q.E.D.

**Proof of Corollary 4.5.** Put $\gamma (s) (1) = \exp_{\omega, \sigma} (s (X (t)))$ for small $s$ and $t \in I$. 

Then $\gamma(-e, e) \to \mathbb{R}^n$ is $C^\infty$ (for some $n > 0$) and $(d/dt)\gamma|_{t=0} = X$. Set $p(s) = P^0|_1(\beta \gamma) = P^0|_1(\gamma|_{t=1})$ and $q(s) = Q^0|_1(\beta \gamma)$ where $Q^0 = \mathbb{R}^n$ is parallel translation on $TM$ defined by the Levi-Civita connection $(D)$ on $M$. Then by definition

\[
(\mathbb{B}^* D_s)_{L^g} \frac{d}{ds}\bigg|_{s=0} L^g\gamma(t) = (P^0|_1(\beta \gamma))^{-1} \frac{d}{ds}\bigg|_{s=0} P^0|_1(\beta \gamma) \gamma(t), \quad q(s) \cdot p(s) \cdot \gamma(t)
\]

\[
= p(s)^{-1} P^0|_1(\beta \gamma) \gamma(t), \quad \gamma(t) \cdot q(s) \cdot P^0|_1(\beta \gamma)
\]

(4.25)

Define the function $C(-e, e) \to \mathbb{R}^n$ by

\[
C(s) = \begin{cases} \gamma(t)|_{t=2s} & 0 \leq s \leq 1/2 \\ \gamma(t)|_{t=1-2s} & 1/2 < s \leq 1 \end{cases}
\]

(4.26)

for $s \in (-e, e)$ and $t \in F$. It is easy to check using Proposition 4.2 and Lemma 3.2 that $C$ is a $C^1$-function, and in fact even $C^\infty$. We now may write out (4.25) more explicitly as

\[
(\mathbb{B}^* D_s)_{L^g} \frac{d}{ds}\bigg|_{s=0} L^g\gamma(t) = (P^0|_1(\beta \gamma))^{-1} \frac{d}{ds}\bigg|_{s=0} \gamma(t) P^0|_1(\beta \gamma)
\]

(4.27)

since $P^0(\gamma(t)) = p(s)^{-1} P^0|_1(\beta \gamma)$ by the "multiplicative" property of parallel translation. But, by definition,

\[
(\nabla \otimes D_s)_{\gamma(t)} = \frac{d}{ds}\bigg|_{s=0} p(s)^{-1} \gamma(t) P^0|_1(\beta \gamma)
\]

(4.28)

Therefore, by the product rule, Corollary 4.5, and equations (4.27), (4.28), and (4.5) one can easily verify equation (4.42). Q.E.D.

In the sequel we will find the following form of Corollary 4.5 to be more useful.

**Corollary 4.6.** Let $\gamma \in \Gamma(TM)$-vector bundle fields on $M$; put $U(\gamma) = u(|\gamma(1)|)$ and $W(\gamma) = w(\gamma(1))$. With this notation $L^g(U, W)$ is an End$(E_m)$ valued path function. The derivative of this function is

\[
(\hat{X} L^g(U, W)) = [L^g(U, W)(\hat{X}), \hat{X}]_g(X)
\]

(4.29)

\[
+ \hat{X} (e^{-1}(\gamma|_{t=1} F^0\gamma(t)), \gamma(t) \cdot \gamma(t) \cdot \gamma(t) \cdot \gamma(t))
\]

where $X \in T_x \mathbb{R}^g$, and $u$, $w$ in (4.29) should be evaluated at $s(1)$.

---

**5. Loop Variable Characterization of $(E, V)$**

This section shows that the set of equivalence classes (defined below) of vector bundle connection pairs are in one to one correspondence with a certain class of functions, $P: \Omega \to G$, defined up to conjugation by an element of $G$. In order to formulate this statement precisely, it is necessary to introduce some more definitions and notation. For the rest of this paper, $(E, V)$ or $(E, V)$ will denote vector bundle over $M$ with connection $\nabla$ or $\mathbb{V}$, respectively, structure group $G$, and fiber model space $V$ as described in Section 2.1. We also fix a distinguished base point $m \in M$.

**Definition 5.1.** A pair $(E, V)$ is said to be equivalent to $(E', V')$ if there is a vector bundle isomorphism $K: E \to E'$ such that $\nabla V = \nabla V$ and $K$ respects the structure group $G$. To be more explicit, the first condition requires that for any $e \in \Gamma(E)$ and $t \in TM$, then $K' E = \nabla (K' E)$ and the second condition requires that $\psi' \circ K \circ \psi$ be in $G$ for all admissible local trivializations $\psi$ of $E$ and $\psi$ of $E'$. Let $(E, V)$ denote the equivalence class containing $(E, V)$, and $\mathbb{E}(M, V)$ will denote the collection of equivalence classes $(E, V)$.

**Definition 5.2.** A function $P: \Omega \to G$ is said to be strongly differentiable if $P$ is $C^\infty$ on $\Omega_m$ and there is a non-negative function $c$ on $\Omega_m$ such that

\[
|\hat{X} P| < c(s)|\hat{X} P|_m(m)
\]

(5.1)

for all $X \in T_x \mathbb{R}^g$, where $|\hat{X} P| = \int_0^1 |\hat{X} P(t)|^2 dt$ for $X \in T_x \mathbb{R}^g$ as in (3.2).
Definition 5.4. A function $P: \Omega_m \rightarrow G$ is said to be parametrization invariant if $P(\tau \cdot r) = P(\tau)$ for all PC\(1\)-functions $r, \tau$ such that $t(0) = 0$ and $t(1) = 1$. Such a function $P$ is called a reparametrization.

Definition 5.4. A function $P_m: \Omega_m \rightarrow G$ is called multiplicative if
\[
P(\tau) = P(\sigma)P(\tau)
\]
for all $\sigma, \tau \in \Omega_m$, where $\tau$ is the concatenation of paths defined in (423a). Note that if $P$ is multiplicative then $P(C_m) = 1$, where $C_m$ is the constant path at $m$, since $C_mC_m = C_m$.

Example 5.1. Let $\psi$ be a local trivialization of a bundle connection pair $(E, \nabla)$, then function $P(\tau) = \psi_m\psi_{m}^{-1}$, for $\tau \in \Omega_m$, is strongly parametrization invariant and multiplicative, wherever $P_m$ is the parallel translation operator on $\Omega_m$.

Given a strongly differentiable, parametrization invariant, and multiplicative (SDPIM) function $P: \Omega_m \rightarrow G$, we can always define another by $P'(\tau) = gP(\tau)g^{-1}$ for any $g \in G$. Two SDPIM functions $P$ and $P'$ related by conjugation in this way will be called equivalent, and the equivalence class will be denoted by $[P]$. Let $\mathcal{M} = \mathcal{M}(M, M, V, G)$ be the collection of all equivalence classes $[P]$ of SDPIM functions.

Definition 5.5. A linear isomorphism $\varphi: E \rightarrow V$ is said to be admissible if $\varphi\psi_{m}^{-1} \in G$ for any admissible local trivialization of $E$ about $m \in M$.

We now state the main theorem of this section. An informal version of this theorem is mentioned in Gills [GI].

Theorem 5.1. There is a one to one correspondence between $\mathcal{M}$ and $\mathcal{M}$ given by
\[
[E, \nabla] \rightarrow [P^\varphi],
\]
where $[P^\varphi]$ is by definition $[\varphi P^M(\tau) \varphi^{-1} \varphi_{m}^{-1}]$ for any admissible $\varphi$ as in Definition 5.5. (Note that $[\varphi P^M(\tau) \varphi^{-1} \varphi_{m}^{-1}]$ is independent of the choice of admissible $\varphi$).

A statement similar to this theorem has been announced by Kobayashi [Kob]. However, the necessary condition of strong differentiability seems to be missing from his statement. The proof of this theorem will be carried out in a sequence of lemmas and propositions to follow.
Given an m-contraction over U and an admissible isomorphism ψ : E_u → Y, there is a natural local trivialization E over U induced by parallel translation as

$$\varphi^\ast_s = \kappa \cdot \varphi^\ast(\psi_s(\cdot, x))^{-1} : E_u \to Y$$

for all x ∈ U. (Clearly φ^\ast is C^{\infty}.)

**Proposition 5.2.** Let φ be an m-contraction over U, and v ∈ T_u U, then

$$A^\phi(v) = x \cdot B^\phi(X) \cdot x^{-1},$$

where X ∈ T_u U is the vector field along the path \( \sigma = (\psi_s(\cdot, x), x) \) given by X(t) = φ(t, 1) * v. A^\phi is the connection 1-form of \( \tilde{\nabla} \) with respect to \( \varphi^\ast \), and B^\phi is the integrated laxis of Definition 4.5.

**Proof.** Set f(\( \sigma \)) = \( \varphi^\ast(\sigma(\cdot, 1), 1) \cdot B^\phi(X) \cdot \sigma(\cdot, 1) \) for all \( \sigma \in \mathcal{E}_\phi \), with \( \sigma(1) = u \in U \). Let \( x \) be a smooth curve in \( \mathcal{M} \) such that \( x(0) = u \) and so that \( X(t) = (d\|d)(\sigma(t)) \cdot x(\sigma(t)) \). By the definition of \( \varphi^\ast \), f(\( \sigma(t) \cdot x \cdot \sigma(t) \cdot 1 \) for all \( x \in U \), so that \( X(t) = (d\|d)(\sigma(t)) \cdot x(\sigma(t)) \). Noting that \( X(0) = 0 \), the proposition now follows directly from Theorem 4.1. Q.E.D.

**Proposition 5.3.** Let \( \mathcal{E} \) be an admissible isomorphism, and let \( \varphi \) and \( \psi \) be two m-contractions over open sets \( U \) and \( W \) of \( \mathcal{M} \). Then the transition function between \( \varphi^\ast \) and \( \psi^\ast \) is

$$\varphi^\ast \circ \psi^{-1} = \kappa \cdot \varphi^\ast(\psi_s(\cdot, x))^{-1} \cdot x \cdot \psi^\ast(\cdot, x) \cdot x^{-1}$$

for \( x \in U \cap W \). (5.7)

Notice that the path \( \psi_s(\cdot, x) \cdot \psi^{-1}(\cdot, x) \) is in \( \mathcal{Q}_\phi \).

**Proof.** A simple matter of unwinding definitions. Q.E.D.

Propositions 5.2 and 5.3 will be the main motivation for the construction of a pair (E, \( \nabla \)) from an SDPDM function F. In order to carry out this construction it is necessary first to develop some properties of SDPDM functions.

**Lemma 5.1.** Let F be an SDPDM function on \( \mathcal{Q}_\phi \). Define a one form B on \( \mathcal{Q}_\phi \) by \( B = P^{-1} \cdot dF \). The following properties hold.

1. For each \( \sigma \in \mathcal{Q}_\phi \), there is a function \( b(\sigma, t) \in L^1 \{T^*M\} (\text{the } L^1\text{-sections of } T^*M \text{ along } \sigma) \) such that

$$B(X) = \int_0^1 b(\sigma, t)(X(t)) \, dt \quad \text{for all } \ X \in T_u \mathcal{Q}_\phi.$$
The next goal is to show that $B = P^{-1}dP$ can be extended naturally to $T_{Q_0}$. First we note that $B$ extends by continuity uniquely to $L^1(T_\tau)(TM)$ for $\tau \in Q_0$. Now suppose that $\tau \in Q_0$ and $X \in T_{T_\tau}Q_0$; the next lemma shows that $B(Q, X)$ is independent of $\tau$ in $Q_0$ such that $\tau(1) = (1)$. Here

$$0, x_0, t = (X(2t)) \text{ for } 0 < t < 1/2 \text{, and } 0 < t < 1 \text{ in } L^1(T_\tau)(TM).$$

(5.10)

Thus we extend $B$ (using the same symbol) to $T_{Q_0}$ by $B(x) = B(0, X)$.

**Lemma 5.2.** Assuming the notation above, $B(0, X)$ is independent of $\tau$ in $Q_0$ such that $\tau(1) = (1)$. Furthermore, the extended 1-form on $Q_0$ is $C^\infty$.

**Proof.** We will show that $B(0, X) = B(0, X)$ in the special case that $X(1) = 0$. This restriction is easily removed by the continuity of $B$ on $L^1(T_\tau)(TM)$. Define, for small $s$,

$$a(s) = \exp(sX) e_{\tau}. \text{ Note that } a(s(1)) = a(1) \text{ for all } s. \text{ By property (3) of Lemma 5.1,}$$

$$P(a(s)) = P(a(s)).$$

(5.11)

Differentiate both sides of (5.11) at $s = 0$ to conclude that

$$P(a(s)) = P(a(s)) = P(a(0), Y).$$

(5.12)

By another application of property (3) of Lemma 5.1, (5.12) implies that $B(0, X) = B(0, X)$.

To show that $B$ is $C^\infty$ on $Q_0$, let $W(\tau, e)$ be a coordinate neighborhood of $Q_0$ and $X \in T_{T_\tau}Q_0$ for some $x \in W(\tau, e)$. Choose an $m$-contraction $(\varphi, U)$ about $(1)$ and $d > 0$ sufficiently small such that $\exp(\varphi(z)) \in U$. Let $f \in C^\infty([0, d])$ such that $f(0) = 1$ for $x$ near 0, and $f(t) = t$ for $t > d$. For all $X$ sufficiently close to 0 such that $\exp(f(0)) \varphi(x) \in U$ for $0 < t < d$, define $F \in C^\infty$ by $f(t)$ and $\exp(f(0)) \varphi(x)$. Then the concatenation

$$P = \{0, \varphi, \varphi(z), \varphi(y) \}$$

is in $T_{Q_0}$. Using Proposition 3.2c, it is easy to show that the map $Y \to P$ from a neighborhood of $X$ in $T_{Q_0}$ to $T_{Q_0}P$ is $C^\infty$. Finally, let $\eta \in Q_0$ such that $P \in T_{T_\eta}Q_0$, then

$$B(Y) = \int_0^{1} h(t, r(\tilde{Y}(t)) dt = \int_0^{1} h(t, r(\tilde{Y}(t)) dt$$

by property (2) of Lemma 5.1 and the fact that $\tilde{Y}(t)$ is proportional to $Y(t)$ for $0 < t < 1$. (Note that $h(\tilde{X}(t), 1) = C^\infty$.) From this last equation it is clear that $B(Y) = B(0, \varphi(z)) > B(0, X)$, since $B(0, X) = B(Y)$ and $Y > \tilde{Y}$ is $C^\infty$, it follows that $B$ is $C^\infty$.

**Conclusion of the Proof of Theorem 5.1.** We will construct a bundle connection pair $(E, \nabla)$ from the SPDIM function $P$ of the pair $(E, V)$ will be described by prescribing a collection of transition functions and connection 1-forms.

Let $(\{\varphi, U\})$ be a collection of $m$-contractions over an open cover $\{U\}$ of $M$. In analog with (5.7) and (5.6), define $g(x, y) = P(x(s), y(x), a(x))$ for $x \in U \cap U_x$, and put $A(x) = B(x, \varphi(s), y(x))$ for $x \in U_x$, where $s(x) = \varphi(s), y(x)$.

Claim 1. The $\{g(x)\}$ form the transition functions for a vector bundle $E$ over $M$. Let $g(x)$ denote the local trivializations of $E$ over the set $U \cap U_x$, satisfying $\nabla^*(\varphi(s)) = g(x)$ for $x \in U \cap U_x$.

Claim 2. The $\{A^*(\varphi(s))\}$ are the connection 1-forms with respect to the local trivializations $(\varphi(s))$ for a connection $\nabla$ on the bundle $E$ constructed in Claim 1.

Claim 3. Let $g(x)$ be any local trivialization of $E$ about $m \in M$. Then $F(x) = P(x(s), y(x), a(x))$ for all $x \in E$.

First note that the $g(x)$ are $C^\infty$ since the map $x \to \varphi(s, x \varphi(s), x)$ is $C^\infty$ from $U \cap U_x$ to $E_x$. Furthermore, by property (3) of Lemma 5.1, the $g(x)$ obey the cocycle condition $g(x)g(y)(x) = g(x)g(y)(x)$ for $x \in U \cap U_x$, so the $g(x)$ are the transition functions for a vector bundle $E$ with local trivializations as described in Claim 1. By Lemma 5.2, each $A^*(\varphi(s))$ is a smooth $\nabla$-valued 1-form on $T_{Q_0}$, since the map $(t \to x) \in T_{Q_0}P$ is $C^\infty$.

By the cocycle condition $g(x)g(y)(x) = g(x)g(y)(x)$ for $x \in U \cap U_x$, so the $g(x)$ are the transition functions for a vector bundle $E$ with local trivializations as described in Claim 1. By Lemma 5.2, each $A^*(\varphi(s))$ is a smooth $\nabla$-valued 1-form on $T_{Q_0}$, since the map $(t \to x) \in T_{Q_0}P$ is $C^\infty$.

Suppose that $m \in C^\infty(T_{Q_0})$, put $\nabla^*(\varphi(s)) = P(t, \tilde{X}(t))$, where $a(t)$ is the concatenation $g(a(s), e)(s) = g(a(s), e)(s) = g(a(s), e)(s)$.

We will now show that $\nabla^*(\varphi(s))$ is parallel translation along $s$ with respect to the connection 1-form $A^*(\varphi(s))$ by showing $d^\nabla = 0$ can correct differential equation. Using the definition of $B$ and $P$ one finds that

$$\frac{d}{ds}\nabla^*(\varphi(s)) = \frac{d}{ds}(P(a(s))) = -B(0, a(s)) P(a(s)).$$

So now compute

$$B(0, a(s)) = B(0, a(s)) - B(0, a(s)) - B(0, a(s))$$

$$= B(0, a(s), a(s)) - B(0, a(s), a(s))$$

$$= B(0, a(s), a(s)) - A^*(\varphi(s)).$$
where in the second equality we have used property (2) of Lemma 5.1. Because of these last two equations, $p_P^*(\alpha) \in C^1(I, M)$ satisfies the desired differential equation. Furthermore $p_P^*(\alpha) = 1$ in $G$ by property (3) of Lemma 5.1.

Now suppose that $\alpha$ is also in $C^1(I, U)$ for some other index $b$, then

$$p_P^*(\alpha) = g_{ab}(\alpha) \cdot p_P^*(\alpha) = g_{ab}(\alpha)(0),$$

(5.13)

using the definitions of $p_P^*, p_P, g_{ab},$ and $g_{ab}$ along with property (3) of Lemma 5.1. Using the above computation, take the derivative of both sides of (5.13) at $s = 0$ to show that

$$A^c(x) = dg_{ab}(\alpha(x)) \cdot g_{ab}(\alpha(x)) A^c(x) = g_{ab}(\alpha(x)) A^c(x) g_{ab}(\alpha(x)),

(5.14)

where $r = \alpha(0)$ in $T \Omega, M$. Since the curve $\cup = \alpha(0)$, equation (5.14) holds for all $r \in T \Omega, r \in U_2$. In view of Remark 2,1, this shows that the connection 1-forms $[A^c]$ are consistently related, and hence define a connection $\nabla$ on the bundle $E$ as in Claim 2.

Furthermore our computations show that

$$P^\alpha(\alpha) = (\Phi_{\alpha})^{-1} P(\Phi_{\alpha}(\alpha) \circ (\alpha(0))(\Phi_{\alpha}(\alpha))^{-1})$$

(5.15)

for all $\alpha \in C^1(I, U)$. Now suppose that $\alpha = \beta \alpha$, where $\alpha \in C^1(I, U)$, $\beta \in C^1(I, U)$, and $\alpha(1) = \beta(1) = x$. Then by (5.15), the multiplicative property of $P^\alpha$, and the relation $g_{ab}(\alpha(x)) = \Phi_{\alpha}(\alpha(x))^{-1} = P(\Phi_{\alpha}(\alpha(x))^{-1} \Phi_{\alpha}(\alpha(x)))$

one shows

$$P^\beta(\alpha) = (\Phi_{\beta})^{-1} P(\Phi_{\beta}(\alpha) \circ (\alpha(0))(\Phi_{\beta}(\alpha))^{-1})$$

$$= (\Phi_{\beta})^{-1} P(\Phi_{\beta}(\alpha) \circ (\alpha(0))(\Phi_{\beta}(\alpha))^{-1})$$

$$= (\Phi_{\beta})^{-1} P(\Phi_{\beta}(\alpha) \circ (\alpha(0))(\Phi_{\beta}(\alpha))^{-1})$$

$$= (\Phi_{\beta})^{-1} P(\Phi_{\beta}(\alpha) \circ (\alpha(0))(\Phi_{\beta}(\alpha))^{-1})$$

(5.16)

where we have used repeatedly the reparametrization invariance of $P$ and the properties of Lemma 5.1. Now any curve $\sigma \in C^1(I, M)$ (if appropriately reparametrized) may be split into a finite product of paths $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, with each $\sigma_i \in C^1(I, M)$ for some $a_i$. So by repeating the above argument $n - 1$ times one finds that

$$P^\alpha(\alpha) = (\Phi_{\beta})^{-1} P(\Phi_{\beta}(\alpha) \circ (\alpha(0))(\Phi_{\beta}(\alpha))^{-1})$$

where $\sigma(0) = 0 \in U_1$ and $\sigma(1) \in U_2$. Since both sides of (5.16) are continuous in $\sigma$, (5.16) holds for $\sigma \in \Omega$. Claim 3 is a special case of (5.16).

Because of Claim 3, $[P^\alpha] = [P]$. Hence the map in (5.1) is onto. This fact along with Proposition 5.1 proves Theorem 5.1.

Q.E.D.

6. CHARACTERIZATION OF $[E, \nabla]$ BY LASIOS

The main result of this section (Theorem 6.1) states that if $M$ is simply connected, then the space $\mathcal{B} = \mathcal{B}(M, V)$ is parameterized by the set of pairs $(\beta, L)$ such that $B^\beta$ has curvature zero on $\Omega_{\beta}$ and $L$ satisfies a monodromy condition associated with $\Pi_\beta(M, m)$. In Theorem 6.2 the zero curvature condition is reformulated so as to be in closer analogy with the condition $df = 0$ for ordinary closed 2-forms. Using Theorem 6.2, it is easy to show that the Weil's integrality condition for $U(1)$-bundle can be recovered from Theorem 6.1. In order to state the main results of this section we need the following two propositions.

PROPOSITION 6.1. Let $\Omega$ be a connected Hilbert manifold and $B$ be a $\mathfrak{g}$-valued 1-form on $\Omega$. Suppose that $P^B = \mathbb{R} + B : \beta = 0$, then the parallel translation operator $P^B$ associated to $B$ (see definition 2.2) induces a homomorphism from $\Pi_\Omega(M, \alpha)$ to $G$, where $\alpha$ is a fixed point in $\Omega$.

Proof. Suppose that $\gamma \in C^0(\Omega, \alpha) = \{ \gamma \in C^0(\Omega, \alpha) : \gamma(1) = \alpha \}$ (the $C^\infty$ loops based at $\alpha$) and that there is a $C^\infty$-homotopy $\gamma$ between $\alpha$ and $\beta$. We may view $\gamma$ as a map from a $I \in C^0(\Omega, \alpha)$ such that $\gamma(0) = \alpha$ and $\gamma(1) = \beta$. Had we developed the loop space of an infinite dimensional Hilbert manifold we could apply Corollary 4.1 to see that $P^\gamma(x)$ is differentiable. Then by Corollary 4.4 it would follow that (that (di) $P^\gamma(x)$ is $C^\infty$, since the integrated lasso associated to $B$ would be zero because $B$ has zero curvature. To make this rigorous one only needs to note that $\gamma$ may be used to pullback all objects to a bundle over $I \times I$ where one can use Corollaries 4.1 and 4.4. (The fact that $I \times I$ has boundaries and corners does not cause any difficulties.)

Therefore, if $\alpha$ and $\beta$ are $C^\infty$-homotopic, then $P^\alpha(\beta) = P^\beta(\alpha)$. Now suppose $\gamma$ is only a continuous homotopy. Since Hilbert manifolds admit partitions of unity (see [1] or [E1]), the standard smoothing techniques (see [M2] or [S4]) can be used to modify $\gamma$ to produce a $C^\infty$ homotopy from $\alpha$ to $\beta$. So again $P^\gamma(x) = P^\beta(x)$. Since, each $\alpha \in C^0(\Omega, \alpha)$ is homotopic to some $x \in C^0(\Omega, \alpha)$, one may define $P^\gamma(x) = P^\beta(x)$, which is well defined by the above comments. Thus, $P^\gamma$ is constant on homotopy classes and is multiplicative, hence defines a homomorphism from $\Pi_\Omega(M, \alpha)$ to $G$.

Q.E.D.

PROPOSITION 6.2. $\Pi_\beta(M, m)$ is in one to one correspondence with the connected components of $\Omega_{\beta}$ as $\beta$ varies on $M$ based at $m$.
Theorem 7.1 that the correspondence in (6.1) would no longer be one to one.

The proof of Theorem 6.1 will be postponed until we have proved a number of useful propositions which will also help to motivate the theorem. Let $B^0$ be the integrated lasso form for a connection $\mathfrak{V}$ as in Definition 4.3. Our main point of view is that $B^0$ should be considered as a globally defined connection one form on the trivial vector bundle $\mathcal{E}_n \times \mathcal{E}_n$. Recall from Corollary 4.9 that $B^0$ is isomorphic to $\mathcal{E}_n \times \mathcal{E}_n$. Since $B^0$ is the pullback of $\mathfrak{V}$ by a connection $\mathfrak{V}$, there is a natural connection $B^0\mathfrak{V}$ on $B^0$. The next proposition shows that under the identification of $\mathcal{E}_n \times \mathcal{E}_n$ with $B^0\mathfrak{V}$, the connection one form for $B^0\mathfrak{V}$ is $B^0$.

Proposition 6.3. Let $B^0\mathfrak{V}$ denote the induced parallel translation operator on the bundle $B^0\mathfrak{V}$ and $H: B^0\mathfrak{V} \to \mathcal{E}_n \times \mathcal{E}_n$ be the global trivialization of Corollary 4.3. Suppose that $\gamma: I \to \mathcal{E}_n$ is a C^0 map, then

$$R^0(y) = H_{11}(B^0\mathfrak{V}(y)) H^{-1}_{11} = F^0(y)(1)^{-1} F^0(\beta \gamma) F^0(y(0)),$$

and $B^0$ is the connection 1-form for $B^0\mathfrak{V}$ with respect to the trivialization ($H$) of the bundle $B^0\mathfrak{V} = \{\{\sigma, \xi\}, \xi \in \mathcal{E}_n\}$ where $\sigma \in \mathcal{E}_n$.

Proof. Recall that $(B^0\mathfrak{V}(y) - \gamma(0), \gamma(1)) = (\gamma(1), B^0(\beta \gamma) - \gamma(0))$, where $\xi \in \mathcal{E}_n$. Hence, the fact that (6.2) holds is an immediate consequence of the definition of $H$ in Corollary 4.3. In order to show that $B^0$ is the connection 1-form for $B^0\mathfrak{V}$ with respect to the trivialization ($H$), it suffices to show

$$\frac{d}{ds} R^0(y(s)) + B^0(\gamma(s)) R^0(y(s)) = 0. 
(6.3a)$$

This is equivalent to saying that

$$\frac{d}{ds} R^0(y(s))^{-1} = R^0(y(s))^{-1} B^0(\gamma(s)). 
(6.3b)$$

To this end define $\Gamma: I \to \mathcal{E}_n$ by

$$\Gamma(t) = \begin{cases} 
\gamma(0) & 0 \leq t \leq 1 \\
\gamma(0)(1 - 2t) & 1 < t < 1 
\end{cases}. 
(6.4)$$

which is seen to be $C^0$ by Lemma 3.2, Proposition 3.3, and Proposition 4.2. In terms of (6.4), $R^0(y(t))^{-1} = P^0(y(0)^{-1} P^0(F(t)))$. We may apply
Corollary 6.4 to this last expression to find \( \frac{d}{d\tau} R^X(\gamma)^{-1} = R^X(\gamma)^{-1} B^X(\Gamma(\tau)^{-1}) \), where by definition

\[
B^X(\Gamma(\tau)) = \int_0^{1/2} L^X(\Gamma(\tau)^{-1}) \delta(\Gamma(\tau)(t), \delta(\Gamma(\tau)(t))) \, dt
\]

The second equality follows from the fact that \( \delta(\Gamma(\tau)(t)) \) and \( \delta(\Gamma(\tau)(t)) \) are parallel for \( \frac{1}{2} \leq t \leq 1 \) as is easily seen from (6.4) (Note: \( L \) is a 2-form and hence zero on parallel vectors.) By a change of variables in (6.5) and the expression (6.4) for \( \Gamma \), one easily shows that \( B^X(\Gamma(\tau)^{-1}) = B^X(\gamma(\tau)^{-1}) \).

Q.E.D.

**Proposition 6.4.** Let \( F^B = dB^B + B^B \times B^B \) denote the curvature of \( B^B \) as in Definition 2.1, then

\[
F^B(\gamma) = L^B(\gamma) \left( B_B \cdot B_B \right)
\]

for all \( \gamma \in \Omega^\infty \). In particular, the curvature of \( B^B \) is path-independent for \( \mu_B \) in Definition 2.2. Clearly \( R(\gamma) = R(y(\gamma)^{-1} \gamma) \) if \( \gamma \) is a loop. So the induced representation on \( \Omega^\infty \) is trivial. We could have deduced the curvature zero condition from the path independence of \( \gamma \).

**Proposition 6.5.** Let \( \Omega \) be a connected Hilbert manifold and \( B \) be a smooth \( \Omega \)-valued 1-form on \( \Omega \). Then there exists a smooth function \( P: \Omega \to G \) such that

\[
\exp(\lambda) \longmapsto \left( B(\lambda)^{-1} + \frac{d}{dt} P(\lambda) B(\lambda(1)^{-1}) \right)
\]

for all \( \lambda \in \Omega \) and \( \lambda \in T\Omega \) if and only if \( F^B = dB^B + B^B \times B = 0 \) and the homomorphism \( \pi: \Omega(\lambda, \sigma) \to G \) described in Proposition 6.1 is trivial. (Such a \( B \) is said to have trivial monodromy.)

This proposition is the "curved" version of Lemma 3.6 of [G1].

**Proposition 6.6.** Suppose that \( P \) is a solution to (6.7). Let \( x = (-x, e) \times (-e, x) \to \Omega \) be a smooth function. Let \( p(x, t) = P(ax(t, t)) \), so that

\[
p(x, t) \in C(\mathbb{R}, (-x, e) \times (-e, x), \mathbb{G}).
\]

Therefore

\[
\delta_0 \delta_p(t, x) = \delta_0 \delta_p(t, x) = \delta_0 \delta_p(t, x).
\]

So compute

\[
\delta_0 \delta_p(t, x) = \delta_0 \left( p(t, x) B(\delta_p(t, x)) \right),
\]

\[
= p(t, x) \left( \delta_0 B(\delta_p(t, x)) + B(\delta_p(t, x)) B(\delta_p(t, x)) \right) (6.8)
\]

Interchange \( s \) and \( t \) in (6.8) and subtract from (6.8) to get

\[
\delta_0 B(\delta_p(t, x)) = \delta_0 B(\delta_p(t, x)) + B(\delta_p(t, x)) B(\delta_p(t, x)) = 0.
\]

But the left hand side of this last equation is \( F^B(\delta_p(t, x), \delta_p(t, x)) \). Since \( \sigma \) was arbitrary we conclude that \( F^B \) is identically zero.

Let \( \gamma \in C(\Omega, \mathbb{D}) \), and set \( R(y) = P(\gamma)^{-1} \). Then an easy computation using equation (6.7) shows that

\[
\frac{d}{dt} R(x) + B(\gamma(x)) R(x) = 0,
\]

so that \( R \) is the parallel translation operator associated to \( B \), see Definition 2.2. Clearly \( R(\gamma) = R(y(\gamma)^{-1}) \) if \( \gamma \) is a loop. So the induced representation on \( \Omega(\mu) \) is trivial. We could have deduced the curvature zero condition from the path independence of \( \mu \).

Conversely, suppose that \( B \) is given with curvature zero and trivial monodromy. In other words, if \( R(\gamma) \) denotes the solution to the equation (6.9) with \( R(y) = 1 \in \mathbb{G} \), then \( R(\gamma) = 1 \in \mathbb{G} \) if \( \gamma \) is a loop based at some fixed \( \sigma \) in \( \Omega \). Suppose that \( x \in \Omega \), choose any C^1 path \( \gamma \) from \( \sigma \) to \( \tau \), and then define \( (\chi \rho) \sigma \) for all \( \gamma \in \Gamma \) is well defined since \( R \) is path independent \( (\rho, \chi) \Gamma \). Suppose that \( \chi \in \Omega(\mu, \mathbb{D}) \), choose \( \gamma \in C(\mathbb{R}, \mathbb{D}) \), and \( \chi(\gamma(1)) = 1 \). Then by (6.9),

\[
\frac{d}{dt}(\exp(\gamma)) = (\exp(\gamma))^\prime = (\exp(\gamma))^\prime(\gamma(1)) = B(\gamma(1)).
\]

Evaluating this last expression at \( t = 1 \) shows that\( \exp(\gamma) = \exp(\gamma(B(\gamma(1))) \) for \( x \in T \Omega \) and \( \mathbb{D} \).

The only thing left to prove is that the function \( P \) is \( C^\infty \). Let \( \gamma \in \Omega \), and choose a 1-contraction \( \sigma \) over neighborhood \( V \in \mathbb{D} \). Since \( P \) satisfies equation (6.7), it follows that

\[
\frac{d}{dt} P(\gamma(1)) = P(\gamma(1)) \cdot B(\gamma(1)) \]
found by solving a differential equation of the form \((6.10)\) once \(P(\sigma)\) is specified. Furthermore, the value at any one point can be specified arbitrarily, and since for any \(g \in G\), \(g P(\cdot)\) is a solution to the system \((6.7)\) provided \(P(\cdot)\) is a solution.

**Lemma 6.1.** Let \(M\) be simply connected so that \(\Omega_M\) is connected. Suppose that \(L = \Psi\)-valued path 2-form on \(\Phi\), with trivial monodromy. Then by Proposition 6.5, there exists a solution \(P = P(\sigma) B^\sigma(X)\) for all \(\sigma \in \Omega_M\) and \(X \in T\Omega_M\). Furthermore, by Remark 6.1, it is possible to choose \(P\) such that \(P(\tau(0)) = 1_G\) for \(\tau\) on \(\Omega_M\), where \(\tau(0)\) is the constant loop at \(m\). This function \(P\) is a strongly differentiable, parameterization invariant, and multiplicative (i.e., SDPIM).

**Proof.** It is obvious that \(P\) is strongly differentiable. Now let \(r\) be a reparametrization of \(\tau\) and set \((\sigma, r) = (\sigma - 1)(r) + 1\) so that \(\sigma \rightarrow \sigma - r\) is a path in \(\Omega_M\) from \(\sigma\) to \(\tau\), provided \(\sigma \in C^0\Omega_M\). Using \((6.7)\),

\[
0 = \frac{d}{dt} P(\sigma - r(t)) = P(\sigma - r(t)) B^{\sigma - r(t)} = \frac{d}{dt} \sigma - r(t)
\]

because \((d/dt)\sigma - r(t)(t) = (d/dt)\sigma + r(t)(t)\) and \((d/dt)\sigma + r(t)(t)\) are both proportional to \(\sigma'(\tau(t))\), so that \(L((d/dt)\sigma + r(t)(t)) = 0\) and hence \(B^{\sigma - r(t)} = P(\sigma - r(t)) = 0\). Therefore, \(P(\sigma) = P(\sigma - r)\) for all \(\sigma\) loops in \(\Omega_M\). By continuity, this holds for all loops \(\sigma\) in \(\Omega_M\).

Let \(s, \tau \in \Omega_M\) and choose a smooth path \(\gamma : I \rightarrow \Omega_M\), starting \(\gamma(0)\) and ending \(\tau\). Since \(\gamma(0)\) is a reparametrization of \(\tau\), \(P(\gamma(0)) = P(\tau)\). The strategy now is to show that both \(P(\gamma(t))\) and \(P(\tau)\) satisfy the same differential equation. By \((6.7)\), this amounts to showing that \(B((d/dt)\gamma(t)) = B((d/dt)\tau(t))\).

\[
B \left( \frac{d}{dt} \gamma(t) \right) = \left[ L(\gamma(t)) \frac{d}{dt} \gamma(t) \right] - \left[ L(\gamma(t)) \frac{d}{dt} \gamma(t) \right] dt
\]

where \((d/dt)\gamma(t)(t) = 0\) for \(t > \frac{1}{2}\) was used in the first equality, and a change of variables was performed to get the fourth equality. The other steps are all a matter of using the definitions. Q.E.D.

The proof of Theorem 6.1 is now an easy matter.

**Proof of Theorem 6.1.** By Proposition 6.3 and 6.4, if \(L = \Psi\), for some lasso \(L\), where \(\kappa : \mathbb{E}_\gamma \rightarrow \mathbb{V}\) is an admissible isomorphism, then \(L\) is closed and has trivial monodromy. Furthermore, suppose that \((\mathbb{E}, \mathbb{V})\) and \((\mathbb{E}', \mathbb{V}')\) are equivalent and \(\mathbb{V} = \mathbb{V}'\). Then \(L\) is also equivalent, so that \((\mathbb{E}, \mathbb{V}) = (\mathbb{E}' , \mathbb{V}')\).

Now suppose that \((\mathbb{L}, \mathbb{V}) \in \mathcal{C}\). By Proposition 6.5 and Lemma 6.1, there is a unique SDPIM function \((\mathbb{P}^0 : \mathbb{P})\) such that \((\mathbb{P}^0\mathbb{L}) = \mathbb{P}\). Thus, there is an admissible isomorphism \(\kappa : \mathbb{E}_\gamma \rightarrow \mathbb{V}\) such that \(\kappa^\ast \mathbb{P}^0 = \mathbb{P}\) on \(\Omega_M\).

Differentiating \((6.11)\) by \(X \in T\Omega_M\) implies that \(\kappa^\ast (\mathbb{P}^0\mathbb{L}) = \mathbb{P}\) on \(\Omega_M\).

Now suppose that \((\mathbb{L}' = \mathbb{L}'(\mathbb{L}))\); that is, suppose there exists an admissible \(\kappa : \mathbb{E}_\gamma \rightarrow \mathbb{V}\) such that \(\kappa^\ast (\mathbb{P}^0\mathbb{L}) = \mathbb{P}\) on \(\Omega_M\). This implies that \(\kappa^\ast (\mathbb{L}') = \mathbb{L}'\). By the uniqueness of solutions to the system of equations \((6.7)\) (see Remark 6.1), it follows that \(\kappa^\ast : \mathbb{P}^0 = \mathbb{P}'\) on \(\Omega_M\). That is \(\mathbb{P}' = \mathbb{P}\). So again by Theorem 6.1, it follows that \(\mathbb{L} = \mathbb{V}'(\mathbb{V})\) and \(\mathbb{L}' = \mathbb{V}'(\mathbb{V})\). Hence the map in \((6.11)\) is one to one. Q.E.D.

From the proof it is fairly evident that Theorem 6.1 could be formulated directly in terms of \(\Psi\)-valued 1-forms on \(\Phi\). That is the path 2-forms \(L\) could be eliminated altogether. I refer the reader to Theorem 3.13 of [G1] for a result along these lines which has an obvious generalization to this setting.

We will finish this section by discussing characterizations for a \(\Psi\)-valued path 2-form \(L\) to be closed. Gross, in [G1], gives a number of characterizations using the notion of "end point derivative." The definitions and characterization used in [G1] all carry over to this more general case with only minor changes. So I will concentrate on one specific characterization.
of closeness given in Corollary 4.12 and Remark 4.13 of [G1]. This is not a serious omission, since the other characterizations can be easily deduced from the results to follow. The following lemma allows us to define the endpoint derivative. The reader is urged to consult [G1] for a more natural definition of this derivative.

**Lemma 6.2.** Let \( u, z \in \Gamma(TM) \), and \( U = \beta \), \( Z = \gamma \) be as in Corollary 4.6. Suppose that \( L : \mathbb{R} \rightarrow \mathbb{R} \) is a \( \mathbb{R} \)-valued path 2-form on \( \mathbb{R}^2 \) which satisfies
\[
X(\beta(0), Z(0)) + [B(X, Z), L(\beta, Z)] = 0
\]
for all \( X \in \mathcal{T}_\mathbb{R} \), for which \( X(\beta(0)) = 0 \), where \( B = B^\varepsilon \). Also suppose that \( \{X_\lambda\} \) is a sequence in \( \mathcal{T}_\mathbb{R} \) for which
\[
X_\lambda(0) = X(0) = \begin{cases} 0 & \text{if } 0 \leq \lambda < 1 \\ \in \mathcal{T}_\mathbb{R} & \text{if } \lambda = 1 \end{cases}
\]
as \( n \to \infty \), and
\[
\{X_\lambda(t)\} = \mathcal{T}_\mathbb{R} \quad \text{as } \lambda \to \infty
\]
for all \( n \), and some constant \( C \). Then the limit \( \lambda \to \infty \), \( X(\beta(0), Z(0)) \) exists and is independent of the particular sequence converging to \( X_\lambda(0) \). This limit will be denoted by \( w(\beta, Z) \), and will be called the endpoint derivative of \( L(\beta, Z) \) with respect to \( w \). The endpoint derivative satisfies the relation
\[
X(\beta(0), Z(0)) = X(\beta(0), Z(0)) + [B(X, Z), L(\beta, Z)](\sigma)
\]
where \( X \in \mathcal{T}_\mathbb{R} \) and \( X(\beta(0), Z(0)) \) need not be zero.

**Remark.** The endpoint derivative is interpreted to be the variation of a function as one varies the endpoint of a path. Equation (6.13) may be interpreted as saying that the end point derivative of \( L(U, V) \) is in the same as the covariant derivative with respect to the connection determined by the \( \mathbb{R}^2 \)-valued 1-form (B) on \( \mathbb{R}^2 \).

**Proof.** Fix \( X \in \mathcal{T}_\mathbb{R} \) such that \( X(0) = w \in \mathcal{T}_\mathbb{R} \). Let \( Y \in \mathcal{X} \) satisfies \( Y(0) = 0 \). By hypothesis
\[
Y(\beta(0), Z(0)) + [B(Y, Z), L(\beta, Z)(\sigma)] = 0
\]
for all \( n \). In other words
\[
X(\beta(0), Z(0)) + [B(X, Z), L(\beta, Z)(\sigma)] = Y(\beta(0), Z(0)) + [B(Y, Z), L(\beta, Z)(\sigma)]
\]
for all \( n \). By the dominated convergence theorem, \( B(X, Z) \) converges to zero as \( n \to \infty \). This shows that the \( \lim_{n \to \infty} X(\beta(0), Z(0)) \) exists and satisfies (6.12).

**Remark 6.2.** If the path 2-form is a lattice, then by Corollary 4.6, \( L^\varepsilon \) satisfies condition (6.12), and furthermore the end point derivative of \( L^\varepsilon \) with respect to \( w \in \mathcal{T}_\mathbb{R} \) is
\[
\{w(\beta(0), Z(0))\} = \{w(\beta(0), Z(0))\} - \{w(\beta(0), Z(0))\}
\]
(6.14)

**Definition 6.3.** Let \( L \) be a path 2-form satisfying condition (6.12). The endpoint differential of \( L \) is a path 3-form defined by
\[
d^2L(\beta(0), Z(0)) = \{w(\beta(0), Z(0))\} + \{w(\beta(0), Z(0))\} + \{w(\beta(0), Z(0))\} + \{w(\beta(0), Z(0))\}
\]
where \( U, W \) are \( \omega, \omega, \omega, \omega \), respectively, and \( U, Z, W \) are vector fields on \( M \). The notation \( \{U, Z, W\} \) denotes \( \{U, Z, W\} \).

**Remark 6.3.** One can easily check that \( d^2L(\beta(0), Z(0)) \) is antisymmetric and tensorial in \( U, V, W \). If \( U = (x_1, \ldots, x_n) \), then \( \{U, V, W\} \) is a chart for \( M \); then write \( L_{ij}(\sigma) = L_{ij}(\beta(0), Z(0)) \) and \( d^2L_{ij}(\sigma) = d^2L_{ij}(\sigma) = d^2L_{ij}(\sigma) \), where \( d_{ij} = (\partial_{ij})_{\beta} \). In these local coordinates (6.15) becomes
\[
d^2L_{ij} = \delta_{ij} + \text{cycle}
\]
valid for \( \sigma \) such that \( \sigma(1) \) in the domain of the chart \( x \).

By Remarks 6.2 and the definition of \( d^2 \), if \( L = L^\varepsilon \) is a lattice, then \( d^2L^\varepsilon \neq 0 \), since \( S_{\varepsilon}^\varepsilon \) satisfies the Bianchi identity \( d^2S_{\varepsilon}^\varepsilon \neq 0 \). This suggests the following theorem, see Theorem 4.5 of [G1] for the case \( M = \mathbb{R}^n \).

**Theorem 6.2.** Let \( L \) be a path 2-form which satisfies condition (6.12) and the "Bianchi" identity \( d^2L = 0 \). Then \( L(\beta(0), Z(0)) \) is closed—that is the curvature of \( B = B^\varepsilon \) is zero on \( \Omega_{w} \).

**Proof.** The proof will be quite similar to the proof of Theorem 4.5 of [G1]. We must show that \( P_{\varepsilon}^\varepsilon(X, Y) = 0 \) for \( X \) and \( Y \) in \( \mathcal{T}_\mathbb{R} \), and \( w \in \Omega_{w} \), where \( P^\varepsilon (dB + B \wedge B) \). By continuity it is sufficient to prove it for the special case where \( X \) and \( Y \) in \( \mathcal{T}_\mathbb{R} \) are \( C^\infty \)-sections along a \( C^\infty \)-path \( \in \Omega_{w} \). Let \( \gamma(\sigma, \tau) \) be the \( C^\infty \)-map given by \( e^{\int_{\beta(0)}^{\tau} \gamma(x(\sigma, \tau))} \), for \( \sigma \) and \( \tau \) near zero. Then \( \gamma(\sigma, \sigma) \) is a \( C^\infty \)-section along \( X = \gamma(0, 0) \) and \( Y = (0, 0) \), where \( \gamma_{1} \) and \( \gamma_{2} \) denote partial derivatives of \( \gamma \) with respect to \( \sigma \) and \( \tau \), respectively. Since the differential of commutes with pullbacks, \( dB(\gamma_{1}(x, \tau), \gamma_{2}(x, \tau)) \) may be computed as \( \delta_{1} B(\gamma_{1}(x, \tau), \gamma_{2}(x, \tau)) \).

In order to simplify notation, the \( \sigma \) and \( \tau \) parameters will be suppressed. If \( f \) is a function of \( f(x, r, t) \), then \( f_{r} \) will denote \( \partial_{r} f \), \( f_{r} \) will denote \( \partial_{r} f \), and \( \partial_{r} f \) will denote \( \partial_{r} f \).
Also let \( \gamma' = \gamma(s, t') \) and \( \gamma'' = \gamma(s, t') \). Then a straightforward but tedious calculation using the fact that \( d' = 0 \) (see the appendix to this section) shows that
\[
\delta_t (L'(\gamma'))(s, \gamma(s)) - \delta_t (L'(\gamma'')(s, \gamma''(s))) \\
= \delta_t (L'(\gamma'))(s, \gamma'(s)) + \left[ L'(s, \gamma(s)), B(\gamma'(s)) \right] - \left[ L'(s, \gamma''(s)), B(\gamma''(s)) \right].
\]
(6.17)

Now observe
\[
\delta_t B(\gamma) = \int_0^1 \delta_t (L'(\gamma))(t, \gamma(t)) \, dt
\]
and
\[
\int_0^1 \delta_t (L'(\gamma))(t, \gamma(t)) \, dt = \delta_t (L'(\gamma))(s, \gamma(s)) = 0,
\]
(6.18)
since \( \gamma(s, t') = 0 \) for \( t = 0 \) and \( t = 1 \). Therefore, by integrating (6.17) with respect to \( r \) over the unit interval \( I \) we find
\[
\delta_t B(\gamma'(s)) - \delta_t B(\gamma''(s)) \\
= \int_0^1 \left[ \left( L''(\gamma')(\gamma(t), \gamma(t)), B(\gamma'(t)) \right) - \left[ L''(\gamma')(\gamma(t), \gamma(t)), B(\gamma''(t)) \right] \right] \, dt.
\]
(6.19)

But by the definition of \( B \) and a simple change of variables, we get
\[
B(\gamma'(s)) = \int_0^1 L''(\gamma')(\gamma(t), \gamma(t)) \, dt
\]
\[
= \int_0^1 L''(\gamma''(s))(\gamma(t), \gamma(t)) \, dt.
\]

So by this last equation, (6.19) may be rewritten as
\[
\delta_t B(\gamma'(s)) - \delta_t B(\gamma''(s)) \\
= \int_0^1 \left[ \left[ L''(\gamma')(\gamma(t), \gamma(t)), L''(\gamma''(s))(\gamma(t), \gamma(t)) \right) \right] \, dt
\]
\[
- \left[ L''(\gamma''(s))(\gamma(t), \gamma(t)), L''(\gamma'(s))(\gamma(t), \gamma(t)) \right] \, dt
\]
\[
\int_0^1 \left[ \left[ L''(\gamma''(s))(\gamma(t), \gamma(t)), L''(\gamma'(s))(\gamma(t), \gamma(t)) \right) \right] \, dt
\]
\[
+ \left[ L''(\gamma'(s))(\gamma(t), \gamma(t)), L''(\gamma''(s))(\gamma(t), \gamma(t)) \right] \, dt.
\]
(6.20)

After interchanging the letters \( s \) and \( r \) and the order of integration in the second term, one shows that the two terms in (6.20) add up to give \( B(\gamma'(s)) - B(\gamma''(s)) \). This proves \( B \) has zero curvature on \( \Omega \). Q.E.D.

As a consequence of this last characterization for a path 2-form to be closed, we can easily see that Theorem 6.1 reduces to the \( U(1) \) structure group to the standard result of Weil's stated in the introduction.

**Corollary 6.1.** Let \( F \) be an imaginary valued 2-form on a simply connected manifold \( M \) for which \( [2n(2n-1)F] \) is integral, then there exists a \( U(1) \)-valued bundle connection pair \((E, \nabla)\) such that the curvature \( F' = -F \). This correspondence characterizes the \( (E, \nabla) \) up to equivalence.

**Proof.** Let \( L(s) = L'(s) = P(s(1)) \), a path 2-form on \( \mathbb{R} \). Condition (6.12) is easily seen to hold in this case and it is easy to check that \( (dL'(s) - dP(s(1))) \) which is zero by assumption. Set \( B(\gamma)(X) = B'(\gamma)(X) = \int_0^1 P(s(\gamma(t)) (\gamma(t), X(t)) \, dt \). Now suppose that \( \gamma : I - \Omega_m \) is a smooth path such that \( \gamma(0) = \gamma(1) = c_m \) the constant path at \( m \). Since \( U(1) \) is commutative, the operators \( B(\gamma)(\cdot) \) all commute among themselves, so that differential equation for parallel translation may be explicitly solved to give
\[
P(t) = \exp \left[ \int_0^1 B(\gamma(t'))(\partial, \gamma(t') X, \partial, \gamma(t') X) \right].
\]

Now thinking of \( \gamma \) as a map from \( I \times I \) to \( M \), the last double integral may be written as \( \int_I \int_{\gamma(t)} \gamma(t') \, F \)—using the standard differential form notation. According to Theorem 6.1, \( L \) is a lasso if and only if \( P(t) = \exp \left[ \int_0^1 F(t) \, dt \right] \) for all such \( \gamma \). That is \( \int_I F \, dx + 2\pi \). Now by the Hurewicz isomorphism theorem (see Bott and Tu [BT] Theorem 17.21), the second fundamental group \( \pi_2(M, m) \) is isomorphic to \( H_2(M, Z) \), since \( M \) is simply connected. Making use of this isomorphism, the map \( \gamma : I - \Omega_m \) with \( \gamma(0) = \gamma(1) = c_m \) may be identified as representative of a homology class \( [\gamma] \) in \( H_2(M, Z) \). Since this mapping from \( \pi_2(M, m) \) to \( H_2(M, Z) \) is onto it follows that \( \int_I F \, dx + 2\pi \) is equivalent to the statement that \( (2n-1)F \) is integral. Hence, \( F \) is closed and integral if and only if \( L' \) is closed and has trivial monodromy. This shows that \( F - L' \) is a map from the space of closed imaginary valued 2-forms such that \( (2n-1)F \) is integral onto the space of lassos. This map is also easily seen to be one to one. The Corollary now follows from Theorem 6.1, and the fact that the conjugacy classes \( [L] \) contain only the single element \( L \) because of the commutativity of \( U(1) \). Q.E.D.
The calculation following (6.13) holds will be presented in this appendix. Introduce a chart \((x', x_1, \ldots, x_9)\) on \(M_0\) in a neighborhood of \((x, \theta(x))\) and let \(\gamma = x' - x\). We continue the notation introduced in the proof of theorem 6.2 with obvious extensions to the functions \(\gamma_i\), namely \(\gamma_i = x_i' - x_i\). With this notation and the notation of Remark 6.3,

\[
L(x', \gamma_i) = \sum L(x_i') \gamma_i^i, \gamma_i^n, \gamma_i^j,
\]

where all sums are on the indices \(i\) and \(j\) from 1 to \(n\). Therefore,

\[
\partial_i L(x') \delta \partial_i L(x') = \sum \left( \left( \partial_i L(x_i') \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \right) \right)
\]

\[
= \sum \left( \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \right) \gamma_i^n + \left( \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) + \left( \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right)
\]

\[
= \sum \left( \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \right) \gamma_i^n + \left( \sum \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \right) \gamma_i^n
\]

where (6.13) was used in the last equality. Taking (6.21) and subtracting the same expression with \(s\) replaced by \(i\) one finds

\[
\partial_i L(x') \delta \partial_i L(x') = \sum \left( \left( \partial_i L(x_i') \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) + \left( L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) + \left( L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) \right)
\]

(6.22)

where the Bianchi identity (6.16) has been used in the form

\[
\gamma_i \gamma_i = (\gamma_i \gamma_i) - \gamma_i \gamma_i + (\gamma_i \gamma_i) - \gamma_i \gamma_i = \gamma_i \gamma_i
\]

to get the first term. (All derivatives in this last expression are end point derivatives.) Now notice that

\[
\partial_i L(x') \delta \partial_i L(x') = \sum \left( \left( \gamma_i^n \gamma_i^j + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) + \left( L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) + \left( L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n + L(x_i') \gamma_i^n \gamma_i^j \gamma_i^n \right) \right)
\]

(6.23)

Where, in the second equality, we have used the fact that

\[
\partial_i L(x') = (\partial_i L) Q_x
\]

which follows from (6.13) using \(B(\partial_i \gamma_i) = 0\), and \(\partial_i \gamma_i = \gamma_i\). \((B(\partial_i \gamma_i = 0, \text{since} \gamma_i \partial_i \gamma_i = 0)\). So (6.23) can be used to replace the sum in (6.22) by \(\partial_i L(x') \delta \partial_i L(x')\), which results in the desired equality (6.17).

7. Remarks for \(M\) Not Simply Connected

In this section, the manifold \(M\) is no longer assumed to be simply connected. For the purposes of this section let \(\mathcal{A} = \mathcal{A}(M, V, G) = \mathcal{L}(M, V)\), where \(M, V\) is a G-bundle connection pair. If \(M\) is not simply connected, we no longer have an intrinsic characterization of \(\mathcal{A}\) even in the case \(G = U(1)\) (see Example 7.1 below). We first show (Theorem 7.1) that in general the space \(\mathcal{A}\) is no longer in 1-1 correspondence with \(\mathcal{A} = \mathcal{A}(M, V, G)\). This "gauge copy" problem is intimately related with our difficulty in characterizing the space \(\mathcal{A}\). However, in Theorem 7.2 and Corollary 7.1, it is shown that the closed path 2-forms (\(E\) modulo conjugation) with trivial monodromy are in one to one correspondence with \(\mathcal{A}^e = \mathcal{A}(M, V, G)\) where \(M^e\) is the universal covering space of \(M\).

**Theorem 7.1.** Suppose that \(E, V, G, \mathcal{A} = \mathcal{A}(M, V, G)\) is an admissible isomorphism. Set \(p(E, V) = \mathcal{L}(M)\), then \(p^{-1}(\mathcal{L}(M))\) is in one to one correspondence with the set of functions \(h : \Pi^1(M, m) \to G\), modulo conjugation by elements of \(G\), which satisfy

\[
h((x, y)) P(x) = h((x)) P(x) h(y)
\]

(7.1)

for all \(x, y \in M, m\). The notation \(\Pi^1(M, m)\) denotes the homotopy class (i.e., path component of \(\mathcal{U}_M\)) containing \(e\). In particular any \(h \in \text{Hom}(\Pi^1(M, m), G(\mathcal{G}))\) (the homomorphisms from \(\Pi^1(M, m)\) to the center \(Z(G)\) of \(G\)) satisfies (7.1). Conversely if \(G\) is connected and \(V\) is irreducible in the sense that the span \(\langle x \rangle \mathcal{L}(\pi)(x, v) \subset \mathcal{L}(\pi)^{-1}(x, v)\) \(e \in \mathcal{P}, u \in T_{x, y}(M) \neq 0\) then every \(g\) satisfying (7.1) is actually in \(\text{Hom}(\Pi^1(M, m), G(\mathcal{G}))\).

**Proof.** Set \(B(\gamma) = x \mathcal{B}(\gamma) x^{-1}\). So by Corollary 4.4

\[
XP = P(x) B(X)
\]

(7.2)

for all \(X \in \mathcal{G}, m, g \in \mathcal{U}_M\). Using Theorem 5.1, it is easy to see that \(p^{-1}(\mathcal{L}(M))\) is in 1-1 correspondence with \(\{ p^e \} \in \mathcal{A}\) for which \(p^e(\pi) = 1\) satisfies (7.2) for some \(g \in G\). This is the same as saying that \(p^{-1}(\mathcal{L}(M))\) is
in 1-1 correspondence with \(\{F\} \in \mathcal{M} : X^F = P(\sigma) B(X)\) for \(X \in T\Omega_m\). Because of Remark 6.1, the most general solution to (7.2) is of the form

\[ P(\sigma) = h(\sigma) P(\sigma) \]  
\[ (7.3) \]

where \(h \in \Pi(M, m) \rightarrow \mathcal{V} \), since \(h\) indexes the path component of \(\Omega_m\). Any function of the form (7.3) is strongly differentiable and parameterization invariant. Thus w can be defined as \(P(\sigma)\) to be multiplicative. It is easily checked, by demanding that \(P(\sigma) = P(\sigma') P(\sigma)\) holds for all \(\sigma, \tau\), that the condition (7.1) is precisely what is needed to make \(P(\sigma)\) defined by (7.3) multiplicative.

Now suppose that \(G \) is connected and \(\mathcal{V} \) is irreducible. If \(\{\tau\} = \Pi(M, m)\) then (7.1) reduces to \(P(\tau(h(\sigma))) = h(\sigma) P(\tau) P(\sigma)\) for all \(\sigma \in \Omega_m\). By the Ambrose Singer Theorem (see for example Corollary 2.17 of [O]) for a proof in a still later paper of this kind, we can conclude that \(P(\tau) : \tau \in \Pi(M, m)\) is a group, hence \(G \) is connected. Therefore, \(h(\sigma)\) must be in the center of \(G\) for all \(\sigma \in \Omega_m\), in which case (7.1) requires \(h\) to be a homomorphism of a group.

Let \(M\) be a connected but non-simply connected manifold, with universal cover \(\overline{M}\) and covering map \(\theta : M \rightarrow \overline{M}\). Choose \(m' \in \overline{M}\) and set \(m = \theta(m')\). Let \(\mathcal{P}_\theta = \mathcal{P}_\theta(M)\) and \(\mathcal{P}_\theta = \mathcal{P}_\theta(M')\). If \(\pi \in \mathcal{P}_\theta\), set \(\theta = \theta\) to be the unique curve such that \(\theta(0) = m', \theta(1) = m\). Suppose that \(L^\theta\) is a path 2-form on \(\mathcal{P}_\theta\) by

\[ L(\sigma)(u, v) = L^\theta(g(\sigma), v), \]  
\[ (7.4) \]

where \(u = \theta u_+|_{\mathcal{P}_\theta}\) (u). In the future, \(u\) may be written as \(u\), since by context it will be clear when \(u\) should be lifted to \(T\mathcal{M}\). One can easily check that

\[ B^\theta(X) = B^{\overline{X}}(X) \]  
\[ (7.5) \]

where \(X\) is the unique element of \(T_\theta\mathcal{P}_\theta\) such that \(\overline{X} = X \in T_\theta\mathcal{P}_\theta\).

**Theorem 7.2.** Let \(\mathcal{L}\) be a \(T\)-valued closed path 2-form on \(\mathcal{P}_\theta(M)\), such that \(L\) has trivial monodromy in the sense that the representation it induces on \(H_2(M, \mathcal{M}) = \Omega_2(\Omega_\mathcal{M}, \mathcal{M})\) is trivial. Then there exists an element \(B(\mathcal{L} = B^\theta = \theta^\theta(\mathcal{M}, M, V, G)\) such that \([\mathcal{L}^\theta] = [B^\theta]\), where

\[(\mathcal{L}^\theta) = \mathcal{L}(\theta = \theta, u, v) \]  
\[ (7.6) \]

for \(\sigma \in \mathcal{P}_\theta\) and \(u, v \in \mathcal{T}_{\mathcal{M}_{\theta}(M)}\).
let \( F' \) be a closed imaginary valued 2-form on \( M' \) for which \( F' \) is not \( \theta \)-\( F \) for some 2-form on \( M \). (All of this is easily accomplished on \( S^1 \times \mathbb{R} \) for instance.) Using \( F' \), construct a path 2-form (\( L \)) on \( \mathbb{R} \) by

\[ L(\epsilon) = F'(\theta(\epsilon)). \]  

(7.11)

So if \( x \in \mathcal{T}_s, \rho_x \) such that \( x(1) = 0 \), then \( \mathcal{T}_x = 0 \), since the variation of \( \sigma \) by \( x \) does not change the homotopy class of \( \sigma \), and hence does not change \( \theta(1) \). So \( L \) satisfies the condition in equation (6.12). Furthermore, it is easy to check that \( dL(\mathcal{T}) = dF'(\theta(\mathcal{T})) \) which is zero by assumption. So by Theorem 6.2, \( L \) is closed. The path 2-form \( L \) also has trivial monodromy, since \( \Pi_1((M, m)) = \{1\} \). Nevertheless, \( L \) need not be a lasso, because if it would it would imply that \( L(\epsilon) = \mathcal{F}(\theta(\epsilon)), \) for some 2-form \( F \). But this would imply that \( F' = \mathcal{F} \).

This last example shows that the conditions of Theorem 6.1 no longer characterize the lassos. Furthermore, using the same example with \( M = S^1 \times \mathbb{R} \), one may easily show that the representations induced by \( B = B' \) on the other fundamental groups \( \Pi_1((G, a)) \) are trivial, where \( a \in \mathcal{O}_U \) is a path not homotopic to \( \mathcal{C}_U \). This shows that requiring trivial monodromy on the fundamental groups \( \Pi_1((G, a)) \) for each path component of \( \mathcal{O}_U \) is still not enough to guarantee that \( L \) is a lasso.

To finish this section we record a result which may be useful for future considerations of non-simply connected \( M \). In order to state it, let \( \text{COV}(M') \) denote the space of covering transformations \( \mu \) of \( M' \) with respect to the covering map \( \theta: M' \to M \). Recall that \( \text{COV}(M') \) is a group under composition which is isomorphic to \( \Pi_1((M, m)) \).

**Theorem 7.3.** Let \( \theta: M' \to M \), and \( m = \theta(m') \) be as above. Suppose that \((E', V')\) is a vector bundle over \( M' \) with connection \( V \), fiber model space \( V \) and structure group \( G \). Then there exists a bundle connection pair \((E, V)\) over \( M \) such that \((E', V')\) is equivalent to the pullback of \((E, V)\) by \( \theta \) if and only if the following conditions hold:

1. For each \( \mu \in \text{COV}(M') \) there exists a lifting to a smooth map \( \check{\mu}: E' \to E \) for which \( \check{\mu}_2 = \mu_2 \) is a linear isomorphism from \( E'_2 \) to \( E_2 \).

Furthermore, the lifts should satisfy \( \check{\mu}_1 = \mu(\check{\mu}_2) \) for all \( \mu, \check{\mu} \in \text{COV}(M') \).

2. If \( \pi \in \Gamma(E) \) is a smooth section, then the connection \( V \) should satisfy

\[ V_\xi S = \check{\pi}^{-1}(V_{\check{\pi}(\xi)}(\check{\mu}_2 S + \check{\mu}_1)), \]  

(7.12)

for any \( \xi \in TM' \).

**Remark.** The theorem gives a criterion for when one can "push forward" the bundle \((E', V')\) over \( M' \) to a bundle connection pair \((E, V)\) over \( M \). Sketch of Proof. First suppose that \((E', V')\) is the pullback of \((E, V)\). Then by definition of the pullback, the fibers of \( E' \) are \( E'_x = \{x \in E_m : \xi \in E_m \} \). So the desired lifts may be defined by \( \check{\mu}(x, \xi) = (\xi, x) \in E_x' \). For this choice of lifts, it is easily checked that conditions (1) and (2) hold.

Conversely assume that \((E', V')\) is a bundle connection pair for which conditions (1) and (2) hold. Define an equivalence relation on \( E' \) by \( z \sim \xi \) if there is a \( \mu \in \text{COV}(M') \) such that \( \check{\mu}_1(\xi) = \mu(\xi) \). It is clear that \( \sim \) should be defined as \( E' \sim E \) with the obvious projection map onto \( M \). To define the local trivializations of \( E \), let \( \psi', U' \) be a local trivialization of \( E' \) over an open set \( M' \) such that \( \psi'_1 \) is a diffeomorphism of \( U' \) onto an open set \( U = \theta(U') \) of \( M \). For any \( \xi \in E' \), let \( \xi \) denote the element of \( E \) given by the equivalence class containing \( \xi \). Let \( \psi(\xi)(x) = \psi'((\xi), x) \) for all \( \xi \in \pi^{-1}(U') \), where \( \pi \) is the projection map from \( E' \) to \( E' \). It is easy to check using property (1) that these local trivializations define a vector bundle structure on \( E \) with structure group \( G \). Now suppose that \( \pi \in \Gamma(E) \) is a section of \( E \), and \( \mu, \check{\mu} \in \text{COV}(M') \). Let \( U \) be a neighborhood of \( m \), which is covered by \( U' \) in such a way that \( \pi^{-1}(U) \mu_1(\xi) \) is a diffeomorphism. The section \( \pi \) can be written as \( \pi(p) = [S[p]] \) for \( p \in U \), where \( S^3 \) is a local section of \( E' \) over \( U' \). With this notation, define

\[ V_\xi S = (V_\xi^3 S^3). \]

One can check using properties (1) and (2) that the last expression is well defined and in fact defines a covariant derivative on \( E \). Furthermore, the bundle pair \((E', V')\) is equivalent to the pullback of the bundle pair \((E, V)\).

Q.E.D.

So in order to characterize the lassos \( \mathcal{L} \) is sufficient to find conditions on a path 2-form \( L \) such that the bundle connection pair \((E', V')\) constructed in Theorem 7.2 satisfies the hypothesis of Theorem 7.3.

**References**


BUNDLE-CONNECTION CLASSIFICATIONS


Printed by Catherine Press, Ltd., Templehof 41, B-8000 Brugge, Belgium.