HOLOMORPHIC FUNCTIONS AND SUBELLIPTIC HEAT KERNELS OVER LIE GROUPS.

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ABSTRACT. A Hermitian form q on the dual space, $\mathfrak{g}^*$, of the Lie algebra, $\mathfrak{g}$, of a Lie group, G, determines a sub-Laplacian, Δ, on G. It will be shown that Hörmander's condition for hypoellipticity of the sub-Laplacian holds if and only if the associated Hermitian form, induced by q on the dual of the universal enveloping algebra, is nondegenerate. The subelliptic heat semigroup, $e^{t\Delta/4}$, is given by convolution by a C^∞ probability density ρ_t. We will show that if G is complex, connected, and simply connected then the Taylor expansion defines a unitary map from the space of holomorphic functions in $L^2(G, \rho_t)$ onto (a subspace of) the dual of the universal enveloping algebra.

CONTENTS

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1. Introduction

Denote by G a real or complex Lie group, by $\mathfrak{g} = T_eG$ its Lie algebra, and by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let q be a non-negative quadratic or Hermitian form on $\mathfrak{g}^*$ according to whether $\mathfrak{g}$ is real or complex. Let $K = \mathrm{Nul}(q) = \{\alpha \in \mathfrak{g}^* : q(\alpha, \alpha) = 0\}$ and

$$H = K^0 := \{\xi \in \mathfrak{g} : \alpha(\xi) = 0 \text{ for all } \alpha \in \mathrm{Nul}(q)\}$$

be the backwards annihilator subspace of K in $\mathfrak{g}$. We say that q satisfies *Hörmander's condition* if H generates $\mathfrak{g}$ as a Lie algebra.

In Section 2 we are going to characterize those q for which Hörmander's condition holds in terms of the following natural seminorms on the dual space of the universal enveloping algebra of $\mathfrak{g}$. Denote by $q^{\otimes k}$ the extension of q to a non-negative quadratic/Hermitian form on $(\mathfrak{g}^*)^{\otimes k}$ where by convention, $(\mathfrak{g}^*)^{\otimes 0}$ is $\mathbb{R}$ or $\mathbb{C}$ according to whether G is real or complex and $q^{\otimes 0}(1) = 1$. If $T(\mathfrak{g})$ is the tensor algebra over $\mathfrak{g}$ then the algebraic dual space of $T(\mathfrak{g})$ is the direct product: $T(\mathfrak{g})' = \Pi_{k=0}^{\infty}(\mathfrak{g}^*)^{\otimes k}$. For each $t > 0$ define

$$q_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} q^{\otimes k} \tag{1.1}$$

on $T(\mathfrak{g})'$, where we allow for the possibility that $q_t(\alpha)$ is infinite. On the subspace where q_t is finite it is the square of a semi-norm. Because of the allowed degeneracy of q the semi-norm may not be a norm. But we are going to restrict the domain of q_t further. Denote by J the two-sided ideal in $T(\mathfrak{g})$ generated by the elements $\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta]$ wherein ξ and η run over $\mathfrak{g}$. We can identify the universal algebra $\mathcal{U}$ of $\mathfrak{g}$ with $T(\mathfrak{g})/J$ and then the algebraic dual space, $\mathcal{U}'$, may be identified with J^0, the annihilator of J in $T(\mathfrak{g})'$. Let

$$J_t^0 := \left\{\alpha \in J^0 : q_t(\alpha) < \infty\right\}.$$

We will show in Section 2 (see Theorem 2.7 and Corollary 2.14) that the following conditions on q are equivalent: 1) Hörmander's condition holds, 2) $T(\mathfrak{g}) = T(H) + J$ ($T(H)$ is the tensor algebra over H), and 3) for any $t > 0$, $q_t|_{J_t^0}$ is the square of a norm, i.e., $q_t|_{J_t^0}$ is the quadratic (Hermitian) form associated to a positive definite inner product on J_t^0.

For each $A \in \mathfrak{g}$, let $\tilde{A}$ denote the unique extension of A to a left invariant vector field on G. If G is real, for any basis $X_1, \ldots, X_M$ of $\mathfrak{g}$ with dual basis $\{X_j'\}$ the second order differential operator

$$\Delta_q = \sum_{j,k=1}^{M} q(X_j', X_k') \tilde{X}_j \tilde{X}_k \tag{1.2}$$

is easily seen to be independent of the choice of basis. Hörmander's theorem [29] states that Δ_q is hypoelliptic if and only if q satisfies Hörmander's condition.

Now suppose that G is a complex connected Lie group and q is a non-negative Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition. Let $\mathrm{Re}\, q$ denote the real part of the Hermitian inner product on $\mathfrak{g}^*$ with the complex structure forgotten. Since $\mathrm{Nul}(q) = \mathrm{Nul}(\mathrm{Re}\, q)$, one easily shows that q satisfies Hörmander's condition iff $\mathrm{Re}\, q$ satisfies Hörmander's condition. We may form the associated hypoelliptic sub-Laplacian, $\Delta_{\mathrm{Re}\, q}$, as in (1.2) and in this case, the heat semigroup, $\exp(\frac{t}{4}\Delta_{\mathrm{Re}\, q})$, is given by convolution by a C^∞ heat kernel ρ_t on G. Let $\mathcal{H}$ denote the space of holomorphic functions on G and for any function f in $\mathcal{H}$ and $x \in G$, let $\hat{f}(x) \in J^0 \cong \mathcal{U}'$ be the "Taylor coefficient" at the point x defined by

$$\langle \hat{f}(x), \beta \rangle = (\tilde{\beta} f)(x) \text{ for all } \beta \in T(\mathfrak{g}), \tag{1.3}$$

where $\tilde{\beta}$ is the left invariant differential operator on G associated to β, see Notation2.4 below. Because of the results of Section 2, we know J_t^0 is a Hilbert space with respect to the norm $(q_t|_{J^0})^{1/2}$. The aim of this paper is to show that the Taylor map, $f \to \hat{f}(e)$, is a unitary isomorphism of $\mathcal{H} \cap L^2(G, \rho_t)$ onto J_t^0 when G is simply connected. See Theorem 5.1 in the case G is nilpotent and Theorem 7.1 for general G.

This kind of unitary isomorphism of holomorphic function spaces with a Hilbert space of "Taylor coefficients" has a long history. A knowledgeable reader could "read out" of the 1932 paper [9] by the physicist V.A. Fock such an isomorphism. But the isomorphism was not actually made clear until the work of Segal [45, 46] and Bargmann [2]. (See also [26] for more history.) In that classical case, the complex group G is just $\mathbb{C}^M$, q is just the usual Hermitian norm on $\mathbb{C}^M$ and the density ρ_t is just a Gaussian. A detailed exposition of this isomorphism along with a discussion of its extensive history may be found in the expository portion of the paper [26]. Inspired by related work of B. Hall [27], the first named author [7] proved such an isomorphism for a wide class of complex Lie groups G, for a strictly positive definite quadratic form q. This was subsequently extended to an arbitrary complex Lie group in [8] but again, for a strictly positive definite quadratic form q. To our knowledge this is the first work dealing with this isomorphism in the degenerate (i.e. subelliptic) case. The Taylor map isomorphism has also been proven for some infinite dimensional groups: in [16] and [15] M. Gordina found a precise analog of this unitary isomorphism for the infinite dimensional complex Hilbert-Schmidt orthogonal group and in [17] she proved the analog for the group of invertible operators in a factor of type II_1. Also M. Cecil, in [4], has shown that a unitary Taylor isomorphism holds for path groups over stratified Lie groups.

Section 4 establishes that $\hat{f}(e) \in J_t^0$ for every $f \in \mathcal{H} \cap L^2(G, \rho_t)$ and that the Taylor map,

$$f \in \mathcal{H} \cap L^2(G, \rho_t) \to \hat{f}(e) \in J_t^0,$$

is isometric. The proof that $f \to \hat{f}(e)$ is an isometry follows closely the proof in [8].

In Section 5, we will prove that the Taylor map is actually surjective and therefore unitary, when G is nilpotent and simply connected. In this case we will first prove that the finite rank tensors are dense in J_t^0 when the Lie algebra is stratified and then use the fact that any nilpotent group is covered by a stratified nilpotent group. This approach cannot work for a general group because finite rank tensors are not dense in J_t^0 when the group is not nilpotent. For general groups we will adapt, in Section 6, the method first introduced in [7] to recover a holomorphic function from its Taylor coefficient $\hat{f}(e)$. Then, in Section 7, we will prove the surjectivity for an arbitrary simply connected group G. See Theorem 7.1 for a precise statement.

In Section 8 we will compare the norms induced on J^0 by two different quadratic forms on $\mathfrak{g}^*$. When such a comparison can be made by direct combinatorial techniques one obtains an alternative proof of surjectivity in the degenerate case based on the known surjectivity in the nondegenerate case. We will implement this approach in the case of the three dimensional Heisenberg group.

In section 9 we will show how the Fourier-Wigner transform leads to examples of holomorphic functions which are in $L^2(G, \rho_t)$ on the complex Heisenberg group $G = H_3^{\mathbb{C}}$.

This paper continues a body of work in which the heat kernel on a Lie group G plays the role of a weight for the study of $L^2(G, w(x)dx)$. If G is complex then such a (rapidly decreasing) weight is required if this space is to contain non-constant holomorphic functions. In addition to a study of these holomorphic function spaces, $\mathcal{H} \cap L^2(G, w(x)dx)$, there are natural transforms into such spaces from function spaces over compact Lie groups. Heat kernel measures play a key role here also in place of Haar measure. For further background the reader may consult the recent surveys [21] and [28].

It may be useful to comment on the term "subelliptic" used in the title of this paper. Consider a second order differential operator $L = \sum \partial_i a_{i,j}(x)\partial_j$ with smooth coefficients in an open set $\Omega \subset \mathbb{R}^M$. The operator L is called *elliptic* if the matrix $(a_{i,j}(x))$ is everywhere positive definite (this is one of the standard usages of the term *elliptic*, see [31, 34]). The operator is called *subelliptic* if the matrix $(a_{i,j}(x))$ is everywhere positive semidefinite and there is a real $s \in (0, 1]$ such that L satisfies the *subelliptic* estimate

$$\forall\, u \in \mathcal{C}_0^\infty(\Omega), \|u\|_{(2s)} \leq C(\|u\| + \|Lu\|), \tag{1.4}$$

where $\|\cdot\|$ stands for the usual L^2-norm and $\|u\|_{(s)} = (\int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi)^{1/2}$ is the Sobolev norm of index s. See [34] and the references therein. Note that any elliptic operator satisfies (1.4) with $s = 1$, locally (See, e.g., Lemma 17.1.2 in [30]) and that any subelliptic operator is hypoelliptic (See Proposition 3.2 in [29]).

Now, if G is a real Lie group and $L = \sum_1^k \tilde{X}_i^2$ is the sum of the squares of left invariant vector fields on G, L is (locally) subelliptic if and only if it satisfies *Hörmander's condition*, i.e., $\{X_1, \ldots, X_k\}$ generates the Lie algebra of G. See [29]. The term *subelliptic* heat kernel on G refers to the minimal solution of the Cauchy problem $\partial_t u = Lu$, $u_0 = \delta_e$, where $L = \sum_1^k \tilde{X}_i^2$ and is subelliptic.

2. Hörmander's condition and nondegeneracy of norms

Notation 2.1. We will denote by $\mathfrak{g}$ a real (respectively complex) finite dimensional Lie algebra. We let q be a nonnegative quadratic (respectively Hermitian) form on the dual space $\mathfrak{g}^*$. Thus

$$q(f) = (f, f)_q \tag{2.1}$$

for some, possibly degenerate, nonnegative bilinear (respectively sesquilinear) form $(\ ,\ )_q$ on $\mathfrak{g}^*$. Let

$$K := \{a \in \mathfrak{g}^* : q(a) = 0\} \tag{2.2}$$

be the null space of q and let

$$H = K^0 := \{\xi \in \mathfrak{g} : \langle a, \xi\rangle = 0 \quad \forall\ a \in K\} \tag{2.3}$$

be the backwards annihilator of K. Here, as elsewhere, we use $\langle\cdot,\cdot\rangle$ for the bilinear pairing between a vector space and its dual, while $(\cdot,\cdot)_q$ denotes the bilinear (or sesquilinear) form induced by q on $\mathfrak{g}^*$. We call H the **Hörmander space** associated to q.

The degenerate case is of primary interest to us. We will assume henceforth that the kernel, K, of q is nontrivial. The next elementary result gives an explicit characterization of q.

Lemma 2.2. *There is a unique inner product,* $(\cdot,\cdot)_H$ *, on* H *such that for any orthonormal base* $\{X_j\}_{j=1}^m$ $(m := \dim(H))$ *of* H *we have*

$$(a,b)_q = \sum_{j=1}^m \langle a, X_j\rangle \overline{\langle b, X_j\rangle} \text{ for all } a,b\in\mathfrak{g}^*.$$

In particular

$$q(a) = (a,a)_q = \sum_{j=1}^m |\langle a, X_j\rangle|^2. \tag{2.4}$$

Proof: The form q descends to a strictly positive definite quadratic form, $\bar{q}$, on $\mathfrak{g}^*/K$ and the map

$$\mathfrak{g}^*/K \ni (a+K) \to a|_H \in H^*$$

is a linear isomorphism of vectors spaces. Using this isometry, $\bar{q}$ induces an inner product, $(\cdot,\cdot)_{H^*}$, on H^* and hence, by the Riesz theorem, an inner product, $(\cdot,\cdot)_H$, on H. Suppose that $\{X_j\}_{j=1}^m$ is any orthonormal basis of $(H,(\cdot,\cdot)_H)$ and $a,b\in\mathfrak{g}^*$. Then

$$(a,b)_q = (a+K,b+K)_{\bar{q}} = (a|_H, b|_H)_{H^*} = \sum_{j=1}^m \langle a, X_j\rangle \overline{\langle b, X_j\rangle}.$$

Q.E.D.

Notation 2.3. The form q induces a degenerate (real or Hermitian) quadratic form $q_k := q^{\otimes k}$ whose inner product, $(\cdot,\cdot)_{q_k}$, on $(\mathfrak{g}^*)^{\otimes k}$ is determined by

$$(a_1\otimes\cdots\otimes a_k, b_1\otimes\cdots\otimes b_k)_{q_k} = \prod_{j=1}^k (a_j,b_j)_q \quad a_i,b_i\in\mathfrak{g}^*,\ i=1,\ldots,k \tag{2.5}$$

for $k\geq 1$. If $\alpha\in(\mathfrak{g}^*)^{\otimes k}$, we will write $q_k(\alpha)$ or $|\alpha|_{q_k}^2$ for $(\alpha,\alpha)_{q_k}$. By convention, $V^{\otimes 0}$ is $\mathbb{R}$ or $\mathbb{C}$ depending on whether V is a real or complex vector space respectively and we define q_0 on $(\mathfrak{g}^*)^{\otimes 0}$ so that $q_0(1) = 1$.

Notation 2.4 (Left Invariant Differential Operators)**.** Denote by $T(\mathfrak{g})$ the tensor algebra over $\mathfrak{g}$. An element of $T(\mathfrak{g})$ is a finite sum:

$$\beta = \sum_{k=0}^N \beta_k \qquad \beta_k\in\mathfrak{g}^{\otimes k}. \tag{2.6}$$

We define a linear map $\left(\beta\to\tilde{\beta}\right)$ from $T(\mathfrak{g})$ to left invariant differential operators on G determined by: 1) $\tilde{1} = Id$ and 2) for $\beta = A_1\otimes\cdots\otimes A_k\in\mathfrak{g}^{\otimes k}$, $\tilde{\beta} := \tilde{A}_1\ldots\tilde{A}_k$.

The algebraic dual space $T(\mathfrak{g})'$ may be identified with the direct product $\prod_{k=0}^\infty (\mathfrak{g}^*)^{\otimes k}$ in the pairing

$$\langle\alpha,\beta\rangle = \sum_{k=0}^\infty \langle\alpha_k,\beta_k\rangle \tag{2.7}$$

where

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \qquad \alpha_k \in (\mathfrak{g}^*)^{\otimes k}. \tag{2.8}$$

Notation 2.5. Let J denote the two sided ideal in $T(\mathfrak{g})$ generated by

$$\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] : \xi, \eta \in \mathfrak{g}\}.$$

The universal enveloping algebra of $\mathfrak{g}$ is the associative algebra $\mathcal{U} := T(\mathfrak{g})/J$ and the algebraic dual space $\mathcal{U}'$ can be identified with

$$J^0 := \{\alpha \in T(\mathfrak{g})' : \langle \alpha, J \rangle = \{0\}\}. \tag{2.9}$$

For $t > 0$ define

$$\|\alpha\|_t^2 := \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|_{q_k}^2 \tag{2.10}$$

when α is given by (2.8).

The function, $\|\cdot\|_t$, defines a seminorm in the subspace of $T(\mathfrak{g})'$ on which $\|\alpha\|_t^2$ is finite. But we will, by restriction, always consider $\|\cdot\|_t$ to be a semi-norm on

$$J_t^0 := \{\alpha \in J^0 : \|\alpha\|_t^2 < \infty\}. \tag{2.11}$$

It was shown in [8] that when $\mathfrak{g}$ is complex and q is nondegenerate then the Hilbert space J_t^0, in the norm $\|\cdot\|_t$, is naturally isomorphic to the Hilbert space of holomorphic functions in $L^2(G, \rho_t(x)dx)$ where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and convolution by $\rho_t(x)$ is the heat kernel operator for the left invariant sub-Laplacian on G induced by q. The isomorphism is given by a Taylor expansion. cf. [8, Theorem 2.6]. This will also be shown in the subelliptic case in Section 4. The goal of the present section is to characterize the circumstance under which the seminorm $\|\cdot\|_t$ on J_t^0 is actually a norm.

Definition 2.6. We say that Hörmander's condition holds for q if the smallest Lie subalgebra, $\mathrm{Lie}\,(H)$, containing H is $\mathfrak{g}$.

Theorem 2.7. *Let $t > 0$. The seminorm $\|\cdot\|_t$ on J_t^0 is a norm if and only if Hörmander's condition holds.*

Proof: The proof of this theorem is the contents of Lemmas 2.12 and 2.13 below whose proofs were motivated by the techniques developed in [24]. Q.E.D.

The Lie subalgebra containing H may be described explicitly as follows. Let H_n denote those elements of $\mathfrak{g}$ which may be written as linear combinations of elements of the form

$$A = ad_{A_1} \dots ad_{A_{k-1}} A_k = [A_1, [A_2, [\dots [A_{k-1}, A_k]] \dots] \tag{2.12}$$

with $A_i \in H$ for $i \le k$ and $k \le n$. Here, for $k = 1$, we interpret $ad_{A_1} \dots ad_{A_{k-1}}$ to be the identity operator in (2.12). In particular, $H_1 = H$.

Lemma 2.8. $\mathrm{Lie}\,(H) = H_n$ *for all sufficiently large n.*

Proof: It is clear that H_n is an increasing sequence of subspaces which are contained in $\mathrm{Lie}\,(H)$ and because $\mathfrak{g}$ is finite dimensional, H_n must be independent of n for large n. So to finish the proof it suffices to show $\cup_n H_n$ is a Lie algebra and for this it suffices to show $[A, B] \in \cup_n H_n$ whenever A is as Eq. (2.12) and

$$B = ad_{B_1} \dots ad_{B_{m-1}} B_m = [B_1, [B_2, [\dots [B_{m-1}, B_m]] \dots] \tag{2.13}$$

for some $B_i \in H$. However this is easily proved by induction on k. The case $k=1$ is trivial. Now suppose that the assertion $[A,B] \in \cup_n H_n$ holds for any $k \le k_0$. Let

$$A' := ad_{A_2} \dots ad_{A_{k_0}} A_{k_0+1}$$

and

$$A = ad_{A_1} \dots ad_{A_{k_0}} A_{k_0+1} = [A_1, A'] .$$

Then, by the Jacobi identity,

$$[A,B] = ad_A B = ad_{[A_1,A']} B = ad_{A_1} ad_{A'} B - ad_{A'} ad_{A_1} B$$

which is in $\cup_n H_n$ by the induction hypothesis and the fact that $\cup_n H_n$ is stable under applying ad_{A_1} with $A_1 \in H$. Q.E.D.

Notation 2.9. Let $r = \min\{n : H_n = \mathrm{Lie}(H)\}$.

The proof of Theorem 2.7 will depend on the following lemmas. Since the theorem has no content if q is nondegenerate we will assume throughout that q is degenerate.

Lemma 2.10. *Let $\alpha \in (\mathfrak{g}^*)^{\otimes k}$ for some $k \ge 1$. Then*

$$q_k(\alpha) > 0 \tag{2.14}$$

if and only if there exist vectors $\xi_1, \dots, \xi_k \in H$ such that

$$\langle \alpha, \xi_1 \otimes \dots \otimes \xi_k \rangle \neq 0. \tag{2.15}$$

Proof: From Eqs. (2.4) and (2.5),

$$|\alpha|_{q_k}^2 = \sum_{j_1,\dots,j_k=1}^{m} |\langle \alpha, X_{j_1} \otimes \dots \otimes X_{j_k} \rangle|^2 \tag{2.16}$$

for any element $\alpha \in (\mathfrak{g}^*)^{\otimes k}$. So $|\alpha|_{q_k}^2 > 0$ if and only if one of the terms on the right side of the last equality is not zero. Q.E.D.

Lemma 2.11. *If Hörmander's condition holds then there exists an $r \in \mathbb{N}$ and an algebra homomorphism, $P : T(\mathfrak{g}) \to T(H)$ such that:*

1. *if $\beta \in T(\mathfrak{g})$ with maximum rank at most n then $P\beta$ has maximum rank at most nr in $T(H)$.*
2. *$P|_{T(H)} = id_{T(H)}$ and in particular P is a projection operator.*
3. *For all $\beta \in T(\mathfrak{g})$, $\beta - P\beta \in J$ and in particular* $\mathrm{Nul}(P) \subset J$ *and*

$$T(\mathfrak{g}) = T(H) \oplus \mathrm{Nul}(P) = T(H) + J. \tag{2.17}$$

4. *$P|_{\mathfrak{g}} : \mathfrak{g} \to \oplus_{k=1}^{r} H^{\otimes k} \subset T(H)$ is a bounded linear operator.*

Proof: Given $\Gamma := (A_1, A_2, \dots, A_n) \in \mathfrak{g}^n$, let

$$[\Gamma] := [A_1, [A_2, [A_3, \dots, [A_{n-1}, A_n] \dots] = ad_{A_1} ad_{A_2} \dots ad_{A_{n-1}} A_n. \tag{2.18}$$

and let

$$\hat{\Gamma} := A_1 \wedge (A_2 \wedge (A_3 \wedge \dots (A_{n-1} \wedge A_n) \dots) \subset T(\mathfrak{g}), \tag{2.19}$$

where $u \wedge v = u \otimes v - v \otimes u$ for any two tensors u and v. A simple induction argument shows that $\hat{\Gamma} = [\Gamma] + j(\Gamma)$ with $j(\Gamma) \in J$. Indeed, if $n=2$,

$$A_1 \wedge A_2 = [A_1, A_2] + j(A_1, A_2)$$

where $j(A_1, A_2) = A_1 \wedge A_2 - [A_1, A_2] \in J$. Similarly if $A_0 \in \mathfrak{g}$, then

$$A_0 \wedge \hat{\Gamma} = A_0 \wedge [\Gamma] + A_0 \wedge j(\Gamma) = [A_0, [\Gamma]] + j(A_0, [\Gamma]) + A_0 \wedge j(\Gamma)$$

which completes the induction argument since J is an ideal. Clearly if $\Gamma \subset H^n$ then $\hat{\Gamma} \in H^{\otimes n}$.

Choose a basis $X_1, \ldots, X_m, Y_1, \ldots Y_\ell$ of $\mathfrak{g}$ ($m + \ell = d = \dim \mathfrak{g}$) such that $X_1, \ldots, X_m$ is a basis for H. By Hörmander's condition each vector Y_k is a finite linear combination of commutators $[\Gamma]$ with $\Gamma \in H^n$ and $n \leq r$. The corresponding linear combination, $\hat{Y}_k$, of such $\hat{\Gamma}$ lies in $\sum_{k=1}^{r} H^{\otimes k}$ while $\hat{Y}_k - Y_k$ lies in J. Define P on $\mathfrak{g}$ by

$$P(\sum_{j=1}^{m} a_j X_j + \sum_{k=1}^{\ell} b_k Y_k) = \sum_{j=1}^{m} a_j X_j + \sum_{k=1}^{\ell} b_k \hat{Y}_k \tag{2.20}$$

where a_j and b_k are in either $\mathbb{R}$ or $\mathbb{C}$ if $\mathfrak{g}$ is real or complex respectively. At this point $P : \mathfrak{g} \to \oplus_{k=1}^{r} H^{\otimes k} \subset T(H)$ is a linear operator such that: a) $P(A) - A \in J$ for all $A \in \mathfrak{g}$, b) $P(A) = A$ for all $A \in H$, and c) P is bounded for any norm on $\mathfrak{g}$ because $\mathfrak{g}$ is finite dimensional.

By the universal property of the tensor algebra, there is a unique extension of P to an algebra homomorphism from $T(\mathfrak{g}) \to T(H)$, which we still denote by P, such that $P\left(1_{T(\mathfrak{g})}\right) = 1_{T(H)}$. Since, for $(A_1, A_2, \ldots, A_n) \in \mathfrak{g}^n$,

$$P(A_1 \otimes \cdots \otimes A_n) = PA_1 \otimes \cdots \otimes PA_n \in (A_1 + J) \otimes \cdots \otimes (A_n + J)$$

and J is an ideal, it follows that $P(A_1 \otimes \cdots \otimes A_n) - A_1 \otimes \cdots \otimes A_n \in J$. With this observation, the remaining stated properties of P are now easily verified. Q.E.D.

Lemma 2.12. *Assume that Hörmander's condition holds. If $\alpha \in J^0$ and $\|\alpha\|_t = 0$ for some $t > 0$ then $\alpha = 0$.*

Proof: If $\alpha \in J^0$ and $\|\alpha\|_t = 0$ for some $t > 0$ then, by Lemma 2.10 and the definition (2.10), $\alpha|_{T(H)} = 0$. By property 3. of Lemma 2.11, $\alpha = \alpha \circ P = \alpha|_{T(H)} \circ P = 0$. Q.E.D.

This proves a half of Theorem 2.7. The next lemma proves the other half.

Lemma 2.13. *If Hörmander's condition fails then there is an element $\alpha \in J^0$ such that $\alpha \neq 0$ but $q_k(\alpha_k) = 0$ for $k = 0, 1, 2, \ldots$, i.e. $\|\alpha\|_t = 0$ for all $t > 0$.*

Proof: Let r be as in Notation 2.9 so that $H_r = \mathrm{Lie}(H) \subsetneq \mathfrak{g}$. Then there exists an element $a \in \mathfrak{g}^*$ such that $a \neq 0$ while $a|_{H_r} \equiv 0$. Let $\tilde{a} \in T(\mathfrak{g})'$ be defined so that $\tilde{a}^j = 0$ if $j \neq 1$ and $\tilde{a}^1 = a$.

By the Poincaré-Birkhoff-Witt theorem $T := T(\mathfrak{g})$ is the direct sum, $T = \mathcal{S} \oplus J$ where $\mathcal{S}$ is the space of symmetric tensors over $\mathfrak{g}$. (See e.g. [53, Lemma 3.3.3].) Let $P_{\mathcal{S}} : T \to \mathcal{S}$ be the projection onto $\mathcal{S}$ along J and let $\alpha := \tilde{a} \circ P_{\mathcal{S}}$. Then $\alpha \in J^0$. Since $\alpha^1 = a \neq 0$, $\alpha \neq 0$. So to finish the proof it suffices to show $q_k(\alpha) = 0$ for all k. Because of Lemma 2.10, this last assertion will be a consequence of the following assertion;

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_k \rangle = 0 \text{ for all } \xi_1, \ldots, \xi_k \in H_r = \mathrm{Lie}(H) \text{ and } k = 1, 2, \ldots. \tag{2.21}$$

We will verify Eq. (2.21) by induction. The case $k = 1$ is trivial since $\alpha^1 = a = 0$ on H_r. Now suppose Eq. (2.21) holds up to some level $k \geq 1$ and let $\xi_i \in H_r$ for $i = 1, 2, \ldots, k+1$. Using the fact that $\alpha \in J^0$, we have for any $i = 1, \ldots, k$ that

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{k+1} \rangle - \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{i+1} \otimes \xi_i \otimes \cdots \otimes \xi_{k+1} \rangle$$

$$= \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{i-1} \otimes [\xi_i, \xi_{i+1}] \otimes \xi_{i+2} \otimes \cdots \otimes \xi_{k+1} \rangle = 0, \tag{2.22}$$

where the induction hypothesis along with the fact that $[\xi_i, \xi_{i+1}] \in H_r$ was used in the second equality. Since any permutation of $\{1, 2, \ldots, k+1\}$ may be written as a product of permutations consisting of interchange of nearest neighbors, it follows from repeated use of Eq. (2.22) that

$$\langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{k+1} \rangle = \langle \alpha, \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \rangle \tag{2.23}$$

for any permutation, σ, of $\{1, 2, \ldots, k+1\}$. Averaging Eq. (2.23) over all permutations of $\{1, 2, \ldots, k+1\}$ gives

$$\begin{aligned} \langle \alpha, \xi_1 \otimes \cdots \otimes \xi_{k+1} \rangle &= \left\langle \alpha, \frac{1}{(k+1)!} \sum_{\sigma} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \right\rangle \\ &= \left\langle \tilde{a} \circ P_{\mathcal{S}}, \left(\frac{1}{(k+1)!} \sum_{\sigma} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \right) \right\rangle \\ &= \left\langle \tilde{a}, \left(\frac{1}{(k+1)!} \sum_{\sigma} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(k+1)} \right) \right\rangle = 0. \end{aligned}$$

Q.E.D.

Corollary 2.14. *Hörmander's condition holds if and only if*

$$T(\mathfrak{g}) = T(H) + J. \quad \text{(not necessarily a direct sum)} \tag{2.24}$$

Proof: We have already seen in Lemma 2.11 that Eq. (2.24) holds under Hörmander's condition. Conversely, if Hörmander's condition fails then, by Lemma 2.13 there is a non-zero element $\alpha \in J^0$ (with $\alpha_0 = 0$) which annihilates $T(H)$. Thus α annihilates $T(H) + J$ which would be impossible if Eq. (2.24) were valid. Q.E.D.

3. The subelliptic heat kernel

Section 2 above gives an algebraic interpretation of Hörmander's condition in the tensor algebra, see Theorem 2.7. The rest of this paper is mostly analytic in nature and depends heavily on heat kernel estimates. This short section reviews the necessary material and gives pointers to the literature concerning subelliptic heat kernels.

Let G be a real connected Lie group equipped with its right Haar measure dx. Let q be a non-negative quadratic form on $\mathfrak{g}^*$ and let $(H, (\cdot, \cdot)_H)$ be the Hörmander space associated to q as defined in Section 2. Assume that $\mathrm{Lie}(H) = \mathfrak{g}$, i.e., assume that the Hörmander condition is satisfied. Let $\{X_i : i = 1, \ldots m\}$ be an orthonormal basis of $(H, (\cdot, \cdot)_H)$. Set

$$\Delta = \Delta_q = \sum_1^m \widetilde{X}_i^2$$

where, as before, $\tilde{X}_i$ denotes the left invariant vector field on G which extends the vector $X_i \in \mathfrak{g} = T_e(G)$. Then Δ depends only on q. See (1.2).

It is straight forward to prove that any sum of squares: $L = \sum_{j=1}^m \tilde{X}_j^2$ of left invariant vector fields, $\tilde{X}_j$, is essentially self-adjoint on $C_c^\infty(G)$ in $L^2(G, dy)$ when dy is right invariant Haar measure. Indeed it is sufficient to prove that $C_c^\infty(G)$ is a core for L^*. To this end one proves that L commutes with left convolution by any function $u \in C_c^\infty(G)$. This in turn implies that $C^\infty(G) \cap D(L^*)$ is a core for L^*. For any function f in this core the truncations $f_n(x) = h_n(x)f(x)$ are in $C_c^\infty(G)$ and converge to f in L^* graph norm if the sequence $h_n \in C_c^\infty(G)$ converges to one on G in a strong enough sense, as e.g. in [8, Lemma 3.6]. A reader pursuing this route will find it necessary to prove the integration by parts identity $\int_G \sum_{j=1}^m (\tilde{X}_j f)^2 dy = -(L^* f, f) < \infty$ for functions $f \in C^\infty(G) \cap D(L^*)$. This can be proved by inserting the sequence h_n in the left side before integrating by parts.

The exponential $e^{t\Delta/4}$ may therefore be defined by the spectral theorem. This semigroup commutes with left translations and the associated quadratic form $\int_G \sum_{j=1}^m (\tilde{X}_j f)^2 dy,\ f \in D(\sqrt{-\Delta})$, is a Dirichlet form, see [12]. It follows that $e^{t\Delta/4}$ admits a transition kernel $\rho_t(x, dy)$ with $\rho(t, A) \geq 0$ for all borel sets A and $\rho_t(x, G) \leq 1$ and such that

$$\left(e^{t\Delta/4} f\right)(x) = \int_G f(y)\rho_t(x, dy)$$

for all $f \in L^2(G, dx)$. We will see shortly that the measure $\rho_t(e, dy)$ admits a smooth positive density $x \mapsto \rho_t(x)$ with respect to the right-invariant Haar measure on G. We call the measure $\rho_t(e, dx) = \rho_t(x)dx$ **the heat kernel measure** on G associated to the sub-Laplacian Δ. It plays a central role in this paper since one of the main objects of interest to us is the scale

$$\mathcal{H}L^2(G, \rho_t(x)dx) = \mathcal{H} \bigcap L^2(G, \rho_t(x)dx),\ t > 0,$$

of Hilbert spaces of holomorphic functions that are in L^2 with respect to the heat kernel measure $\rho_t(e, dx) = \rho_t(x)dx$. In order to study these spaces, one needs information concerning the heat kernel ρ_t. In particular, the properties of ρ_t collected below play a key technical part in the proof of Theorem 4.1, outlined in Section 4.

The properties of the transition kernel $\rho_t(x, dy)$ are mostly derived through an understanding of the basic geometry associated to the operator Δ (i.e., the quadratic form q). More precisely, define the **intrinsic sub-Riemannian distance** d associated to Δ by setting

$$d(x, y) = \sup \left\{ f(y) - f(x) : f \in C_0^\infty(G),\ \sum_1^m |\widetilde{X}_i f|^2 \leq 1 \right\}. \tag{3.1}$$

It is well-known that

$$d(x, y) = d_H(x, y)$$

where d_H is the horizontal distance obtained by minimizing the horizontal length of absolutely continuous curves as spelled out precisely in the next definition. See, e.g., [34] and [51, 50]. In what follows, θ will denote the **Maurer-Cartan form** on G; i.e., θ is the $\mathfrak{g}$ − valued 1-form on G defined by $\theta(v) \equiv L_{g^{-1}*} v$ when $v \in T_g G$.

Definition 3.1. Let $(H, (\cdot, \cdot)_H)$ be the Hörmander space associated to q as defined in Section 2 and set $|u|_H^2 = (u, u)_H$, $u \in H$.

(i) A path $g : [a, b] \to G$ is said to be **horizontal** if g is absolutely continuous and $\theta(g'(s)) \in H$ for a.e. s.

(ii) The **horizontal length** or H **- length** of a horizontal path $g : [a, b] \to G$ is defined to be

$$\ell_H(g) = \int_a^b |\theta(g'(s))|_H ds. \tag{3.2}$$

If g is not horizontal we define $\ell_H(g) = \infty$.

(iii) The **horizontal distance** between x and y is defined by

$$d_H(x,y) = \inf\{\ell_H(g) : g(0) = x,\ g(1) = y\}. \tag{3.3}$$

Chow's theorem asserts (in a more general context, see, e.g., [36]) that Hörmander's condition implies that any two points in G can be joined by a horizontal path of finite H - length. Thus $d(x,y)$ is finite for all x,y. The Ball-Box Theorem, see for example [36, Theorem 2.10] or [20, Section 0.5.A], asserts that there exists $a>0$ such that for any left invariant Riemannian distance function, d_{Riem}, $C_1 d_{\mathrm{Riem}}(x,y) \le d(x,y) \le C_2 d_{\mathrm{Riem}}(x,y)^a$ for all x,y such that $d(x,y)\le 1$. Theorem 7.7 below implies the weaker result that $\{d(e,x)<r\}$ is an open neighborhood of e in the natural topology of G. By either of these results, it follows that d is continuous and that d induces the manifold topology of G.

Set

$$B(x,r) = \{y \in G : d(x,y) < r\}$$

and let $|B(x,r)|$ denote the right Haar measure of $B(x,r)$. One of the most basic results concerning the local analysis of the sub-Laplacian Δ is the following.

Theorem 3.2. *Referring to the above setting and notation, there are constants C_1, C_2 such that for any $x \in G$ and any $r \in (0,1)$ we have:*

(1) $|B(x,2r)| \le C_1 |B(x,r)|$.

(2) $\int_B |f(z) - f_B|^2 dz \le C_2 r^2 \int_B \sum_1^m |\widetilde{X}_i f(z)|^2 dz, \quad B = B(x,r),\ f \in Lip(\overline{B})$

where $f_B := |B|^{-1} \int_B f(z)\,dz$ is the mean of f over B.

Proof: For the doubling property (1) we refer to [34, 37, 54]. In fact, there are constants $c_3, C_3 \in (0,\infty)$ and an integer $\nu = \nu_q$ such that

$$\forall\, r \in (0,1),\quad c_3 r^\nu \le |B(e,r)| \le C_3 r^\nu. \tag{3.4}$$

The integer ν plays a role in the heat kernel estimates given below.

For the Poincaré inequality (2), see [33, 34, 43, 44]. Q.E.D.

By the general results of [43, 48], Theorem 3.2 yields a powerful local parabolic Harnack inequality and the heat kernel bounds stated in the following two theorems.

Theorem 3.3. (Parabolic Harnack inequality.) *There exists a constant $C>0$ such that, for any $T>0$, if $(0,T)\times G \ni (t,x) \mapsto u(t,x)$ is any non-negative solution of $\partial u/\partial t = (1/4)\Delta u$ then*

$$u(s,x) \le u(t,y)\cdot \exp\left(C\left[\frac{t}{s} + \frac{d(x,y)^2}{t-s}\right]\right) \tag{3.5}$$

for all $x \in G$ and $0<s<t<T$.

Proof: See [43, Theorem 3.1] and the arguments in [44, Sec. 5.4.3]. See also [48, Theorem 3.5] and [54, Proposition IX.1.1]. Q.E.D.

One of the many consequences of Theorem 3.2 is that the transition kernel $\rho_t(x,dy)$ of the semigroup $e^{t\Delta/4}$ admits a continuous density $\rho_t(x,dy) = h_t(x,y)dy$ with respect to the right-invariant Haar measure on G. The function $(t,x,y)\mapsto h_t(x,y)$ is called **the heat kernel** associated to the sub-Laplacian Δ on G. Moreover, h_t is a fundamental solution of the heat equation on G, i.e., is a solution of the initial value problem

$$\begin{cases} \partial h_t(x,\cdot)/\partial t = (1/4)\Delta h_t(x,\cdot) \\ h_t(x,y)dy \to \delta_x(dy) \text{ (weakly) as } t \to 0. \end{cases} \tag{3.6}$$

A further consequence of Theorem 3.2 is that uniqueness holds for the non-negative Cauchy problem associated with the heat equation (3.6). See [1].

By construction, the operator $e^{t\Delta/4}$ commutes with left translations whereas the Haar measure dy is right invariant. It follows that

$$h_t(x,y) = h_t(e, x^{-1}y)m(x)$$

where m denotes the modular function defined by $\int_G f(gx)dx = m(g)\int_G f(x)dx$ (the function m is a continuous multiplicative function). A reader may consult [41] for further details. In what follows we set

$$\rho_t(x) = h_t(e,x)$$

so that $\rho_t(x)$ is the density of the heat kernel measure

$$\rho_t(e,dx) = \rho_t(x)dx.$$

We will often refer to ρ_t, somewhat improperly, as the heat kernel.

Theorem 3.4. *Referring to the above setting and notation, the heat kernel $\rho_t(x)$ has the following properties:*

1. (Regularity.) $(t,x) \mapsto \rho_t(x)$ *is a smooth positive function on* $(0,\infty)\times G$.
2. (Conservation of heat.) $\int_G \rho_t(x)dx = 1$.
3. (Gaussian upper bound.) *For any $\kappa \in (0,1)$, there exists $C_\kappa \in (0,\infty)$ such that for all $x \in G$ and all $t > 0$,*

$$\rho_t(x) \le C_\kappa \left(1+\frac{1}{t}\right)^{\nu/2} e^{C_\kappa t} e^{-\kappa d(e,x)^2/t}. \tag{3.7}$$

4. (Gaussian lower bound.) *There are constants $C, c \in (0,\infty)$ such that, for all $x \in G$ and all $t > 0$,*

$$\rho_t(x) \ge c \left(1+\frac{1}{t}\right)^{\nu/2} e^{-Ct} e^{-Cd(e,x)^2/t}. \tag{3.8}$$

In the last two statements, ν is the integer introduced at (3.4).

Proof: (outline)

1. That the heat kernel is smooth is a basic consequence of Hörmander's hypoellipticity theorem. That it is positive easily follows, for instance, from (3.5) although it can be obtained more directly.
2. This property (conservativeness) is again a consequence of Theorem 3.2 by way of a local Harnack inequality, see [49]. It also follows by the remark made above concerning uniqueness of solutions to the positive Cauchy problem, see [1]. Alternatively, one can use Grigor'yan's volume criterion (see [19] and [49]). Indeed, on any group, $r \mapsto |B(x,r)|$ grows at most exponentially fast.
3. This heat kernel upper bound is in [54, Theorem IX.1.2.] It also follows from the local parabolic Harnack inequality and the volume estimate (3.4). see [43].
4. This heat kernel lower bound is stated in [54, Theorem IX.1.2.] for $0 < t < 1$. The global Harnack type inequality (3.5) easily gives the desired result for $t \ge 1$.

Q.E.D.

Remark 3.5. Note that as κ tends to 1, the Gaussian factor $e^{-\kappa d(e,x)^2/t}$ in item 3. of Theorem 3.4 tends to its optimal value $e^{-d(e,x)^2/t}$ (recall that our heat semigroup is $e^{-t\Delta/4}$). The fact that such an approximately optimal heat kernel upper bound holds is crucial for the analysis developed in this paper.

4. The Taylor map

Let G be a complex Lie group with Lie algebra $\mathfrak{g}$. Suppose we are given a non-negative Hermitian form q on the complex vector space $\mathfrak{g}^*$. As in Notation 2.1 and Lemma 2.2, let K be the kernel of q, $H = K^0$ be the backward annihilator of K in $\mathfrak{g}$, and let $\{X_j\}_{j=1}^m$ be an orthonormal basis for the complex inner product space H. Then the vectors $\{X_j, iX_j : j = 1, \ldots, m\}$, where $i = \sqrt{-1}$, form an orthonormal basis of H as a real vector space with inner product $\mathrm{Re}\,(\cdot,\cdot)_H$. The subspace $H \subset \mathfrak{g}$ generates the full Lie algebra $\mathfrak{g}$ over the complex numbers if and only if it generates $\mathfrak{g}$ as a real Lie algebra. Define

$$\Delta = \sum_{j=1}^{m} (\tilde{X}_j^2 + \widetilde{(iX_j)}^2). \tag{4.1}$$

where as before, for $A \in \mathfrak{g}$, $\tilde{A}$ is the left invariant vector field on G such that $\tilde{A}(e) = A$. It is easy to see that the second order differential operator, Δ, is independent of the choice of orthonormal basis $X_1, \ldots, X_m$. By Hörmander's theorem the operator Δ is subelliptic if and only if H generates $\mathfrak{g}$. Throughout this section we will assume that H does generate $\mathfrak{g}$. Let ρ_t in $C^\infty(G)$ be the heat kernel introduced in (3.6)

Recall for each $\beta \in T(\mathfrak{g})$, $\tilde{\beta}$ is the corresponding left invariant partial differential operator on G as in Notation 2.4. If f is a holomorphic function defined in a neighborhood of the identity element of G then, as in Eq. (1.3), f defines a linear functional $\hat{f}(e)$ on $T(\mathfrak{g})$. Notice that $\hat{f}(e)$ is complex linear and that $\hat{f}(e) \in J^0$ where J^0 is the annihilator of $J \subset T(\mathfrak{g})$, defined in Notation 2.5. The complex linearity is a consequence of the fact that f is holomorphic. To see that $\hat{f}(e) \in J^0$, observe that $\tilde{\beta}_1 \tilde{h} \tilde{\beta}_2$ annihilates all functions if β_1 and β_2 are in $T(\mathfrak{g})$ and $h = A \otimes B - B \otimes A - [A,B]$ is a generator of J. Since J is the linear span of such elements, $\langle \hat{f}(e), \beta \rangle = \left(\tilde{\beta} f\right)(e) = 0$ for all $\beta \in J$.

We denote by $\mathcal{H}$ the space of holomorphic functions on G. Our main theorem in this section is the following.

Theorem 4.1. *Let G be a connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds, (cf. Definition. 2.6). Let ρ_t denote the heat kernel associated to Equation (3.6). Then the* ***Taylor map,***

$$f \to \hat{f}(e), \tag{4.2}$$

is an isometry from $\mathcal{H}L^2(G, \rho_t(x)dx)$ into J_t^0.

Proof: The proof follows the pattern of proof given in [8] for the case of nondegenerate q. We are going therefore just to sketch the proof, emphasizing the issues that present a possible difference. The tensor $D^n f(x)$, of n^{th} - order derivatives of f at x, is defined by

$$\langle (D^n f)(x), \xi_1 \otimes \cdots \otimes \xi_n \rangle = \left(\tilde{\xi}_1 \ldots \tilde{\xi}_n f\right)(x). \tag{4.3}$$

Let us first observe that the identity

$$(\Delta/4)|D^k f(x)|^2_{q_k} = |D^{k+1} f(x)|^2_{q_{k+1}} \text{ when } \quad f \in \mathcal{H}(G) \tag{4.4}$$

holds in our degenerate case when the norms that appear are those induced on k tensors by q. The proof is identical to that for the nondegenerate case. (cf. [8], Remark 3.7.) Suppose now that $f \in \mathcal{H}L^2(G, \rho_t(x)dx)$ and define

$$F(s) = \int_G |f(x)|^2 \rho_s(x) dx \quad 0 \leq s \leq t. \tag{4.5}$$

We are going to proceed, at first, entirely informally and then discuss what needs to be done to justify the following computations. By definition ρ_s satisfies $\partial_s \rho_s(x) = (\Delta/4)\rho_s(x)$. Differentiate Equation (4.5) and use (4.4) to find

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_G |f(x)|^2 \rho_s(x) dx \tag{4.6}$$

$$= \int_G |f(x)|^2 \frac{\partial}{\partial s}\rho_s(x) dx \tag{4.7}$$

$$= \int_G |f(x)|^2 (\Delta/4)\rho_s(x) dx \tag{4.8}$$

$$= \int_G \{(\Delta/4)|f(x)|^2\}\rho_s(x) dx \tag{4.9}$$

$$= \int_G \{|Df(x)|^2_q\}\rho_s(x) dx. \tag{4.10}$$

A similar derivation shows by induction that

$$F^{(k)}(s) = \int_G |D^k f(x)|^2_{q_k} \rho_s(x) dx, \quad k = 0, 1, 2, \ldots. \tag{4.11}$$

Were it possible to use these derivatives to expand F as a power series around $s = 0$ we would find from (4.11) and the expected relation, $F^{(k)}(0) = \lim_{s \downarrow 0} F^{(k)}(s) = |D^k f(e)|^2$, that

$$F(s) = \sum_{k=0}^{\infty} (s^k/k!) F^{(k)}(0) = \sum_{k=0}^{\infty} (s^k/k!) |(D^k f)(e)|^2_{q_k} = |\hat{f}\,(e)\,|^2_{J^0_s}. \tag{4.12}$$

Therefore

$$\|f\|^2_{L^2(G,\rho_s(x)dx)} = \|\hat{f}\,(e)\,\|^2_{J^0_s}, \tag{4.13}$$

which, for $s = t$, is the isometry we wish to prove.

Among the previous steps the following clearly need justification:

a) the interchange of d/ds with $\int_G$ in (4.7),
b) the integration by parts in (4.9) (and in the similar derivation of (4.11)), and
c) the validity of the expansion in (4.12).

The only hypothesis available to us for these justifications is the assumption that $f \in \mathcal{H}L^2(G, \rho_t(x)dx)$. We do not have, for a general complex group, a method of approximating such rapidly growing holomorphic functions by more slowly growing holomorphic functions. Justification of the three items in a), b), c) must therefore be done directly for the rapidly growing function f. The justification of these steps, developed in [8], consists in establishing expansion coefficient bounds, $\|\hat{f}\,(e)\,\|_s \leq$

$\|f\|_{L^2(\rho_s)}$, (cf. Proposition 3.3 in [8]), and pointwise bounds, (cf. Corollary 3.10 in [8]),

$$|f(x)|^2 \leq \|\hat{f}(e)\|_s^2 e^{d^2(e,x)/s} \text{ for all } x \in G, \tag{4.14}$$

as well as similar pointwise bounds on the derivatives of f, $|D^k f(x)|_{q_k}$, where in [8], $d(e,x)$ refers to the Riemannian distance associated to q in the non-degenerate case. These estimates go over to the subelliptic case with no changes except that $d(e,x)$ should now be interpreted as the sub-Riemannian distance associated to q defined in either of Eqs. (3.1) or (3.3). (The estimate in Eq. (4.14) and the analogous estimates for $|D^k f(x)|_{q_k}$ will be re-derived in Corollary 6.15 below.) One combines growth rates, such as (4.14), with known decay rates for subelliptic heat kernels (see Theorem 3.4) to justify the steps listed in a), b) and c). To prove (4.12), it is shown in [8, Section 4], using these rather detailed bounds on the derivatives $D^k f(x)$ and the consequent bounds on the derivatives $F^{(k)}(s)$ for $0 < s < t$, that F has a complex analytic extension to a complex neighborhood of $[0,t)$. The result of this procedure is to establish Eq. (4.13) for $s < t$. For $s = t$ one then uses a monotonicity argument on both sides of (4.13) as $s \uparrow t$. (cf. [8], [Section 5 or Appendix 8]. One should replace the Li-Yau Harnack inequality used in [8] by the Harnack inequality stated in Eq. (3.5).) Q.E.D.

The following proposition complements Theorem 4.1 and makes use of the estimate in Eq. (4.14). In words, it says that the inverse image of J_t^0 by the Taylor map $f \mapsto \hat{f}(e)$ from $\mathcal{H}(G)$ into J^0 is contained in $\mathcal{H}L^2(G, \rho_t(x)dx)$.

Proposition 4.2. *Let $f \in \mathcal{H}(G)$ and assume that $\hat{f}(e) \in J_t^0$ (see Eq. (1.3)) for some $t > 0$. Then $f \in \mathcal{H}L^2(G, \rho_t(x)dx)$.*

Proof: As noted above, (4.14) and known heat kernel estimates (cf. Eq. (3.7) show that if $\hat{f}(e) \in J_t^0$ then $f \in \mathcal{H}L^2(G, \rho_s)$ for $s < t$. By Theorem 4.1 we have

$$\|f\|_{L^2(G,\rho_s(x)dx)} = \|\hat{f}(e)\|_s \leq \|\hat{f}(e)\|_t.$$

The desired conclusion follows because

$$\lim_{s \uparrow t} \|f\|_{L^2(G,\rho_s(x)dx)} = \|f\|_{L^2(G,\rho_t(x)dx)}.$$

See [8, Sect. 5 or Appendix 8]. The Li-Yau Harnack inequality used in [8] should be replaced by (3.5). Q.E.D.

5. Taylor expansion over complex nilpotent groups

In this section we are going to prove, for a connected, simply connected complex nilpotent Lie group, the surjectivity of the isometry described in Theorem 4.1. Of course this yields unitarity of the Taylor map for such groups. In Section 7 below, we will give a different (and more complicated) argument which works for general simply connected complex Lie groups.

Theorem 5.1. *Let G be a connected, simply connected, nilpotent complex Lie group. Suppose that q is a nonnegative Hermitian form on the dual space $\mathfrak{g}^*$ of the complex Lie algebra of G. Assume that q satisfies Hörmander's condition (cf. Definition 2.6.) Let $t > 0$. If f is in $\mathcal{H} \cap L^2(G, \rho_t(x)dx)$ then $\hat{f}(e)$ is in J_t^0 and the map*

$$\left(f \to \hat{f}(e)\right) : \mathcal{H} \cap L^2(G, \rho_t(x)dx) \to J_t^0 \tag{5.1}$$

is unitary.

The proof of Theorem 5.1 will follow the proof of Lemma 5.6, which asserts that Theorem 5.1 holds under the additional assumption that $\mathfrak{g}$ is a "graded" Lie algebra and q is nicely related to the gradation.

Notation 5.2. A Lie algebra $\mathfrak{g}$ is *graded* if it is representable as a direct sum:

$$\mathfrak{g} = \oplus_{j=1}^{\infty} V_j \tag{5.2}$$

where all but finitely many of the the subspaces $\{V_j\}_{j=1}^{\infty}$ equal $\{0\}$ and

$$[V_i, V_j] \subset V_{i+j}, \qquad i, j = 1, 2, \ldots. \tag{5.3}$$

A graded algebra is necessarily nilpotent.

Notation 5.3. A Lie algebra $\mathfrak{g}$ is *stratified* if it is graded and V_1 generates $\mathfrak{g}$. In this case, we have

$$[V_1, V_k] = V_{k+1}, \qquad k = 1, \ldots, \infty, \tag{5.4}$$

and there exists an integer r such that $V_r \neq \{0\}$, $V_{r+1} = \{0\}$, and

$$\mathfrak{g} = \oplus_{j=1}^{r} V_j. \tag{5.5}$$

If $\mathfrak{g}$ is stratified and r is as in notation 5.3 then $\mathfrak{g}$ is r - step nilpotent.

An important example illustrating these definitions comes from the complex Heisenberg algebra $\mathfrak{h}_3^{\mathbb{C}}$. This is a 3 - dimensional complex vector space with basis X, Y, Z equipped with a Lie bracket verifying $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$. In this case, V_1 is the vector subspace spanned by X, Y and V_2 is spanned by Z. Obviously, $\mathfrak{h}_3^{\mathbb{C}}$ is graded and, in fact, stratified.

Notation 5.4 (Dilations)**.** Let $\mathfrak{g}$ be a graded Lie algebra with $\mathfrak{g} = \oplus_{j=1}^{\infty} V_j$ as in (5.2). For $\lambda \in \mathbb{C}$ and $v = \sum_1^{\infty} v_i \in \mathfrak{g}$, $v_j \in V_j$, $j = 1, \ldots, \infty$, define

$$\delta_\lambda(v) = \sum_{k=1}^{\infty} \lambda^k v_k \tag{5.6}$$

It is straightforward to verify that

$$\delta_{\lambda\mu} = \delta_\lambda \delta_\mu \qquad \lambda, \mu \in \mathbb{C}. \tag{5.7}$$

and that, for $\lambda \neq 0$, δ_λ is an automorphism of the Lie algebra $\mathfrak{g}$. See [10, Chapter 1] for details.

Lemma 5.5. *Let $\mathfrak{g}$ be a complex graded Lie algebra. Let q be a nonnegative Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition (Definition 2.6). Assume that q is invariant under the action of the transposed dilations $(\delta_{e^{i\theta}})'$. Then the finite rank tensors in J_t^0 are dense in J_t^0 for each $t > 0$.*

Proof: Let $\Gamma_\theta : T(\mathfrak{g}) \to T(\mathfrak{g})$ be the automorphism of the tensor algebra over $\mathfrak{g}$ induced by the automorphism $\delta_{e^{i\theta}}$ of $\mathfrak{g}$, i.e.

$$\Gamma_\theta = \overbrace{\delta_{e^{i\theta}} \otimes \cdots \otimes \delta_{e^{i\theta}}}^{k\text{ - times}} \text{ on } \mathfrak{g}^{\otimes k}.$$

For any ξ and η in $\mathfrak{g}$ we have

$$\begin{aligned}\Gamma_\theta(\xi \wedge \eta - [\xi, \eta]) &= (\delta_{e^{i\theta}}\xi) \wedge (\delta_{e^{i\theta}}\eta) - \delta_{e^{i\theta}}[\xi, \eta] \\ &= (\delta_{e^{i\theta}}\xi) \wedge (\delta_{e^{i\theta}}\eta) - [\delta_{e^{i\theta}}\xi, \delta_{e^{i\theta}}\eta].\end{aligned}$$

Thus Γ_θ takes J into, and in fact onto, J. The transpose, Γ'_θ, on $T(\mathfrak{g})'$ therefore takes J^0 onto itself. Since

$$\Gamma'_\theta = \overbrace{(\delta_{e^{i\theta}})' \otimes \cdots \otimes (\delta_{e^{i\theta}})'}^{k\text{ - times}} \text{ on } (\mathfrak{g}^*)^{\otimes k},$$

it follows that

$$q_k(\Gamma'_\theta u) = q_k(u) \text{ for all } u \in (\mathfrak{g}^*)^{\otimes k}. \tag{5.8}$$

Since $\delta_{e^{i\theta}}\xi$ is continuous in θ for any norm on $\mathfrak{g}$, $\Gamma_\theta\beta$ is continuous in θ for all k tensors β and for any product norm on $\mathfrak{g}^{\otimes k}$. Similarly $\Gamma'_\theta\alpha$ is continuous in θ for any element α in $(\mathfrak{g}^*)^{\otimes k}$.

Let

$$F_n(\theta) = \frac{1}{2\pi n}\sum_{k=0}^{n-1}\sum_{\ell=-k}^{k} e^{i\ell\theta} = \frac{1}{2\pi n}\,\frac{\sin^2(k\theta/2)}{\sin^2(\theta/2)}$$

denote Fejer's kernel [52, p. 413]. Then

$$\int_{-\pi}^{\pi} F_n(\theta)\varphi(\theta)d\theta = 1$$

in case $\varphi \equiv 1$ and if φ is continuous on $[-\pi,\pi]$ then the integral converges to $\varphi(0)$ as $n \to \infty$.

If $\beta = \xi_1 \otimes \cdots \otimes \xi_k$ with $\xi_p \in V_{j_p}$ for $p = 1,\ldots,k$ then

$$\Gamma_\theta\beta = (e^{i\sum_{p=1}^{k} j_p\theta})\beta.$$

So

$$\int_{-\pi}^{\pi} F_n(\theta)\Gamma_\theta\beta d\theta = 0 \quad \text{if} \quad \sum_{j=1}^{k} j_p > n.$$

Since all $j_p \geq 1$ we have

$$\int_{-\pi}^{\pi} F_n(\theta)\Gamma_\theta d\theta = 0 \quad \text{on} \quad \mathfrak{g}^{\otimes k} \quad \text{if} \quad k > n.$$

Consequently

$$\int_{-\pi}^{\pi} F_n(\theta)\Gamma'_\theta\alpha d\theta = 0 \quad \text{if} \quad \alpha \in (\mathfrak{g}^*)^{\otimes k} \quad \text{and} \quad k > n.$$

Now an elementary argument using (5.8) and the strong continuity of $\theta \mapsto \Gamma'_\theta$ on each $(\mathfrak{g}^*)^{\otimes k}$ shows that $\theta \mapsto \Gamma'_\theta$ is strongly continuous on J^0_t in the norm (2.10). Hence if $\alpha \in J^0_t$ then

$$\gamma_n := \int_{-\pi}^{\pi} F_n(\theta)\Gamma'_\theta\alpha d\theta$$

is also in J^0_t and is zero in all ranks $> n$. Moreover

$$\begin{aligned}\|\gamma_n - \alpha\|_t &= \left\| \int_{-\pi}^{\pi} F_n(\theta)(\Gamma'_\theta\alpha - \alpha)d\theta \right\|_t \\ &\leq \int_{-\pi}^{\pi} F_n(\theta)\|\Gamma'_\theta\alpha - \alpha\|_t d\theta \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Q.E.D.

Lemma 5.6 (Theorem 5.1–Graded case)**.** *In addition to the hypotheses of Theorem 5.1, assume that $\mathfrak{g}$ is a graded algebra and that for any complex number λ with $|\lambda| = 1$, q is invariant under the transposes, $(\delta_\lambda)'$, of the dilations introduced in Notation 5.4. Then the conclusions of Theorem 5.1 hold.*

Proof: By Theorem 4.1 the map $f \to \hat{f}(e)$ is isometric from $\mathcal{H} \cap L^2(G, \rho_t)$ into J_t^0. To prove the surjectivity it suffices therefore to prove that the image is dense. To this end it suffices, by Lemma 5.5, to show that if α is a finite rank tensor in J_t^0 then there exists a function $u \in \mathcal{H}L^2(G, \rho_t)$ such that $\alpha = \hat{u}$. Since, in our case, the exponential map is a holomorphic diffeomorphism onto G we may identify G with $\mathbb{C}^N$ and define u as the holomorphic function on G given by $u(\exp \xi) = \sum_{n=0}^{\infty} (n!)^{-1} \langle \alpha_n, \xi^{\otimes n} \rangle$. This is a finite sum because α is of finite rank.

One now easily concludes (see [7, Proposition 6.3]) that $\hat{u} = \alpha$. Indeed, for any $\xi \in \mathfrak{g}$,

$$\begin{aligned} \langle \hat{u}(e), \xi^{\otimes k} \rangle &= \left(\frac{d}{dt}\right)^k |_{t=0} u\left(e^{t\xi}\right) = \left(\frac{d}{dt}\right)^k |_{t=0} \sum_0^\infty \frac{1}{n!} \left\langle \alpha_n, (t\xi)^{\otimes n} \right\rangle \\ &= \left\langle \alpha_k, \xi^{\otimes k} \right\rangle . \end{aligned} \tag{5.9}$$

By polarization, the linear span of $\left\{\xi^{\otimes k} : \xi \in \mathfrak{g}\right\}$ is the set of all symmetric $\mathbb{R}$ - tensors, $\mathcal{S}$. It follows that $\hat{u}(e) = \alpha$ on $\mathcal{S}$. But, by the Poincaré-Birkhoff-Witt theorem, [53], Lemma 3.3.3, we know that $T(\mathfrak{g}) = \mathcal{S} \oplus J$, and, since $\hat{u} - \alpha$ annihilates J, we conclude that $\hat{u}(e) = \alpha$ on $T(\mathfrak{g})$.

Since u is a holomorphic function such that $\hat{u}(e) = \alpha \in J_t^0$, it follows from Proposition 4.2 that $u \in \mathcal{H}L^2(G, \rho_t)$.

Alternatively, one may conclude that $u \in L^2(G, \rho_t)$ (or in fact that $u \in L^p(G, \rho_t)$ for all $0 < p < \infty$) on the grounds that any polynomial is in $L^p(G, \rho_t)$. The latter assertion is proved using the heat kernel upper bound in Theorem 3.4 and the fact that for any polynomial P on G (i.e., $P(x) = \widetilde{P} \circ \exp^{-1}(x)$ where $\widetilde{P}$ is a polynomial on $\mathfrak{g}$) there exist $C, \alpha \geq 0$ such that $|P(x)| \leq C(1 + d(e,x))^\alpha$ with $d(x,y)$ is the sub-Riemannian distance associated with q and defined at (3.1). See, e.g., [54, Sect. IV.5]. Q.E.D.

Remark 5.7. Let $\mathfrak{g}$ be a graded algebra with decomposition $\mathfrak{g} = \oplus_{i=1}^\infty V_i$ and equipped with the dilations introduced in Notation 5.4. Let q be a Hermitian form on $\mathfrak{g}^*$ satisfying Hörmander's condition. Let H be the Hörmander subspace of $\mathfrak{g}$ equipped with its scalar product $(\cdot,\cdot)_H$ induced by q. See (2.3) and Lemma 2.2. It is not hard to check that a necessary and sufficient condition for q to be invariant under the dilation $(\delta_\lambda)'$, $|\lambda| = 1$, is that H be the orthogonal direct sum of the non-trivial $H \cap V_i$, $i = 1, \ldots,$ under $(\cdot,\cdot)_H$. This is equivalent to saying that there exists an orthonormal basis $(X_j)_1^m$ of $(H, (\cdot,\cdot)_H)$ such that each X_j belongs to V_i for some $i = i(j)$. In particular, if $\mathfrak{g}$ is stratified and $H = V_1$, the Hermitian form q is invariant under the dilations $(\delta_\lambda)'$ with $|\lambda| = 1$. But this is far from the only example. For instance, in the Heisenberg algebra $\mathfrak{h}_3^{\mathbb{C}}$ described above, consider the following two cases:

(a) The model subelliptic case where $H = V_1 = \text{span}(X, Y)$ with X, Y being an orthonormal basis (this is equivalent to a description of q);

(b) The non-degenerate case where $H = \mathfrak{h}_3^{\mathbb{C}} = \text{span}(X, Y, Z)$ with X, Y, Z being an orthonormal basis.

Note that the dilatations δ_λ on $\mathfrak{h}_3^{\mathbb{C}}$ are given by

$$\delta_\lambda(X) = \lambda X, \;\; \delta_\lambda(Y) = \lambda Y, \;\; \delta_\lambda(Z) = \lambda^2 Z.$$

Although only structure (a) above is "homogeneous" with respect to all dilations δ_λ, $\lambda \in \mathbb{C}$, the structures (a) and (b) are both invariant under these dilations when $|\lambda| = 1$. Thus Lemma 5.5 applies to both and shows that the finite rank tensors are dense in J_t^0, $t > 0$, for the Hermitian forms in both cases (a) and (b).

Remark 5.8. Lemma 5.5 asserts that the finite rank tensors in J^0 are dense in J_t^0 for each $t > 0$ if $\mathfrak{g}$ is graded nilpotent and the Hermitian form q is automorphism invariant, as in the hypotheses of Lemma 5.5. We don't know whether such density holds if $\mathfrak{g}$ is nilpotent but not graded or even if $\mathfrak{g}$ is graded but q is not invariant. On the other hand, in view of [24, Theorem 4.15], we know that when q is nondegenerate the finite rank tensors cannot be dense in J_t^0 for any $t > 0$ unless $\mathfrak{g}$ is nilpotent.

Proof of Theorem 5.1: Let G be a connected, simply connected, nilpotent complex Lie group with Lie algebra $\mathfrak{g}$ whose dual is equipped with a quadratic form q satisfying Hörmander's condition. Let $X_1, \dots, X_m$ be an orthonormal basis of $(\ker q)^0$. Choose $r \in \mathbb{N}$ sufficiently large so that $\mathfrak{g}$ is nilpotent of step r. Let $\mathfrak{n}(m,r)$ denote the step r free nilpotent complex Lie algebra on m generators $\eta_1, \dots, \eta_m$. (see [42], [13, p. 37] and also [3, Ch. 2 §2]). By definition of $\mathfrak{n}(m,r)$, there exists a Lie algebra homomorphism

$$\pi : \mathfrak{n}(m,r) \to \mathfrak{g}$$

such that $\pi(\eta_i) = X_i$. (This property holds for any step r nilpotent Lie algebra $\mathfrak{g}$ generated by m elements $X_1, \dots, X_m$.) The algebra $\mathfrak{n}(m,r)$ is a stratified Lie algebra with

$$\mathfrak{n}(m,r) = V_1 + \cdots + V_r$$

where V_1 is the linear span of $\eta_1, \dots, \eta_m$ and $V_i = [V_1, V_{i-1}]$, $i = 2, \dots, r$. For a description of a basis of V_i, see [3, 13]. The natural dilation structure on $\mathfrak{n}(m,r)$ is defined by setting $\delta_\lambda(\xi) = \lambda^i \xi$ for $\xi \in V_i$, $i = 1, 2, \dots, r$, $\lambda \in \mathbb{C}$.

Further let $N(m,r)$ be the simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}(m,r)$ and let φ be the unique complex surjective Lie group homomorphism from $N(m,r)$ to G such that $\varphi_{*e} = \pi$. It is known that

$$G_0 := \ker \varphi \subset N(m,r)$$

is connected if and only if G is simply connected, see [14, Theorem 4.8]. Since we have assumed that G is simply connected, G_0 is connected in our case. Moreover, G_0 is a complex Lie group because $\mathrm{Lie}\,(G_0) = \ker \pi$ is a complex Lie algebra.

On the dual $\mathfrak{n}(m,r)^*$ of $\mathfrak{n}(m,r)$, set

$$\widetilde{q}(a) = \sum_1^m |\langle a, \eta_i \rangle|^2. \tag{5.10}$$

By construction, we have

$$(\ker \widetilde{q})^0 = \mathrm{span}(\eta_1, \dots, \eta_m) = V_1.$$

Hence $\widetilde{q}$ satisfies the Hörmander condition and is invariant with respect to the dilations $\delta_{e^{i\theta}}$. Moreover, by (2.4) and (5.10), $\widetilde{q} = \pi^* q = q \circ \pi$.

Suppose now that $t > 0$ and that $\alpha \in J_t^0(\mathfrak{g})$. The surjective homomorphism π extends to a surjective homomorphism from $T(\mathfrak{n}(m,r))$ onto $T(\mathfrak{g})$. We denote the extension again by π. Then π^* maps from $T(\mathfrak{g})'$ into $T(\mathfrak{n}(m,r))'$. Moreover

$\pi(J(\mathfrak{n}(m,r)) = J(\mathfrak{g})$ and so $\pi^*(J^0(\mathfrak{g})) \subset J^0(\mathfrak{n}(m,r))$. Let $\alpha' = \pi^*\alpha = \alpha \circ \pi$. It follows from (5.10) and (2.4) that $\alpha' \in J_t^0(\mathfrak{n}(m,r))$ and

$$\|\alpha'\|_t = \|\alpha\|_t < \infty. \tag{5.11}$$

An application of Lemma 5.6 allows us to conclude that there exists a holomorphic function, v, on $N(m,r)$ such that $\hat{v} = \alpha'$. We further know that v is square integrable relative to the time t heat kernel measure associated to $\tilde{q}$, but we will not need this fact here.

We assert that the function v is right (and therefore left) invariant under the normal subgroup G_0 and consequently factors through a holomorphic function (f) on G, i.e. $v = f \circ \varphi$. To prove this assertion, it suffices to show $\tilde{\eta}v \equiv 0$ on $N(m,r)$ for any vector $\eta \in T_e(G_0)$. Since $\tilde{\eta}v$ is holomorphic on $N(m,r)$ it is enough to show that $(\tilde{\beta}\tilde{\eta}v)(e) = 0$ for all $\beta \in \mathfrak{n}(m,r)$. But because $\pi\eta = 0$,

$$(\tilde{\beta}\tilde{\eta}v)(\tilde{e}) = \langle \alpha', \beta \otimes \eta \rangle = \langle \alpha, \pi(\beta \otimes \eta) \rangle = \langle \alpha, (\pi\beta) \otimes (\pi\eta) \rangle = 0$$

and the assertion is proved.

To each $A_i \in \mathfrak{g}$, we may use the surjectivity of π to find a $B_i \in \mathfrak{n}(m,r)$ such that $\pi B_i = A_i$. It is well known and easy to show that $\tilde{B}_i (F \circ \varphi) = \left(\tilde{A}_i F\right) \circ \varphi$ for any smooth function (F) on G.

By repeated use of this identity, we find

$$\begin{aligned} \langle \hat{v}(e), B_1 \otimes \cdots \otimes B_n \rangle &= \left(\tilde{B}_1 \ldots \tilde{B}_n v\right)(e) = \left(\tilde{B}_1 \ldots \tilde{B}_n (f \circ \varphi)\right)(e) \\ &= \left(\tilde{A}_1 \ldots \tilde{A}_n f\right)(\varphi(e)) = \left\langle \hat{f}(e), A_1 \otimes \cdots \otimes A_n \right\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \hat{v}(e), B_1 \otimes \cdots \otimes B_n \rangle &= \langle \alpha', B_1 \otimes \cdots \otimes B_n \rangle = \langle \pi^*\alpha, B_1 \otimes \cdots \otimes B_n \rangle \\ &= \langle \alpha, \pi B_1 \otimes \cdots \otimes \pi B_n \rangle = \langle \alpha, A_1 \otimes \cdots \otimes A_n \rangle. \end{aligned}$$

Comparing the previous two equations allows us to conclude that f is a holomorphic function on G such that $\hat{f}(e) = \alpha$. In light of Theorem 4.1 and Proposition 4.2, this fact is sufficient to complete the proof of Theorem 5.1. Q.E.D.

6. Power series along a curve in a Lie group

If z is a point in $\mathbb{C}^n$ and $z^{\otimes k}$ is its k^{th} tensor power in $(\mathbb{C}^n)^{\otimes k}$ then the conventional power series representation of a holomorphic function f in a neighborhood of 0 may be written $f(z) = \langle \alpha, \Phi(z) \rangle$, where $\Phi(z) := \sum_{k=0}^{\infty} (k!)^{-1} z^{\otimes k}$ is an element of the (suitably completed) tensor algebra over $\mathbb{C}^n$ and α is in the dual space. In order to recover a holomorphic function f on a complex Lie group G from a knowledge of its Taylor coefficient $\alpha = \hat{f}(e)$, cf. (1.3), we will need to represent f locally and globally on G by an analogous kind of power series. Of course we do not have a global coordinate system as on $\mathbb{C}^n$. Consider a piecewise smooth curve $g : [0,1] \to G$ beginning at the identity, $e \in G$, and ending at a point $z \in G$. We are going to replace the tensor valued function $\Phi(z)$ above by a path dependent tensor valued function $\Psi(g)$ so that f is again given by $f(z) = \langle \alpha, \Psi(g) \rangle$, both locally and globally. When $G = \mathbb{C}^n$ and g is the straight-line path joining 0 to z our function $\Psi(g)$ reduces to $\Phi(z)$ and, in addition, $\langle \alpha, \Psi(g) \rangle = \langle \alpha, \Phi(z) \rangle$ for all paths, g, joining 0 to z.

In order to carry out the replacement we will first develop, in Section 6.1, the needed estimates in the space where $\Psi(g)$ will lie. In Section 6.2 we will describe the path dependent power series expansion associated to a local holomorphic function. And in Section 6.3 we will show that the seemingly path dependent series associated to a presumed Taylor coefficient α of limited size, actually depends only on the homotopy class (with fixed endpoints) of the path. For the elliptic case this machinery has been carried out in [7] and [8].

6.1. **The Frechét Tensor Algebra.**

Definition 6.1 (Frechét Tensor Algebra)**.** Let V be a real or complex finite dimensional vector space with an inner product $(\ ,\ )$ and associated norm $|\cdot|$.

Let

$$T_\infty(V) = \prod_{n=0}^{\infty} V^{\otimes n}$$

and for $A = \sum_{n=0}^{\infty} A_n \in T_\infty(V)$ and $B = \sum_{n=0}^{\infty} B_n \in T_\infty(V)$ with $A_n, B_n \in V^{\otimes n}$ for all n, define

$$AB := \sum_{n=0}^{\infty} (\sum_{k=0}^{n} A_k \otimes B_{n-k}) \in T_\infty(V),$$

$$\|A\|_t^2 := \sum_{n=0}^{\infty} \frac{n!}{t^n} |A_n|^2,$$

$$T_t(V) := \{A \in T_\infty(V) : \|A\|_t < \infty\},$$

and

$$T_+(V) = \lim_{t \downarrow 0} T_t(V) := \bigcap_{t>0} T_t(V).$$

Observe that

$$T(V) \subset T_+(V) \subset T_s(V) \subset T_t(V) \subset T_\infty(V) \text{ for } 0 < s < t < \infty.$$

The containment $T(V) \subset T_+(V)$ asserts that any finite rank tensor is in $T_t(V)$ for all $t > 0$, which is clear. $T_+(V)$ also contains some tensors of infinite rank. For example if $A \in T_\infty(V)$ then $A \in T_+(V)$ if $|A_n| = O\left((n!)^{-\delta}\right)$ for some $\delta > 1/2$. See Proposition 6.10 for more examples of elements of $T_+(V)$.

The following Lemma is a technical improvement on [7, Lemma 2.18].

Lemma 6.2 ($T_+(V)$ is an algebra)**.** *If $s, t > 0$, $A \in T_t(V)$ and $B \in T_s(V)$, then $AB \in T_{s+t}(V)$ and*

$$\|AB\|_{s+t} \le \|A\|_t \|B\|_s. \tag{6.1}$$

In particular, $T_+(V)$ is an algebra.

Proof: Write $A = \sum_{k=0}^{\infty} A_k$ and $B = \sum_{k=0}^{\infty} B_k$, where $A_k, B_k \in V^{\otimes k}$. Then

$$\begin{aligned}
|(AB)_n|^2 &= \left| \sum_{k=0}^{n} A_k \otimes B_{n-k} \right|^2 \le \left(\sum_{k=0}^{n} |A_k||B_{n-k}| \right)^2 \\
&= \left(\sum_{k=0}^{n} |A_k| \sqrt{\frac{k!}{t^k}} |B_{n-k}| \sqrt{\frac{(n-k)!}{s^{n-k}}} \cdot \sqrt{\frac{t^k s^{n-k}}{k! \cdot (n-k)!}} \right)^2
\end{aligned}$$

$$\leq \sum_{k=0}^{n}\left(|A_k|^2\frac{k!}{t^k}|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}}\right)\sum_{k=0}^{n}\frac{t^k s^{n-k}}{k!\cdot(n-k)!}$$
$$= \frac{(t+s)^n}{n!}\sum_{k=0}^{n}|A_k|^2\frac{k!}{t^k}\cdot|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}}$$

and therefore,

$$\|AB\|_{s+t}^2 \leq \sum_{n=0}^{\infty}\frac{n!}{(t+s)^n}\cdot\frac{(t+s)^n}{n!}\sum_{k=0}^{n}|A_k|^2\frac{k!}{t^k}\cdot|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}}$$
$$= \sum_{0\leq k\leq n<\infty}|A_k|^2\frac{k!}{t^k}\cdot|B_{n-k}|^2\frac{(n-k)!}{s^{n-k}} = \|A\|_t^2\|B\|_s^2.$$

Q.E.D.

Remark 6.3. Lemma 6.2 is sharp, in the sense that

$$\sup_{A,B\in T_+(V)\setminus\{0\}}\frac{\|AB\|_r}{\|A\|_t\|B\|_s} = \infty \text{ if } r < s+t.$$

This can be seen by fixing $\xi \in V$ with $|\xi| = 1$ and then taking

$$A = \operatorname{Exp}(a\xi) := 1 + \sum_{n=1}^{\infty}\frac{a^n}{n!}\xi^{\otimes n}$$

and $B = \operatorname{Exp}(b\xi)$ while letting a and b range over $\mathbb{R}$.

Corollary 6.4. *Denote by L_β and R_β left and right multiplication in $T_\infty(V)$. Thus $R_\beta\eta = \eta\beta$ and $L_\beta\eta = \beta\eta$ for all $\eta \in T_\infty(V)$. If $\beta \in T(V)$ then L_β and R_β are bounded operators from $T_s(V)$ into $T_t(V)$ whenever $0 < s < t$.*

Proof: By (6.1) $\|R_\beta\eta\|_t \leq \|\eta\|_s\|\beta\|_{t-s}$. with the same inequality for L_β. Since $\|\beta\|_{t-s} < \infty$ the assertion follows. Q.E.D.

In the remainder of this section we will let G be a complex Lie group and let $\mathfrak{g} := \operatorname{Lie}(G)$ be its Lie algebra. Denote by q a non-negative Hermitian form on $\mathfrak{g}^*$. We will assume throughout this section that q satisfies Hörmander's condition (cf. Definition 2.6). Let $(H, (\cdot,\cdot)_H)$ be the inner product space described in (2.3) and Lemma 2.2 and let J_t^0 be as in (2.11). Lemma 6.2 will be often applied with $V = H$.

Remark 6.5. If $\alpha \in J_t^0$ and $\beta \in T(H)$, then, in view of (2.10) and (2.16),

$$|\langle\alpha,\beta\rangle| \leq \|\alpha\|_t\cdot\|\beta\|_t$$

and therefore, $\alpha|_{T(H)}$ extends uniquely to an element of $T_t(H)^*$. We will continue to denote this extension by α. Moreover, using this identification, we have $\|\alpha\|_t = \|\alpha\|_{T_t(H)^*}$. But α is also a linear functional on $T(\mathfrak{g})$. No norms have been specified so far on $T(\mathfrak{g})$ and moreover there appears to be no natural norm on $T(\mathfrak{g})$ with respect to which α is continuous. Nevertheless we will need to make use of $\langle\alpha,\beta\rangle$ for certain elements β which do not lie over H. To this end, in order to estimate the size of $\langle\alpha,\beta\rangle$, we will need to project these tensors into $T_\infty(H)$ along J. The key tool will be the projection constructed in Lemma 2.11.

Proposition 6.6. *Suppose that* $0 < s < \sigma$ *and that* $\alpha \in J_\sigma^0$. *If* $\beta \in T(\mathfrak{g})$ *then* $\alpha \circ R_\beta$ *and* $\alpha \circ L_\beta$ *are in* J_s^0. *Moreover, denoting by* P *the projection from* $T(\mathfrak{g})$ *onto* $T(H)$ *as constructed in Lemma 2.11, we have*

$$\|\alpha \circ R_\beta\|_s \le \|P\beta\|_{\sigma-s}\|\alpha\|_\sigma. \tag{6.2}$$

Proof: First observe that if $u \in J$ then $(\alpha \circ R_\beta)(u) = \alpha(u \otimes \beta) = 0$ if $\alpha \in J^0$. So $\alpha \circ R_\beta \in J^0$. Thus we need only focus on the size issue. Since J is an ideal, $\beta = P\beta \bmod J$, and α annihilates J, we may write, for any element $\eta \in T(H)$,

$$\begin{aligned} |\langle \alpha \circ R_\beta, \eta\rangle| &= |\langle \alpha, \eta \otimes \beta\rangle| = |\langle \alpha, \eta \otimes P\beta\rangle| \\ &\le \|\alpha\|_\sigma \|\eta \otimes P\beta\|_\sigma \le \|\alpha\|_\sigma \|\eta\|_s \|P\beta\|_{\sigma-s} \end{aligned}$$

This proves (6.2). The proof for L_β is similar. Q.E.D.

Corollary 6.7. *Suppose that* $\alpha \in J_\sigma^0$ *for some* $\sigma > 0$. *Let* $\psi := \sum_{n=0}^\infty \psi_n \in T_+(H)$. *Define* $\psi_{\le N} = \sum_{n=0}^N \psi_n$. *Then* $\alpha \circ L_\psi := \lim_{N\to\infty} \alpha \circ L_{\psi_{\le N}}$ *exists in* J_s^0 *for any* $s \in (0,\sigma)$. *Moreover*

$$\|\alpha \circ L_\psi\|_s \le \|\psi\|_{\sigma-s}\|\alpha\|_\sigma. \tag{6.3}$$

Proof: By Proposition 6.6,

$$\left\|\alpha \circ L_{\psi_{\le N}}\right\|_s \le \|\psi_{\le N}\|_{\sigma-s}\|\alpha\|_\sigma \text{ and} \tag{6.4}$$

$$\|\alpha \circ L_{\psi_{\le N}} - \alpha \circ L_{\psi_{\le K}}\|_s \le \|\psi_{\le N} - \psi_{\le K}\|_{\sigma-s}\|\alpha\|_\sigma. \tag{6.5}$$

Since $\psi_{\le N}$ converges to ψ in the sense of $T_{\sigma-s}(H)$, it follows from (6.5) that $\left\{\alpha \circ L_{\psi_{\le N}}\right\}_{N=1}^\infty$ is convergent in $T_s(H)$. Passing to the limit in (6.4) proves (6.3). Q.E.D.

6.2. A Generalized Power Series.

Definition 6.8. A function, $g : [0,1] \to G$, is a **piecewise** C^k **– path** if: 1) g is continuous and 2) there exists a partition,

$$D := \{0 = r_0 < r_1 < \cdots < r_l = 1\}, \tag{6.6}$$

of $[0,1]$ and functions, $g_i \in C^k([r_{i-1}, r_i], G)$ such that $g|_{[r_{i-1},r_i]} = g_i$ for $i = 1, 2, \ldots, l$. We further say that a collection of paths $\{g_t\}_{t\in\mathbb{R}}$ are **piecewise** C^2 **– paths depending differentiably on** t, if: 1) $(s,t) \to g_t(s) \in G$ is continuous and 2) there exists a partition D as in Eq. (6.6) and functions $g_i \in C^2([r_{i-1}, r_i] \times \mathbb{R}, G)$ such that $g_t(s) = g_i(s,t)$ when $(s,t) \in [r_{i-1}, r_i] \times \mathbb{R}$. In particular we are assuming that $\dot{g}_t(s) := \frac{d}{dt} g_t(s)$ exists for all $s \in [0,1]$.

For $0 \le r < s \le 1$, let

$$\Delta_n(r,s) := \{(s_1, s_2, \ldots, s_n) : r \le s_1 < s_2 < \cdots < s_n \le s\}$$

and let $d\mathbf{s} = ds_1 ds_2 \cdots ds_n$.

Notation 6.9. For $c \in L^1([0,1], \mathfrak{g})$ and $0 \le r \le s \le 1$, define

$$\psi_{r,s}(c) = \sum_{n=0}^\infty \psi_{r,s}^n(c) \in T_\infty(\mathfrak{g}) \tag{6.7}$$

where $\psi_{r,s}^0(c) = 1$ and for $n \ge 1$,

$$\psi_{r,s}^n(c) = \int_{\Delta_n(r,s)} c(s_1) \otimes \cdots \otimes c(s_n)\ d\mathbf{s}. \tag{6.8}$$

Given a piecewise C^1 – path, $g:[0,1]\to G$, and $0\le r<s\le 1$, let

$$\Psi^n_{r,s}(g):=\psi^n_{r,s}(c) \text{ and } \Psi_{r,s}(g)=\psi_{r,s}(c) \tag{6.9}$$

where $c(s):=\theta(g'(s))$ and θ is the Maurer-Cartan form, cf. Section 3. For notational simplicity we will write $\psi_{0,s}(c)$ and $\Psi_{0,s}(g)$ simply as $\psi_s(c)$ and $\Psi_s(g)$ respectively. Its important to observe that if the path g is horizontal, cf. Definition 3.1, then $\Psi^n_{r,s}(g)$ lies in $H^{\otimes n}$.

The following proposition provides a key quantitative control on $\Psi_{r,s}$.

Proposition 6.10. *Suppose that $g:[0,1]\to G$ is a piecewise C^1 – horizontal path, and $0\le r<s\le 1$. Then*

$$|\Psi^n_{r,s}(g)|_{H^{\otimes n}}\le\frac{1}{n!}\left(\int_r^s|\theta(g'(\sigma))|\,d\sigma\right)^n=\frac{\ell^n_H\left(g|_{[r,s]}\right)}{n!}, \tag{6.10}$$

For any $t>0$,

$$\|\Psi_{r,s}(g)\|^2_{T_t(H)}\le\exp\left(\frac{1}{t}\left[\int_r^s|\theta(g'(\sigma))|\,d\sigma\right]^2\right)=\exp\{\frac{1}{t}\ell^2_H(g|_{[r,s]})\}. \tag{6.11}$$

Moreover, if $0<\sigma<t$, and $\beta\in T(H)$ then

$$\|\Psi_1(g)\otimes\beta\|^2_{T_t(H)}\le\|\beta\|_{t-\sigma}\cdot e^{\ell^2_H(g)/\sigma}. \tag{6.12}$$

Proof: Letting $c(s):=\theta(g'(s))$, we may estimate Eq. (6.8) by

$$\left|\Psi^n_{r,s}(g)\right|=\left|\psi^n_{r,s}(c)\right|\le\int_{\Delta_n(r,s)}|c(s_1)|\dots|c(s_n)|\,d\mathbf{s}=\frac{1}{n!}\left(\int_r^s|c(\sigma)|\,d\sigma\right)^n \tag{6.13}$$

which proves Eq. (6.10). Equation (6.11) follows by squaring both sides of Eq. (6.10), multiplying the resulting equation through by $n!/t^n$, and then summing on n. Equation (6.12) now follows from (6.1). Q.E.D.

The following lemma summarizes some elementary properties of the various Ψ functions. We leave the proofs to the reader.

Lemma 6.11. *Let g be a piecewise C^1 –horizontal path in G and let $\Psi(g)$ and $\Psi^n(g)$ be defined as in Notation 6.9.*

(1) *For $n\in\mathbb{N}$ and $0\le r\le s\le 1$ with $r,s\notin D$ (cf. (6.6)),*

$$\frac{d}{ds}\Psi^n_{r,s}(g)=\Psi^{n-1}_{r,s}(g)\otimes c(s) \tag{6.14}$$

and

$$\frac{d}{dr}\Psi^n_{r,s}(g)=-c(r)\otimes\Psi^{n-1}_{r,s}(g). \tag{6.15}$$

(2) *$\Psi_{r,s}(g)$ satisfies*

$$\frac{d}{ds}\Psi_{r,s}(g)=\Psi_{r,s}(g)\otimes c(s) \text{ with } \Psi_{r,r}(g)=1 \tag{6.16}$$

and

$$\frac{d}{dr}\Psi_{r,s}(g)=-c(r)\otimes\Psi_{r,s}(g) \text{ with } \Psi_{s,s}(g)=1 \tag{6.17}$$

where the derivatives exist in $T_t(H)$ for all $t>0$.

The following proposition explains the role of the path dependent Ψ function in the "power series" expansion of a local holomorphic function on G and motivates our reconstruction, in the next section, of a holomorphic function f_α from its "Taylor coefficient," $\alpha \in J^0$.

Notation 6.12. For any $\varepsilon > 0$, let

$$U_\varepsilon^H := \{x \in G : d(e,x) = d_H(e,x) < \varepsilon\}.$$

As we have already mentioned, U_ε^H is an open neighborhood of e.

Proposition 6.13. *Let $\varepsilon > 0$, and $a \in G$. If $f \in \mathcal{H}\left(aU_\varepsilon^H\right)$ and $g : [0,1] \to G$ is a piecewise C^1 – path, such that $g(0) = a$ and $\ell_H(g) < \varepsilon$, then*

$$f(g(1)) = \langle \hat{f}(a), \Psi_1(g)\rangle := \sum_{k=0}^{\infty} \langle \hat{f}(a), \Psi_1^k(g)\rangle \tag{6.18}$$

where the sum in Eq. (6.18) is absolutely convergent. More generally, if $\beta \in T(\mathfrak{g})$,

$$\left(\tilde{\beta} f\right)(g(1)) = \langle \hat{f}(a), \Psi_1(g) \otimes \beta\rangle := \sum_{k=0}^{\infty} \langle \hat{f}(a), \Psi_1^k(g) \otimes \beta\rangle. \tag{6.19}$$

Proof: See [7, Proposition 5.1] where this same result is proved in the case $\varepsilon = \infty$ (i.e. $U_\varepsilon^H = G$) and $a = e$. The proof used there works here as well (when $a = e$) provided the parameter $z \in \mathbb{C}$ which appears in the proof of [7, Proposition 5.1] is always required to satisfy $|z| < \varepsilon/\ell_H(g)$. The main point being; if we define $g_z(s) \in G$ as the solution to the ODE,

$$\theta(g_z'(s)) = zc(s) \text{ with } g_z(0) = e,$$

then $\ell_H(g_z) = |z|\,\ell_H(g) < \varepsilon$ provided that $|z| < \varepsilon/\ell_H(g)$. In particular this implies that $g_z([0,1]) \subset U_\varepsilon^H$ and this is what is required to run the argument in [7, Proposition 5.1]. At the end of this argument we set $z = 1$ which is permissible since $\varepsilon/\ell_H(g) > 1$.

When $a \neq e$, apply the result with f replaced by $w(y) := f(ay)$ and $g(s)$ being replaced by $a^{-1}g(s)$ in which case we learn that

$$f(g(1)) = w\left(a^{-1}g(1)\right) = \langle \hat{w}(e), \Psi_1\left(a^{-1}g\right)\rangle = \langle \hat{f}(a), \Psi_1(g)\rangle;$$

where the last equality holds because $\hat{f}(a) = \hat{w}(e)$ and $\theta\left(\left(a^{-1}g\right)'(s)\right) = \theta(g'(s))$ so that $\Psi_1\left(a^{-1}g\right) = \Psi_1(g)$.

Applying Eq. (6.18) with f being replaced by $\tilde{\beta} f$ implies

$$\left(\tilde{\beta} f\right)(g(1)) = \sum_{k=0}^{\infty} \langle \widehat{\tilde{\beta} f}(a), \Psi_1^k(g)\rangle$$

which completes the proof since

$$\langle \widehat{\tilde{\beta} f}(a), \Psi_1^k(g)\rangle = \left(\widetilde{\Psi_1^k(g)}\tilde{\beta} f\right)(a) = \left(\left(\Psi_1^k(g) \otimes \beta\right)^\sim f\right)(a) = \langle \hat{f}(a), \Psi_1^k(g) \otimes \beta\rangle.$$

Q.E.D.

Remark 6.14. It should be observed that the power series in (6.18) converges, not because of some size restriction imposed on $\hat{f}(a)$, but because f is assumed to be holomorphic in a neighborhood of a, cf. [7, Proposition 5.1]. A size restriction, such as $\hat{f}(a) \in J_t^0$, yields strong bounds on the growth rate of the derivatives of f at a, much stronger than those that hold for a locally defined holomorphic function. The following corollary shows what kind of bounds on the derivatives of f are implied by such a strong condition. The inequalities (6.20) and (6.21) in the following corollary represent a generalization and an improvement over the corresponding inequality, (3.25), in [8]. We thank M. Gordina for her proof of the improvement.

Corollary 6.15. *Let $a \in G,\ f \in \mathcal{H}\left(aU_{\varepsilon}^{H}\right)$ be such that $\alpha := \hat{f}\,(a) \in J_t^0$. Suppose that $r, s > 0$ are such that $r + s \le t$. Then for every piecewise C^1 – horizontal path, $g : [0,1] \to aU_{\varepsilon}^{H}$ such that $g\,(0) = a$ and $\ell_H\,(g) < \varepsilon$,*

$$|D^k f(g\,(1))|_{q_k}^2 \le \frac{k!}{r^k}\|\alpha\|_t^2 e^{\ell_H^2(g)/s} \text{ for } k = 0, 1, 2, \dots. \tag{6.20}$$

Moreover if $x \in aU_{\varepsilon}^{H}$, then

$$|D^k f(x)|_{q_k}^2 \le \frac{k!}{r^k}\|\alpha\|_t^2 e^{d_H^2(e,x)/s} \text{ for } k = 0, 1, 2, \dots. \tag{6.21}$$

Proof: From Proposition 6.13,

$$\left(\tilde{\beta} f\right)(g\,(1)) = \langle \alpha, \Psi_1(g) \otimes \beta\rangle \text{ for all } \beta \in H^{\otimes k}. \tag{6.22}$$

This identity along with the estimate in Eq. (6.12) yields,

$$|\left(\tilde{\beta} f\right)(g\,(1))|^2 \le \|\alpha\|_t^2 \|\Psi_1(g) \otimes \beta\|_t^2 \le \|\alpha\|_t^2 \frac{k!|\beta|^2}{r^k} e^{\ell_H^2(g)/s}. \tag{6.23}$$

Since

$$\begin{aligned} |D^k f(g\,(1))|_{q_k}^2 &= \sup\left\{\left|\langle D^k f(g\,(1)), \beta\rangle\right|^2 : \beta \in H^{\otimes k} \text{ with } |\beta| = 1\right\} \\ &= \sup\left\{\left|\left(\tilde{\beta} f\right)(g\,(1))\right|^2 : \beta \in H^{\otimes k} \text{ with } |\beta| = 1\right\} \\ &\le \|\alpha\|_t^2 \frac{k!}{r^k} e^{\ell_H^2(g)/s}, \end{aligned}$$

Eq. (6.20) is proved. If $x \in aU_{\varepsilon}^{H}$, by definition there exists piecewise C^1 – horizontal path, $g : [0,1] \to aU_{\varepsilon}^{H}$ such that $g\,(0) = a,\ g\,(1) = x$ and $\ell_H\,(g) < \varepsilon$. Therefore. from Eq. (6.20) we learn that

$$|D^k f(x)|_{q_k}^2 \le \frac{k!}{r^k}\|\alpha\|_t^2 \inf\left\{e^{\ell_H^2(g)/s} : g\,(0) = a,\ g\,(1) = x\right\} = \frac{k!}{r^k}\|\alpha\|_t^2 e^{d_H^2(e,x)/s}.$$

Q.E.D.

6.3. Dependence of power series on the endpoint.

Theorem 6.16. *Let $s \to g_t\,(s) \in G$ be a piecewise C^2 -* horizontal *path depending smoothly on a parameter t such that $g_t\,(0) = e \in G$ for all t. Suppose that $\alpha \in J_T^0$ for some $T > 0$. Then*

$$\frac{d}{dt}|_0 \langle \alpha, \Psi_1\,(g_t)\rangle = \left\langle \alpha, \Psi_1\,(g_t) \otimes \theta\left(\frac{d}{dt}\Big|_{t=0} g_t\,(1)\right)\right\rangle. \tag{6.24}$$

where $\Psi_1\,(g_t)$ is defined in Notation 6.9.

Let us first give an informal but illustrative argument for Eq. (6.24). Let $c_t(s) := \theta(g_t'(s)) \in H$ and $h_t(s) := \theta(\dot{g}_t(s)) \in \mathfrak{g}$ where " $'$ " and " $\cdot$ " are shorthand for $\partial/\partial s$ and $\partial/\partial t$ respectively. Then

$$\begin{aligned} \dot{c}_t(s) &= \frac{d}{dt}\theta\left(g_t'(s)\right) = \frac{d}{ds}\theta\left(\dot{g}_t(s)\right) + d\theta\left(\dot{g}_t(s), g_t'(s)\right) \\ &= h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}, \end{aligned} \tag{6.25}$$

wherein we have used the structure equation, $d\theta(v,w) + [\theta(v), \theta(w)] = 0$. Recall from Lemma 6.11 that $\Psi_s(g_t)$ and $\Psi_{s,1}(g_t)$ solve

$$\frac{d}{ds}\Psi_s(g_t) = \Psi_s(g_t) \otimes c_t(s) \text{ with } \Psi_0(g_t) = 1 \tag{6.26}$$

and

$$\frac{d}{ds}\Psi_{s,1}(g_t) = -c_t(s) \otimes \Psi_{s,1}(g_t) \text{ with } \Psi_{1,1}(g_t) = 1. \tag{6.27}$$

Differentiating Eq. (6.26) in t implies

$$\frac{d}{ds}\frac{d}{dt}\Psi_s(g_t) = \frac{d}{dt}\Psi_s(g_t) \otimes c_t(s) + \Psi_s(g_t) \otimes \dot{c}_t(s)$$

and using this identity along with Eqs. (6.27) and (6.25) allows us to conclude

$$\begin{aligned} \frac{d}{ds}\left[\frac{d}{dt}\Psi_s(g_t) \cdot \Psi_{s,1}(g_t)\right] &= \Psi_s(g_t) \otimes \dot{c}_t(s) \otimes \Psi_{s,1}(g_t) \\ &= \Psi_s(g_t) \otimes \left(h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}\right) \otimes \Psi_{s,1}(g_t). \end{aligned}$$

Integrating this equation on s then gives,

$$\frac{d}{dt}\Psi_1(g_t) = \int_0^1 \Psi_s(g_t) \otimes \left(h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}\right) \otimes \Psi_{s,1}(g_t)\, ds. \tag{6.28}$$

An integration by parts along with Eqs. (6.26) and (6.27) shows

$$\begin{aligned} &\int_0^1 \Psi_s(g_t) \otimes h_t'(s) \otimes \Psi_{s,1}(g_t)\, ds \\ &\qquad = \Psi_s(g_t) \otimes h_t(s) \otimes \Psi_{s,1}(g_t)\,|_{s=0}^{s=1} \\ &\qquad\quad - \int_0^1 \Psi_s(g_t) \otimes c_t(s) \otimes h_t(s) \otimes \Psi_{s,1}(g_t)\, ds \\ &\qquad\quad + \int_0^1 \Psi_s(g_t) \otimes h_t(s) \otimes c_t(s) \otimes \Psi_{s,1}(g_t)\, ds \\ &\qquad = \Psi_1(g_t) \otimes h_t(1) - \int_0^1 \Psi_s(g_t) \otimes c_t(s) \wedge h_t(s) \otimes \Psi_{s,1}(g_t)\, ds. \end{aligned}$$

Using this identity in Eq. (6.28) gives

$$\frac{d}{dt}\Psi_1(g_t) = \Psi_1(g_t) \otimes h_t(1) + Z(t) \tag{6.29}$$

where

$$Z(t) = \int_0^1 \Psi_s(g_t) \otimes \left([c_t(s), h_t(s)]_{\mathfrak{g}} - c_t(s) \wedge h_t(s)\right) \otimes \Psi_{s,1}(g_t)\, ds.$$

By truncating Ψ_s and $\Psi_{s,1}$, we may write $Z(t)$ as a limit of elements in J and therefore argue that $\langle \alpha, Z_t \rangle = 0$ for $\alpha \in J_T^0$. Hence by applying $\alpha \in J_T^0$ to Eq. (6.29) implies the desired result;

$$\frac{d}{dt} \langle \alpha, \Psi_1(g_t) \rangle = \langle \alpha, \Psi_1(g_t) \otimes h_t(1) \rangle = \left\langle \alpha, \Psi_1(g_t) \otimes \theta \left(\frac{d}{dt} g_t(1) \right) \right\rangle.$$

The remainder of this section will be devoted to making the above argument rigorous. The proof of Theorem 6.16 will be completed after Lemma 6.18 below.

Lemma 6.17. *Suppose that $h(\cdot) \in C([0,1], \mathfrak{g})$ is piecewise C^1 and that $c(\cdot) \in \mathfrak{g}$ is piecewise continuous. Let $v(s) = h'(s) + [c(s), h(s)]$. For integer $N > 1$ define*

$$R_N = R_N(c,h) = \int_0^1 \sum_{m=0}^{N-1} \psi_s^m(c) \otimes [c(s), h(s)] \otimes \psi_{s,1}^{N-1-m}(c). \tag{6.30}$$

There exists an element $Z_N = Z_N(c,h) \in J$ such that

$$\partial_v \left(\sum_{n=0}^{N} \psi_1^n \right)(c) = \sum_{n=0}^{N-1} \psi_1^n(c) \otimes h(1) + Z_N + R_N. \tag{6.31}$$

Proof: Let $\Delta_n := \Delta_n(0,1)$. Since $\psi_1^n(c)$ is a multi-linear form in c, it is easy to see that $\psi_1^n(c)$ is smooth in c and that

$$\begin{aligned}
\partial_v \psi_1^n(c) &= \sum_{k=1}^{n} \int_{\Delta_n} c(s_1) \otimes \cdots \otimes c(s_{k-1}) \otimes v(s_k) \otimes c(s_{k+1}) \otimes \cdots \otimes c(s_n) \ d\mathbf{s} \\
&= \sum_{k=1}^{n} \int_0^1 \psi_s^{k-1}(c) \otimes v(s) \otimes \psi_{s,1}^{n-k}(c)\, ds.
\end{aligned}$$

Thus the derivative of the n - linear functional $\psi_1^n(c)$ in the direction v may be written

$$(\partial_v \psi_1^n)(c) = \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes v(s) \otimes \psi_{s,1}^k ds \tag{6.32}$$

where, to simplify notation, we are writing $\psi_{r,s}^n$ for $\psi_{r,s}^n(c)$ and we have defined $\psi_{r,s}^k \equiv 1$ if $k = 0$ and $\psi_{r,s}^k \equiv 0$ if $k < 0$. Consider first the terms in (6.32) arising from the summand h' in v. An integration by parts, using (6.26) and (6.27), yields

$$\begin{aligned}
\partial_{h'} \psi_1^n &= \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes h'(s) \otimes \psi_{s,1}^k ds \\
&= \sum_{m+k=n-1} \left(\begin{array}{c} \psi_s^m \otimes h(s) \otimes \psi_{s,1}^k \Big|_{s=0}^{1} \\ - \int_0^1 \left\{ \psi_s^{m-1} \otimes c(s) \otimes h(s) \otimes \psi_{s,1}^k - \psi_s^m \otimes h(s) \otimes c(s) \otimes \psi_{s,1}^{k-1} \right\} ds \end{array} \right).
\end{aligned}$$

Since $h(0) = 0$ and $\psi_{1,1}^k = 0$ if $k \neq 0$, the boundary terms contain at most one nonzero term, $\psi_1^{n-1} \otimes h(1)$. Replace m by $m+1$ in the first integral and replace k by $k+1$ in the second integral. We may then write

$$\partial_{h'} \psi_1^n = \psi_1^{n-1} \otimes h(1) - \sum_{m+k=n-2} \int_0^1 \psi_s^m \otimes (c(s) \wedge h(s)) \otimes \psi_{s,1}^k ds.$$

Adding now the contribution to v from the term $[c(s), h(s)]$ we then find, with the help of (6.32),

$$\partial_v \psi_1^n = \psi_1^{n-1} \otimes h(1) + \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes [c(s), h(s)] \otimes \psi_{s,1}^k ds - \sum_{m+k=n-2} \int_0^1 \psi_s^m \otimes (c(s) \wedge h(s)) \otimes \psi_{s,1}^k ds$$

Summing this equation on n from 0 to N, keeping in mind that $\psi_1^0 = 1$ and $\psi_{r,s}^{-1} = 0$, we find

$$\begin{aligned}\partial_v \sum_{n=0}^{N} \psi_1^n &= \sum_{n=0}^{N-1} \psi_1^n \otimes h(1) \\ &\quad + \sum_{n=1}^{N-1} \sum_{m+k=n-1} \int_0^1 \psi_s^m \otimes \{[c(s), h(s)] - c(s) \wedge h(s)\} \otimes \psi_{s,1}^k ds \\ &\quad + \int_0^1 \sum_{m+k=N-1} \psi_s^m \otimes [c(s), h(s)] \otimes \psi_{s,1}^k ds\end{aligned}$$

Since the middle line is in J the lemma is proved. Q.E.D.

Lemma 6.18. *Suppose that $\alpha \in J_T^0$ for some $T > 0$, $h(\cdot) \in C([0,1], \mathfrak{g})$ is piecewise C^1, $g(\cdot)$ is a piecewise C^2 horizontal path over $[0,1]$, and let $c(s) := \theta(g'(s))$. Let $R_N = R_N(c, h)$ be as in Lemma 6.17 and $\|h\|_\infty = \sup_{s\in[0,1]} |h(s)|$ where $|\cdot|$ is any given fixed norm on $\mathfrak{g}$ such that $|A|^2 = (A, A)_H$ for all $A \in H$. Then there exists constants, $\{C_N(T)\}_{N=1}^\infty$ such that $\lim_{N\to\infty} C_N(T) \lambda^N = 0$ for all $\lambda > 0$ and*

$$|\langle \alpha, R_N \rangle| \le \|\alpha\|_T \cdot \|h\|_\infty C_N(T) \ell_H^n(g). \tag{6.33}$$

Proof: Let $u_m(s) = \psi_s^m(c)$ and $v_m(s) = \psi_{s,1}^{N-m-1}(c)$. Because g is horizontal, $u_m(s) \in H^{\otimes m}$ and $v_m(s) \in H^{\otimes(N-m-1)}$ for each $s \in [0,1]$. If $w(s) := [c(s), h(s)]$, then

$$\langle \alpha, R_N \rangle = \sum_{m=0}^{N-1} \int_0^1 \langle \alpha, u_m(s) \otimes w(s) \otimes v_m(s) \rangle ds \tag{6.34}$$

and we may find $K < \infty$ such that

$$\int_0^1 |w(s)|_{\mathfrak{g}} ds \le K \|h\|_\infty \int_0^1 |c(s)|_H ds = K \|h\|_\infty \ell_H(g) < \infty. \tag{6.35}$$

The integrability of w guarantees the integrals in (6.34) and the integrals appearing in the argument below all exist. Although $u_m(s)$ and $v_m(s)$ lie over H the factor $w(s)$ may not lie in H. Since α is only continuous on tensor spaces over H we must replace the factor $w(s)$ before making estimates.

Let $P : T(\mathfrak{g}) \to T(H)$ be the projection operator constructed in Lemma 2.11 and let $L := P|_{\mathfrak{g}} : \mathfrak{g} \to \bigoplus_{k=1}^r H^{\otimes k}$. We may write, for all $A \in \mathfrak{g}$, $L(A) = \sum_{k=1}^r L_k(A)$ with each L_k being a linear map from $\mathfrak{g}$ into $H^{\otimes k}$. Since $\mathfrak{g}$ is finite dimensional, there exists $K_1 < \infty$ such that $|L_k(A)|_{H^{\otimes k}} \le K_1 |A|_{\mathfrak{g}}$ for $k = 1, 2 \ldots, r$ and $A \in \mathfrak{g}$.

With this notation, (6.34) may be written as

$$\begin{aligned}\langle \alpha, R_N \rangle &= \sum_{m=0}^{N-1} \int_0^1 \langle \alpha, u_m(s) \otimes L(w(s)) \otimes v_m(s) \rangle \, ds \\ &= \sum_{k=1}^{r} \sum_{m=0}^{N-1} \int_0^1 \langle \alpha, u_m(s) \otimes L_k(w(s)) \otimes v_m(s) \rangle \, ds.\end{aligned}$$

Using the estimate (6.10) and writing $|\alpha_j|$ for $|\alpha_j|_{q_j}$, we find

$$\begin{aligned}&\sum_{m=0}^{N-1} | \langle \alpha, u_m(s) \otimes L_k(w(s)) \otimes v_m(s) \rangle | \\ &\quad \leq \sum_{m=0}^{N-1} |\alpha_{(N-1+k)}| |u_m(s)|_{H^{\otimes m}} |L_k(w(s))|_{H^{\otimes k}} |v_m(s)|_{H^{\otimes (N-m-1)}} \\ &\quad \leq \sum_{m=0}^{N-1} |\alpha_{(N-1+k)}| \frac{\ell_H(g|_{[0,s]})^m}{m!} K_1 |w(s)|_{\mathfrak{g}} \frac{\ell_H(g|_{[s,1]})^{(N-m-1)}}{(N-m-1)!} \\ &\quad \leq |\alpha_{(N-1+k)}| K_1 |w(s)|_{\mathfrak{g}} \frac{\ell_H(g))^{(N-1)}}{(N-1)!}\end{aligned}$$

where the binomial formula was used to obtain the last inequality. After integrating on s, summing on k, and using Eq. (6.35) in the previous estimate, we find

$$|\langle \alpha, R_N \rangle| \leq K_1 K \frac{\ell_H^N(g)}{(N-1)!} \|h\|_\infty \sum_{k=1}^{r} |\alpha_{(N-1+k)}|. \tag{6.36}$$

By the definition (2.10) we see that $|\alpha_j|_{q_j} \leq (j!/T^j)^{1/2} \|\alpha\|_T$ which combined with (6.36) gives,

$$|\langle \alpha, R_N \rangle| \leq K_1 K \frac{\ell_H^N(g)}{(N-1)!} \|h\|_\infty \|\alpha\|_T \sum_{k=1}^{r} \sqrt{\frac{(N-1+k)!}{T^{N-1+k}}}$$

which proves the Lemma with

$$C_N(T) := K_1 K \frac{1}{(N-1)!} \sum_{k=1}^{r} \sqrt{\frac{(N-1+k)!}{T^{N-1+k}}}.$$

Q.E.D.

We are now in a position to complete the proof of Theorem 6.16.

Proof of Theorem 6.16: As at the beginning of this section, let $c_t(s) := \theta(g_t'(s)) \in H$ and $h_t(s) := \theta(\dot{g}_t(s)) \in \mathfrak{g}$ and recall from Eq. (6.25) that

$$\dot{c}_t(s) = h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}.$$

Let $f(t) := \langle \alpha, \Psi_1(g_t) \rangle$ and

$$f_N(t) = \left\langle \alpha, \sum_{n=0}^{N} \Psi_1^n(g_t) \right\rangle = \left\langle \alpha, \sum_{n=0}^{N} \psi_1^n(c_t) \right\rangle$$

so that $f(t) = \lim_{N\to\infty} f_N(t)$. By Lemma 6.17 with $v(s) := h_t'(s) + [c_t(s), h_t(s)]_{\mathfrak{g}}$, $f_N(t)$ is differentiable and

$$\frac{df_N(t)}{dt} = \langle \alpha, \sum_{n=0}^{N} \psi_1^n(c_t) \otimes h_t(1)\rangle + \langle \alpha, R_N(c_t, \dot{c}_t)\rangle. \tag{6.37}$$

Because $\ell_H(g_t)$ and $\|\dot{c}_t(\cdot)\|_\infty$ are bounded for t near zero, Lemma 6.18 may be used to conclude the remainder term, $\langle \alpha, R_N(c_t, \dot{c}_t)\rangle$, goes to zero as $N \to \infty$ uniformly in a neighborhood of $t = 0$. Moreover it is easily verified that

$$\lim_{N\to\infty} \langle \alpha, \sum_{n=0}^{N} \psi_1^n(c_t) \otimes h_t(1)\rangle = \langle \alpha, \psi_1(c_t) \otimes h_t(1)\rangle$$

with the above limit being uniform in t near zero. Hence we may conclude that $f(t)$ is differentiable near zero and that $\dot{f}(t) = \langle \alpha, \Psi_1(c_t) \otimes h_t(1)\rangle$. Q.E.D.

7. Reconstruction of f from its Taylor Coefficients

The purpose of this section is to complete the proof of the following theorem, which is the main theorem of this paper.

Theorem 7.1. *Let G be a connected, simply connected complex Lie group. Suppose that q is a non-negative Hermitian form on the dual space $\mathfrak{g}^*$ and assume that Hörmander's condition holds, (cf. Definition. 2.6). Let ρ_t denote the heat kernel associated to Equation (3.1). Then the Taylor map, $f \to \hat{f}(e)$ is a unitary map from $\mathcal{H}L^2(G, \rho_t(x)dx)$ onto J_t^0.*

Proof: Since we have already proved the isometry property of the Taylor map in Theorem 4.1, it suffices to prove the map is surjective. In light of Proposition 4.2, to prove the surjectivity of the Taylor map it suffices to show to each $\alpha \in J_t^0$ there exists $f \in \mathcal{H}(G)$ such that $\hat{f}(e) = \alpha$. But this is the content of Theorem 7.13 below. Q.E.D.

The remainder of this section is devoted to the proof of Theorem 7.13.

7.1. Holomorphic Horizontal Coordinates and Paths.

In this section, let G be a complex Lie group and $\mathfrak{g} := \mathrm{Lie}(G)$ be its complex Lie algebra.

Notation 7.2. For $g, h \in G$, let $[g, h] := g^{-1}h^{-1}gh$.

Lemma 7.3. *For $\Gamma := (A_1, \ldots, A_n) \in \mathfrak{g}^n$ and $\varepsilon \in \mathbb{C}\setminus\{0\}$ define*

$$v_\Gamma(\varepsilon) = \frac{d}{dt}|_0 [e^{\frac{t}{\varepsilon^{n-1}} A_1}, [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \ldots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}] \ldots]. \tag{7.1}$$

Then

$$\lim_{\varepsilon\to 0} v_\Gamma(\varepsilon) = [\Gamma] \tag{7.2}$$

where $[\Gamma]$ is defined as in Equation (2.18).

Proof: If $X \in \mathfrak{g}$ and $b \in G$ then

$$\frac{d}{dt}|_0 [e^{tX}, b] = \frac{d}{dt}|_0 e^{-tX} e^{t(Ad_{b^{-1}})X} = (Ad_{b^{-1}} - I)X \tag{7.3}$$

Let

$$b = [e^{\varepsilon A_2}, [e^{\varepsilon A_3}, \ldots [e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}] \ldots]$$

and let $\Gamma' = (A_2, \ldots, A_n)$. We assert that, for ε near 0, $b = e^{B(\varepsilon)}$ where

$$B(\varepsilon) = \varepsilon^{n-1}[\Gamma'] + \varepsilon^n C(\varepsilon) \tag{7.4}$$

and $C(\varepsilon)$ is an analytic $\mathfrak{g}$ valued function of ε for ε near 0. This may be proven by induction on n with the help of the Baker-Campbell-Hausdorff formula [53, Theorem 2.15.4] as follows. An application of [53, Theorem 2.15.4] shows,

$$\left[e^X, e^Y\right] = e^{-X}e^{-Y}e^Xe^Y = e^{[X,Y]+R_2(X,Y)} \tag{7.5}$$

where $R_2(X,Y)$ is an analytic function of X and Y defined in a neighborhood of 0 in $\mathfrak{g} \times \mathfrak{g}$ and which satisfies,

$$|R_2(X,Y)| \le C_2 (|X| + |Y|)\,|X|\,|Y|.$$

We further assert that

$$[e^{B_2}, [e^{B_3}, \ldots [e^{B_{n-1}}, e^{B_n}] \ldots] = e^{[B_2,[B_3,\ldots[B_{n-1},B_n]\ldots]+R_{n-1}(B_2,\ldots,B_{n-1},B_n)} \tag{7.6}$$

where $R_{n-1}(B_2, \ldots, B_{n-1}, B_n)$ is an analytic function of $(B_2, \ldots, B_{n-1}, B_n)$ in a neighborhood of $0 \in \mathfrak{g}^{n-1}$ which satisfies,

$$|R_{n-1}(B_2, \ldots, B_{n-1}, B_n)| \le C_{n-1}\left(\sum_{i=2}^n |B_i|\right)|B_2| \ldots |B_n|. \tag{7.7}$$

Indeed, assuming Eqs. (7.6) and (7.7) hold, it follows by using Eq. (7.5) that

$$\left[e^{B_1}[e^{B_2}, [e^{B_3}, \ldots [e^{B_{n-1}}, e^{B_n}] \ldots]\right] = e^{[B_1,[B_2,[B_3,\ldots[B_{n-1},B_n]\ldots]]+R_n(B_1,B_2,\ldots,B_{n-1},B_n)}$$

where

$$\begin{aligned}
&R_n(B_1, B_2, \ldots, B_{n-1}, B_n) \\
&\quad = [B_1, R_{n-1}(B_2, \ldots, B_{n-1}, B_n)] \\
&\qquad + R_2(B_1, [B_2, [B_3, \ldots [B_{n-1}, B_n] \ldots] + R_{n-1}(B_2, \ldots, B_{n-1}, B_n)).
\end{aligned}$$

The function R_n is analytic for $(B_1, \ldots, B_{n-1}, B_n)$ in a neighborhood of $0 \in \mathfrak{g}^n$ and is easily seen to satisfy

$$|R_n(B_1, B_2, \ldots, B_{n-1}, B_n)| \le C_n\left(\sum_{i=1}^n |B_i|\right)|B_1|\,|B_2| \ldots |B_n|.$$

Equation (7.6), with $B_i = \varepsilon A_i$, implies $B(\varepsilon)$ is an analytic function of ε in a neighborhood of 0 such that

$$B(\varepsilon) = \varepsilon^{n-1}[\Gamma'] + O(\varepsilon^n).$$

Taking $b = e^{B(\varepsilon)}$ and $X = \varepsilon^{-(n-1)}A_1$ in Eq. (7.3) implies

$$\begin{aligned}
v_\Gamma(\varepsilon) &= (e^{-ad_{B(\varepsilon)}} - I)\varepsilon^{-(n-1)}A_1 \\
&= \sum_{k=1}^\infty \frac{1}{k!}(-ad_{B(\varepsilon)})^k \varepsilon^{-(n-1)}A_1 \\
&= -ad_{[\Gamma']}A_1 + O(\varepsilon) = [\Gamma] + O(\varepsilon).
\end{aligned}$$

Q.E.D.

Equation (7.4) can be viewed as a version of [37, Lemma 2.2.1]. Such a commutator identity frequently plays a role in subelliptic estimates and goes back at least to Hörmander [29].

Notation 7.4. For $\Gamma := (A_1, A_2, \ldots, A_n) \in \mathfrak{g}^n$ and $\varepsilon > 0$, let $\phi_{\Gamma,\varepsilon} : \mathbb{C} \to G$ be defined by

$$\phi_{\Gamma,\varepsilon}(z) := \left[e^{\frac{z}{\varepsilon^{n-1}}A_1}, \left[e^{\varepsilon A_2}, \left[e^{\varepsilon A_3}, \ldots \left[e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}\right]\right] \ldots\right]\right]. \tag{7.8}$$

The function, $\phi(z) := \phi_{\Gamma,\varepsilon}(z) \in G$, is a holomorphic function of z whose derivative at $z = 0$ in the direction w is given by

$$\begin{aligned}\phi_*(w_0) &= w\phi_*(1_0) = w\frac{d}{dt}|_0 \phi(t) \\ &= w\frac{d}{dt}|_0 \left[e^{\frac{t}{\varepsilon^{n-1}}A_1}, \left[e^{\varepsilon A_2}, \left[e^{\varepsilon A_3}, \ldots \left[e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}\right]\right] \ldots\right]\right] \\ &= w \cdot v_\Gamma(\varepsilon) \to [\Gamma]\, w \text{ as } \varepsilon \to 0.\end{aligned}$$

For each $\Gamma := (A_1, A_2, \ldots, A_n) \in \mathfrak{g}^n$, $\varepsilon \neq 0$, and $z \in \mathbb{C}$ we are going to define a piecewise C^∞ - horizontal path, $\sigma_{\Gamma,\varepsilon,z}$, which depends holomorphically on z as follows. First observe that

$$\left[e^{z\varepsilon^{-(n-1)}A_1}, \left[e^{\varepsilon A_2}, \left[e^{\varepsilon A_3}, \ldots \left[e^{\varepsilon A_{n-1}}, e^{\varepsilon A_n}\right]\right] \ldots\right]\right]$$

is the product of $N_n := 3 \cdot 2^{n-1} - 2$ exponentials of the form $e^{B_1}e^{B_2} \ldots e^{B_{N_n}}$ with each B_i being an element from the set,

$$S(\Gamma, \varepsilon, z) := \{\pm\varepsilon A_1, \pm\varepsilon A_2, \cdots \pm \varepsilon A_{n-1}\} \bigcup \left\{\pm\frac{z}{\varepsilon^{n-1}}A_n\right\}. \tag{7.9}$$

Hence if $k = 1, 2, \ldots, N_n$ and $s \in \left[\frac{k-1}{N_n}, \frac{k}{N_n}\right]$, let

$$\sigma_{\Gamma,\varepsilon,z}(s) := e^{B_1} \ldots e^{B_{k-1}} e^{(N_n s - k + 1)B_k}. \tag{7.10}$$

The following proposition summarizes what we have done.

Proposition 7.5. *Assume that* $\Gamma \in H^n$. *The path* $\sigma_{\Gamma,\varepsilon,z}$ *in Eq. (7.10) is a piecewise* C^∞ *- horizontal path from* e *to* $\psi_{\Gamma,\varepsilon}(z)$ *which depends holomorphically on* $z \in \mathbb{C}$. *Moreover, for* $s \in \left(\frac{k-1}{N_n}, \frac{k}{N_n}\right)$, $\theta\left(\sigma'_{\Gamma,\varepsilon,z}(s)\right) = B_k \in S(\Gamma, \varepsilon, z)$ *(see Eq. (7.9)) and hence* $\theta\left(\sigma'_{\Gamma,\varepsilon,z}(s)\right)$ *is either constant in* z *or depends on* z *linearly in each of the intervals,* $\left\{\left(\frac{k-1}{N_n}, \frac{k}{N_n}\right) : k = 1, 2, \ldots, N_n\right\}$.

Let $\mathcal{X} := \{X_j\}_{j=1}^m$ be an orthonormal basis for the Hörmander subspace, H. For $l = m+1, \ldots, M := \dim \mathfrak{g}$, let $n_l \in \mathbb{N}$ and $\Gamma_l \in \mathcal{X}^{n_l}$ be chosen so that

$$\{X_j\}_{j=1}^m \cup \{[\Gamma_l] : l = m+1, \ldots, M\}$$

is a basis for $\mathfrak{g}$. We may apply Lemma 7.3 to find (and fix once and for all) an $\varepsilon \in \mathbb{C} \setminus \{0\}$ sufficiently close to zero such that

$$\{X_j\}_{j=1}^m \cup \{[X_l := v_{\Gamma_l}(\varepsilon)] : l = m+1, \ldots, M\} \tag{7.11}$$

is still a basis for $\mathfrak{g}$. For $z \in \mathbb{C}$, let

$$\phi_j(z) := \begin{cases} e^{zX_j} & \text{if} \quad 1 \le j \le m \\ \phi_{\Gamma_j,\varepsilon}(z) & \text{if} \quad m+1 \le j \le M, \end{cases} \tag{7.12}$$

where $\phi_{\Gamma_j,\varepsilon}$ has been defined in Eq. (7.8).

Notation 7.6 (Horizontal Charts and Paths)**.** For $z = (z_1, \ldots, z_M) \in \mathbb{C}^M$, let

$$\varphi(z) := \phi_1(z_1)\,\phi_2(z_2)\ldots\phi_M(z_M) \in G$$

and let $\sigma_z(s) \in G$ be the horizontal path defined, for $s \in \left[\frac{j-1}{M}, \frac{j}{M}\right]$, by

$$\sigma_z(s) = \phi_1(z_1)\,\phi_2(z_2)\ldots\phi_{j-1}(z_{j-1})\,\sigma_{\Gamma_j,\varepsilon,z_j}(Ms - j + 1).$$

Theorem 7.7. *The function $\varphi : \mathbb{C}^M \to G$ is a local bi-holomorphism from an open neighborhood, Ω, of $0 \in \mathbb{C}^M$ to an open neighborhood, U, of $e \in G$. The path $\sigma_z(s)$ is a piecewise C^∞ - horizontal path in G from e to $\varphi(z)$ which depends holomorphically on $z \in \mathbb{C}^M$. Moreover, there is a partition of $[0,1]$, $D = \{0 = s_0 < s_1 < \cdots < s_N = 1\}$, such that for $s \in (s_{l-1}, s_l)$, either $\theta(\sigma_z'(s)) = B_l$ or $\theta(\sigma_z'(s)) = z_{j_l} B_l$ for some $B_l \in H$ and some $j_l \in \{1, 2, \ldots, M\}$.*

Proof: Since

$$\varphi_*\left([e_j]_0\right) = X_j \text{ for } j = 1, 2, \ldots, M$$

where X_j are defined in Eq. (7.11), the first assertion is a consequence of the inverse function theorem. The remaining assertions have already been proved prior to the statement of the theorem. Q.E.D.

7.2. **Local Existence of f_α .**

Notation 7.8. We will want to consider α and its "translates" in various of the spaces J_t^0. Noting that $J_\sigma^0 \subset J_s^0$ if $0 < s < \sigma$, we define

$$J_+^0 := \bigcup_{t>0} J_t^0. \tag{7.13}$$

A consequence of Corollary 6.7 is that if $\alpha \in J_+^0$ and $\psi \in T_+(H)$ then $\alpha \circ L_\psi \in J_+^0$.

Theorem 7.9 (Local Existence)**.** *Let $\Omega \subset \mathbb{C}^M$ and $U \subset G$ be as in Theorem 7.7. For each $\alpha \in J_+^0$ and $x \in G$ there exists $f = f_\alpha \in \mathcal{H}(xU)$ such that $\hat{f}(x) = \alpha$. This function has the additional property that*

$$\hat{f}(x\varphi(z)) = \alpha \circ L_{\Psi_1(\sigma_z)} \text{ for all } z \in \Omega. \tag{7.14}$$

In particular, $\hat{f}(y) \in J_+^0$ for all $y \in xU$.

Proof: The proof will consist of showing that the function $f : xU \to \mathbb{C}$ defined by

$$f(x\varphi(z)) = f(x\sigma_z(1)) := \langle \alpha, \Psi_1(\sigma_z)\rangle =: u(z) \text{ for all } z \in \Omega$$

is the desired function. By Proposition 6.10,

$$|\langle \alpha, \Psi_1^n(\sigma_z)\rangle| \le |\alpha_n|_{q_n} |\Psi_1^n(\sigma_z)|_{H^{\otimes n}} \le |\alpha_n|_{q_n} \frac{1}{n!} K^n$$

where $K := \sup_{z\in\Omega} \ell_H(\sigma_z)$ and therefore,

$$\sum_{n=0}^\infty |\alpha_n|_{q_n} K^n \frac{1}{n!} \le \left(\sum_{n=0}^\infty |\alpha_n|_{q_n}^2 \frac{t^n}{n!}\right)^{\frac{1}{2}} \left(\sum_{n=0}^\infty \frac{K^{2n}}{n! t^n}\right)^{\frac{1}{2}} = \|\alpha\|_t \, e^{\frac{K^2}{2t}}.$$

Therefore the sum, $\sum_{n=0}^\infty \langle \alpha_n, \Psi_1^n(\sigma_z)\rangle$, defining $u(z)$ is uniformly and absolutely convergent. Moreover it is easy to verify that each summand, $u_n(z) := \langle \alpha_n, \Psi_1^n(\sigma_z)\rangle$, is a holomorphic polynomial in z of degree n and thus $u(z)$ is holomorphic as well.

Using Theorem 6.16, we learn that

$$\begin{aligned} u_* w_z &= \frac{d}{dt}|_0 u(z+tw) = \frac{d}{dt}|_0 \langle \alpha, \Psi_1(\sigma_{z+tw})\rangle \\ &= \left\langle \alpha, \Psi_1(\sigma_z) \otimes \theta\left(\frac{d}{dt}|_0 \sigma_{z+tw}(1)\right)\right\rangle \\ &= \left\langle \alpha, \Psi_1(\sigma_z) \otimes \theta\left(\frac{d}{dt}|_0 \varphi(z+tw)\right)\right\rangle \\ &= \langle \alpha, \Psi_1(\sigma_z) \otimes \theta(\varphi_* w_z)\rangle \end{aligned}$$

while on the other hand

$$u_* w_z = \frac{d}{dt}|_0 f(x\varphi(z+tw)) = \langle Df(x\varphi(z)), \theta(\varphi_* w_z)\rangle.$$

Comparing these two equations shows that

$$\langle Df(x\varphi(z)), \theta(\varphi_* w_z)\rangle = \langle \alpha, \Psi_1(\sigma_z) \otimes \theta(\varphi_* w_z)\rangle$$

for all $w \in \mathbb{C}^M$ and $z \in \Omega$ which implies

$$\langle Df(x\varphi(z)), A\rangle = \langle \alpha, \Psi_1(\sigma_z) \otimes A\rangle \text{ for all } z \in \Omega \text{ and } A \in \mathfrak{g}. \tag{7.15}$$

By Proposition 6.6, $\alpha_A := \alpha \circ R_A$ is in J_+^0. With this notation Eq. (7.15) reads

$$\left(\tilde{A}f\right)(x\varphi(z)) = \langle \alpha_A, \Psi_1(\sigma_z)\rangle \text{ for all } A \in \mathfrak{g}.$$

Applying the above results with f replaced by $\tilde{A}f$ and α replaced by α_A, we learn that

$$\left(\tilde{B}\tilde{A}f\right)(x\varphi(z)) = \langle (\alpha_A)_B, \Psi_1(\sigma_z)\rangle = \langle \alpha_A, \Psi_1(\sigma_z) \otimes B\rangle = \langle \alpha, \Psi_1(\sigma_z) \otimes B \otimes A\rangle.$$

Moreover, a simple induction argument now shows that

$$\left(\tilde{A}_1 \dots \tilde{A}_n f\right)(x\varphi(z)) = \langle \alpha, \Psi_1(\sigma_z) \otimes A_1 \otimes \cdots \otimes A_n\rangle \text{ for all } A_i \in \mathfrak{g}$$

which is equivalent to Eq. (7.14). In light of Corollary 6.7, the proof is complete. Q.E.D.

7.3. **Global Construction of f_α .** In what follows, we will fix an inner product on $\mathfrak{g}$. Such a choice induces a unique left invariant Riemannian metric on G. Fix $\delta > 0$ such that the Riemannian ball, $U = U_\delta^{\text{Riem}}$, of radius δ and centered at e is geodesically convex, and such that there exists an open neighborhood, Ω, of 0 in $\mathbb{C}^M$ for which the results of Theorems 7.7 are valid. In particular, for every $\alpha \in J_+^0$ and $x \in G$ there exists $f \in \mathcal{H}(xU)$ such that $\hat{f}(x) = \alpha$ by Theorem 7.9. The following two simple observations will be used repeatedly below. 1) A point $x \in G$ is in yU iff $y \in xU$. 2) If S is a non-empty finite subset of G such that

$$\operatorname{diam}(S) := \sup\{d_{\text{Riem}}(x,y) : x, y \in S\} < \delta,$$

then $\bigcap_{a \in S}(aU)$ is a non-empty, (pathwise) connected open subset of G containing S. The latter holds because $\bigcap_{a \in S}(aU)$ is a non-empty, geodesically convex, open subset of G containing S.

Theorem 7.10 (Analytic Continuation)**.** *Suppose that $g \in C([0,1], G)$ is a path such that $g(0) = e$. Then to each $\alpha \in J_+^0$, there exists a unique family of functions,*

$$\{f_t \in \mathcal{H}(g(t)U) : 0 \le t \le 1\}, \tag{7.16}$$

satisfying:

(1) $\hat{f}_0(e) = \alpha$ *and*
(2) *if* $0 \leq a \leq b \leq 1$ *with* $\operatorname{diam}(g([a,b])) < \delta$, *then* $f_s = f_t$ *on* $g(s)U \cap g(t)U$ *for all* $s, t \in [a,b]$.

Moreover, $\hat{f}_t(x) \in J_+^0$ *for all* $x \in g(t)U$ *and all* $t \in [0,1]$.

Proof: Uniqueness. Suppose that $\{k_t \in \mathcal{H}(g(t)U) : 0 \leq t \leq 1\}$ is another collection of holomorphic functions satisfying the same properties as $\{f_t : t \in [0,1]\}$ and let

$$T_0 := \sup\{T \in [0,1] : f_t = k_t \text{ for } 0 \leq t \leq T\}.$$

Since holomorphic functions are determined by their Taylor coefficients and $\hat{f}_0 = \alpha = \hat{k}_0$, it follows that $f_0 = k_0$ on U. Moreover if $T > 0$ is chosen so that $g([0,T]) \subset U$, then for $0 \leq t \leq T$, we have $f_t = f_0 = k_0 = k_t$ on $g(t)U \cap U$ which is a non-empty open subset of $g(t)U$. Since $g(t)U$ is a connected open set it follows that $f_t = k_t$ on all of $g(t)U$. Hence we have shown $T_0 > 0$.

Choose $0 < a < T_0 \leq b \leq 1$ such that $\operatorname{diam}(g([a,b])) < \delta$ and $b > T_0$ if $T_0 < 1$. Then for $t \in [a,b]$, $f_t = f_a$ and $k_t = k_a$ on $g(a)U \cap g(t)U$ and $f_a = k_a$ on $g(a)U$. Therefore $f_t = k_t$ on $g(a)U \cap g(t)U$ which implies $f_t = k_t$ on the connected open set, $g(t)U$. If $T_0 < 1$, we would conclude that $T_0 \geq b > T_0$ which is absurd. Hence $T_0 = b = 1$ and we conclude $f_1 = k_1$ and by the definition of T_0 that $f_t = k_t$ for $0 \leq t < 1$.

Existence. From Theorem 7.9, there exists $f_0 \in \mathcal{H}(U)$ such that $\hat{f}_0(e) = \alpha$ having the property that $\hat{f}(x) \in J_+^0$ for all $x \in J_+^0$. If $T > 0$ is chosen so that $\operatorname{diam}(g([0,T])) < \delta$, another application of Theorem 7.9 shows there exists $f_t \in \mathcal{H}(g(t)U)$ such that $\hat{f}_t(g(t)) = \hat{f}_0(g(t))$ for all $t \in [0,T]$. Since f_t and f_0 have the same derivatives at $g(t)$, it follows that $f_t = f_0$ in a neighborhood of $g(t)$ and therefore $f_t = f_0$ on the connected open set, $g(t)U \cap U$. Hence if $s, t \in [0,T]$, then $f_s = f_0$ on $g(s)U \cap U$, $f_t = f_0$ on $g(t)U \cap U$ which implies $f_s = f_t$ on the non-empty open set, $g(s)U \cap g(t)U \cap U$. So again $f_s = f_t$ on the connected open set, $g(s)U \cap g(t)U$.

Let T_0 be the supremum over all $T \in [0,1]$ such that there exists a (unique) family of functions, $f_t \in \mathcal{H}(g(t)U)$ for $0 \leq t \leq T$ with the properties listed in the statement of the theorem (with "1" being replaced by T everywhere) including the assertion that $\hat{f}_t(x) \in J_+^0$ for all $x \in g(t)U$ and all $t \in [0,T]$. The previous paragraph shows that $T_0 > 0$.

Suppose, for sake of contradiction, that $T_0 < 1$. Choose $0 \leq T_- < T_0 < T_+ \leq 1$ such that $\operatorname{diam}(g([T_-, T_+])) < \delta$. Applying Theorem 7.9 as above, we may find $f_t \in \mathcal{H}(g(t)U)$ such that $\hat{f}_t(g(t)) = \hat{f}_{T_-}(g(t))$ for all $t \in [T_-, T_+]$. Let us now suppose that $0 \leq a \leq b \leq T_+$ with $\operatorname{diam}(g([a,b])) < \delta$ and that $a \leq s \leq t \leq b$. If $t \leq T_-$ then $f_s = f_t$ on $g(s)U \cap g(t)U$ by definition of T_0. If $s, t \in [T_-, T_+]$, then arguing as above, we see that $f_s = f_{T_-}$ on $g(s)U \cap g(T_-)U$, $f_t = f_{T_-}$ on $g(t)U \cap g(T_-)U$, and therefore $f_s = f_t$ on $g(s)U \cap g(t)U \cap g(T_-)U$ which implies $f_s = f_t$ on $g(s)U \cap g(t)U$. Finally if $a \leq s \leq T_- \leq t \leq b$, then $f_s = f_{T_-}$ on $g(s)U \cap g(T_-)U$, $f_t = f_{T_-}$ on $g(t)U \cap g(T_-)U$ and so again $f_s = f_t$ on $g(s)U \cap g(t)U$. But this shows $T_0 \geq T_+ > T_0$ which is the desired contradiction and hence $T_0 = 1$.

So far we have constructed a family of functions, $\{f_t : 0 \leq t < 1\}$, with the desired properties. It only remains to extend this family to all $t \in [0,1]$ by defining $f_1 \in \mathcal{H}(g(1)U)$ so that $\hat{f}_1(g(1)) = \hat{f}_T(g(1))$, where $T \in (0,1)$ is chosen so that

$\operatorname{diam}(g([T,1])) < \delta$. Arguing as above, the reader may verify that the family of functions, $\{f_t : 0 \le t \le 1\}$, so constructed fulfill the conclusions of the theorem. Q.E.D.

Notation 7.11. When $g \in C([0,1], G)$ is a path such that $g(0) = e$ and $\alpha \in J_+^0$ write f_t^g for $f_t \in \mathcal{H}(g(t)U)$ as described in Theorem 7.10.

Theorem 7.12 (Monodromy Theorem)**.** *Let $\alpha \in J_+^0$ and $g, h \in C([0,1], G)$ such that $g(0) = h(0) = e$, $g(1) = x = h(1)$, and $d_{Riem}(g(t), h(t)) < \delta/2$ for all t. Then $f_1^g = f_1^h$ on xU.*

Proof: Let $v_t := f_t^g$, $w_t := f_t^h$, and

$$T_0 := \sup\{T \in [0,1] : v_t = w_t \text{ on } g(t)U \cap h(t)U \text{ for all } 0 \le t \le T\}. \tag{7.17}$$

Since $\hat{v}_0(e) = \hat{w}_0(e)$ we know that $v_0 = w_0$. Suppose $T > 0$ such that $\operatorname{diam}(g([0,T]) \cup h([0,T])) < \delta$ and that $t \in [0,T]$. Then $v_t = v_0$ on $g(t)U \cap U$, $w_t = w_0$ on $h(t)U \cap U$ and hence $v_t = w_t = v_0$ on $g(t)U \cap h(t)U \cap U$ and thus $v_t = w_t$ on $g(t)U \cap h(t)U$. This shows that $T_0 > 0$.

Choose $0 \le a < T_0 \le b \le 1$ such that $\operatorname{diam}(g([a,b])) < \delta/2$, $\operatorname{diam}(h([a,b])) < \delta/2$, and $b > T_0$ if $T_0 < 1$. Because $v_t = v_a$ on $g(t)U \cap g(a)U$, $w_t = w_a$ on $h(t)U \cap h(a)U$, and $v_a = w_a$ on $g(a)U \cap h(a)U$, it follows that $v_t = w_t$ on $\mathcal{O}_t := g(t)U \cap g(a)U \cap h(t)U \cap h(a)U$. Since, for $t \in [a,b]$,

$$d_{\text{Riem}}(h(t), g(a)) \le d_{\text{Riem}}(h(t), h(a)) + d_{\text{Riem}}(h(a), g(a)) < \delta/2 + \delta/2 = \delta,$$

$g(a) \in \mathcal{O}_t$ so that $\mathcal{O}_t$ is a non-empty open set contained in the connected open set, $g(t)U \cap h(t)U$. So again we conclude $v_t = w_t$ on $g(t)U \cap h(t)U$ for all $t \in [a,b]$. Hence if $T_0 < 1$, we have shown $T_0 \ge b > T_0$ which is a contraction. Thus $T_0 = b = 1$ and we have shown $v_1 = w_1$ on $g(1)U \cap h(1)U = xU$. Q.E.D.

Theorem 7.13. *Suppose that G is a simply connected complex Lie group. Then to each $\alpha \in J_+^0$, there exits a unique function $f_\alpha \in \mathcal{H}(G)$ such that $\hat{f}_\alpha(e) = \alpha$.*

Proof: For any $x \in G$, we may choose a path $g \in C([0,1], G)$ joining e to x, i.e. such that $g(0) = e$ and $g(1) = x$. We then define $f_\alpha(x) := f_1^g(x)$. If $h \in C([0,1], G)$ is another such path joining e to x, there is a homotopy, $g_t \in C([0,1], G)$ of paths joining e to x, which interpolates between g and h. By the Monodromy Theorem 7.12, one easily sees that $f_1^{g_t}$ is independent of t and in particular, $f_1^g = f_1^{g_0} = f_1^{g_1} = f_1^h$. This shows the function, f_α, is well defined.

Let $V := U_{\delta/2}^{\text{Riem}}$, $y \in xV$, $h \in C([0,1] \to xV)$ be a path joining x to y, and

$$(h * g)(t) = \begin{cases} g(2t) & \text{if } t \in \left[0, \frac{1}{2}\right] \\ h(2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Since $\operatorname{diam}\left[(h * g)\left(\left[\frac{1}{2}, 1\right]\right)\right] = \operatorname{diam}[h([0,1])] < \delta$, we know by property (2) of Theorem 7.10 that

$$f_1^{h*g} = f_{1/2}^{h*g} = f_1^g \text{ on } xU \cap yU \tag{7.18}$$

wherein we have used the (easily proved) fact that $f_t^{h*g} = f_{2t}^g$ for $t \in \left[0, \frac{1}{2}\right]$. Evaluating Eq. (7.18) at y shows, $f_\alpha(y) = f_1^g(y)$. Since $y \in xV$ was arbitrary, we have $f_\alpha = f_1^g$ on xV and hence f_α is holomorphic on xV. Since $x \in G$ was arbitrary, we have shown that f_α is holomorphic on all of G.

Taking $x = e$ and $g(t) \equiv e$ for all $t \in [0,1]$ in the above argument shows $f_\alpha = f_1^g = f_0^g$ on $V = eV$. Therefore, by construction (see Theorem 7.10), $\hat{f}_\alpha(e) = \widehat{f_0^g}(e) = \alpha$. Q.E.D.

8. Algebraic control of $\alpha \in J_t^0$ and comparison of norms

Suppose $\mathfrak{g}$ is a complex Lie algebra, G is the simply connected complex Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$, and that we are given two nonnegative Hermitian quadratic forms, say q^j, $j = 1,2$, on the complex vector space, $\mathfrak{g}^*$. Associated to these Hermitian forms by (4.1) and (3.6) are second order differential operators, Δ^j, and heat kernels, ρ_t^j, on G for $j = 1,2$.

By (2.10), each of the Hermitian forms q^j, $j = 1,2$, induces on J^0 the seminorms (possibly infinite)

$$\|\alpha\|_{q^j,t} = \left(\sum (t^k/k!)|\alpha_k|^2_{q^j_k}\right)^{1/2}, \quad t > 0, \tag{8.1}$$

and yields the family of Hilbert spaces

$$J^0_{q^j,t} = \{\alpha \in J^0 : \|\alpha\|_{q^j,t} < \infty\}, \quad t > 0.$$

We will say that the q^2 family controls the q^1 family if for each $s > 0$ there is a $t > 0$ and a constant C such that

$$\|\alpha\|_{q^1,s} \le C\|\alpha\|_{q^2,t} \tag{8.2}$$

In Section 8.1 we will show (see Theorem 8.2) that for any two quadratic forms satisfying Hörmander's condition each family controls the other. The proof depends on Theorem 7.1 and in particular on the surjectivity of the Taylor map.

In Section 8.2 we will explore combinatorial proofs of the inequalities (8.2) in case $\mathfrak{g}$ is the three dimensional complex Heisenberg Lie algebra. Such inequalities can be used to provide an alternative proof of surjectivity as explained in the following remark.

Remark 8.1. Suppose that the q^2 family controls the q^1 family as in (8.2) and that q^1 is nondegenerate. In this case we know from [8] that the Taylor maps $f \to \hat{f}(e) : \mathcal{H}L^2(G, \rho_s^1 dx) \to J^0_{q^1,s}$ are surjective. Thus if $\alpha \in J^0_{q^2,t}$ then we learn from (8.2) that $\alpha \in J^0_{q^1,s}$ and consequently there exists a function $f \in \mathcal{H}(G)$ such that $\hat{f}(e) = \alpha$. It now follows from Proposition 4.2 that $f \in \mathcal{H}L^2(G, \rho_t^2 dx)$. Therefore the Taylor map is also surjective from $\mathcal{H}L^2(\rho_t^2 dx)$ onto $J^0_{q^2,t}$.

8.1. Indirect control via the heat kernel.

In this section we will show that, as a consequence of the unitarity theorem, Theorem 7.1, the family of inequalities (8.2) holds.

Theorem 8.2. *Let $\mathfrak{g}$ be a complex Lie algebra and let q^1, q^2 be nonnegative Hermitian forms on the dual space $\mathfrak{g}^*$ which satisfy Hörmander's condition. Then there exists $\varepsilon \in (0,1)$ and $C \in (0,\infty)$ such that for any $0 < s < \varepsilon t < \infty$ and $\alpha \in J^0$, we have*

$$\|\alpha\|_{q^1,s} \le C\|\alpha\|_{q^2,t}. \tag{8.3}$$

The proof depends on the following lemma, which compares the size of the two heat kernels, and on the succeeding proposition.

Lemma 8.3. *Assume that q^1, q^2 satisfy Hörmander's condition (cf. Definition 2.6.) Then there exists $\varepsilon \in (0,1)$ such that for all $0 < s < \varepsilon t < \infty$ we have*

$$\sup_{x \in G} \left\{ \frac{\rho_s^1(x)}{\rho_t^2(x)} \right\} = C(s,t) < \infty.$$

Proof: This follows from [54, Prop. III.4.2], which states that any two proper left invariant length distances are comparable on a large scale on G, and from Theorem 3.4. These yield

$$C(s,t) \leq \exp(c_1 s^{-1} + c_2 t).$$

For nilpotent groups, the better heat kernel estimates of [54, Chap. IV] give

$$C(s,t) \leq (1 + t/s)^{c_2} \exp(c_1 s^{-1}).$$

The positive finite constants ε, c_1, c_2 depend on the group and the forms q^1, q^2 but not on s and t as long as $0 < s < \varepsilon t$. Q.E.D.

As an immediate corollary of Lemma 8.3 we have the following proposition.

Proposition 8.4. *Let G be a connected, complex Lie group. Suppose that q^1, q^2 are nonnegative Hermitian forms on the dual space $\mathfrak{g}^*$ of the complex Lie algebra of G. Assume that q^1, q^2 satisfy Hörmander's condition. Then there exist $\varepsilon \in (0,1)$ and $C \in (0,\infty)$ such that, for any $0 < s < \varepsilon t < \infty$ and for any $f \in \mathcal{H}(G)$, we have*

$$\|\hat{f}(e)\|_{q^1,s} \leq C \|\hat{f}(e)\|_{q^2,t}.$$

Proof: By Lemma 8.3, there exists $\varepsilon \in (0,1)$ such that for $0 < s < \varepsilon t$ and $f \in \mathcal{H}$,

$$\int_G |f(x)|^2 \rho_s^1(x) dx \leq C(s,t) \int_G |f(x)|^2 \rho_t^2(x) dx.$$

Hence the desired bound (with a constant $C = C(s,t)$ given by Lemma 8.3) follows from the fact that $f \mapsto \hat{f}(e)$ is an isometry from $\mathcal{H}L^2(G, \rho_\tau^j(x)dx)$ to $J^0_{q^j,\tau}$, $\tau > 0$, $j = 1,2$. To see that the constant C can be taken uniform over all $0 < s < \varepsilon t$, simply observe that the norms $\|\alpha\|_{q^i,\tau}$ are increasing functions of τ. Q.E.D.

As an application, one may take q^1 to be a positive definite Hermitian form, i.e., a form inducing a Riemannian metric on G and q^2 to be a nonnegative Hermitian form satisfying Hörmander's condition but not positive definite. Then the proposition gives control of the series

$$\sum_{k=0}^{\infty} (s^k/k!) |\hat{f}(e)|^2_{q^1,k}$$

which involves "all" Taylor coefficients of f in terms of the series

$$\sum_{k=0}^{\infty} (t^k/k!) |\hat{f}(e)|^2_{q^2,k}$$

which only involves "horizontal" Taylor coefficients of f.

Proof of Theorem 8.2: We may assume $\|\alpha\|_{q^2,t} < \infty$, for otherwise there is nothing to prove. Let G be the complex, connected, simply connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$ and let ρ_t^j, $j = 1,2$ be the heat kernels on G associated to q^j for $j = 1,2$. By Theorem 7.1, there exists $f \in \mathcal{H}(G)$ such that $\hat{f}(e) = \alpha$ and $\int_G |f(x)|^2 \rho_t^2(x) dx < \infty$. The result now follows directly form Proposition 8.4. Q.E.D.

As we shall see in Section 8.2, proving Eq. (8.3) by a direct computation involving commutators appears to be a combinatorial challenge, even under very strong assumptions.

8.2. **Direct control: Heisenberg algebra.** In this section we will give a direct combinatorial proof of the comparison inequalities (8.2) when $\mathfrak{h}_3^{\mathbb{C}}$ is the three dimensional complex Heisenberg Lie algebra, q^2 is the natural degenerate form and q^1 is a particular nondegenerate form. As already noted in Remark 8.1, the inequalities (8.2) then yield a proof of surjectivity of the Taylor map in our degenerate case quite different from the proof given in Sections 6 and 7.

Theorem 8.5. *Suppose $\mathfrak{g}$ is the complex 3 dimensional Heisenberg Lie algebra, so that $\mathfrak{g}$ is the span of X, Y, and Z with Z in the center of $\mathfrak{g}$ and $[X, Y] = Z$. Given $\alpha \in \mathfrak{g}^*$, define*

$$q^1(\alpha) = |\langle\alpha, X\rangle|^2 + |\langle\alpha, Y\rangle|^2 + |\langle\alpha, Z\rangle|^2$$

and

$$q^2(\alpha) = |\langle\alpha, X\rangle|^2 + |\langle\alpha, Y\rangle|^2 .$$

For $\alpha \in T'$, define $\|\alpha\|_{q^j,s}$ as in (8.1). Then, for $\alpha \in J^0$,

$$\|\alpha\|^2_{q^1,s} \le C(s,t)\|\alpha\|^2_{q^2.t}$$

where

$$C(s,t) = \sum_{k=0}^{\infty} (es/t)^k \left(\frac{4}{t} + 1\right)^k ,$$

which is finite provided that $(es/t)\left(\frac{4}{t}+1\right) < 1$.

Let $\mathfrak{g}$ be a Lie algebra, $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra, and for $x \in \mathfrak{g}$, let R_x and L_x denote right and left multiplication by x on $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ and $ad_x := L_x - R_x$. Hence for $\alpha \in \mathcal{U}$ we have

$$L_x\alpha = x\alpha,\ R_x\alpha = \alpha x, \text{ and } ad_x\alpha := x\alpha - \alpha x.$$

Let us recall the following basic result.

Proposition 8.6. *For each $x \in \mathfrak{g}$, ad_x acts as a derivation on $\mathcal{U}$ and e^{tad_x} is an automorphism on $\mathcal{U}$.*

Proof: The first assertion is a consequence of the following computation,

$$\begin{aligned} ad_x(\alpha\beta) &= x\alpha\beta - \alpha\beta x \\ &= (ad_x\alpha)\,\beta + \alpha x\beta - \alpha\beta x \\ &= (ad_x\alpha)\,\beta + \alpha ad_x\beta. \end{aligned}$$

For the second assertion, let $\alpha, \beta \in \mathcal{U}(\mathfrak{g})$ and let

$$\sigma(t) := e^{tad_x}\alpha \cdot e^{tad_x}\beta,$$

Then

$$\begin{aligned} \frac{d}{dt}\sigma(t) &= ad_x e^{tad_x}\alpha \cdot e^{tad_x}\beta + e^{tad_x}\alpha \cdot ad_x e^{tad_x}\beta \\ &= ad_x\sigma(t) \text{ with } \sigma(0) = \alpha\beta \end{aligned}$$

and this implies

$$e^{tad_x}\alpha \cdot e^{tad_x}\beta = \sigma(t) = e^{tad_x}(\alpha\beta),$$

i.e. e^{tad_x} is an automorphism on $\mathcal{U}$. Q.E.D.

Lemma 8.7. *Let $\mathfrak{g}$ be a Lie algebra, $x, y \in \mathfrak{g}$ be linearly independent, and suppose*

$$z := ad_x y = [x, y]$$

commutes with x and y. Then

$$z^n = \frac{1}{n!} ad_x^n y^n = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^n (-x)^{n-k}. \tag{8.4}$$

Proof: Notice that $ad_x^n y = ad_x^{n-1} z = 0$ for all $n \geq 2$ so that

$$e^{tad_x} y = y + tz$$

and hence by Proposition 8.6,

$$e^{tad_x} y^n = \left(e^{tad_x} y\right)^n = (y + tz)^n .$$

Comparing the coefficients of t^n on both sides of this equations proves the first equality in Eq. (8.4). Since L_x and R_x commute,

$$ad_x^n = (L_x - R_x)^n = \sum_{k=0}^{n} \binom{n}{k} {L_x}^k (-R_x)^{n-k} \tag{8.5}$$

and hence the second equality in Eq. (8.4) is an immediate consequence of the first and of Eq. (8.5). Q.E.D.

Proof of Theorem 8.5: Since Z is in the center of $\mathfrak{g}$ and $\alpha \in J^0$

$$|\alpha_k|_{q^1}^2 = \sum_{l=0}^{k} \binom{k}{l} \sum_{A_1, A_2, \dots A_{k-l} \in \{X,Y\}} \left| \langle \alpha, Z^{\otimes l} \otimes A_1 \otimes A_2 \otimes \cdots \otimes A_{k-l} \rangle \right|^2 . \tag{8.6}$$

Moreover

$$Z = X \otimes Y - Y \otimes X \bmod J,$$

Thus, dropping the tensor product symbol from the notation, by Lemma 8.7,

$$Z^l = \frac{1}{l!} ad_X^l Y^l = \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} X^j Y^l (-X)^{l-j} \bmod J.$$

Using this equation in Eq. (8.6) implies that

$$\begin{aligned} |\alpha_k|_{q^1}^2 &= \sum_{l=0}^{k} \binom{k}{l} \sum_{A_1, A_2, \dots A_{k-l} \in \{X,Y\}} \left| \frac{1}{l!} \sum_{j=0}^{l} \binom{l}{j} \langle \alpha, X^j Y^l (-X)^{l-j} A_1 A_2 \dots A_{k-l} \rangle \cdot 1 \right|^2 \\ &\leq \sum_{l=0}^{k} \binom{k}{l} \sum_{A_1, A_2, \dots A_{k-l} \in \{X,Y\}} \frac{2^l}{(l!)^2} \sum_{j=0}^{l} \binom{l}{j} |\langle \alpha, X^j Y^l X^{l-j} A_1 A_2 \dots A_{k-l} \rangle|^2 \\ &\leq \sum_{l=0}^{k} \binom{k}{l} \frac{2^l}{(l!)^2} \sum_{j=0}^{l} \binom{l}{j} |\alpha_{k+l}|_{q^2}^2 \end{aligned} \tag{8.7}$$

wherein the second line we have used the Cauchy Schwarz inequality with the measure $\binom{l}{j}$ along with the binomial formula.

Now suppose that

$$\|\alpha\|_{q^2,t}^2 = \sum_{k=0}^{\infty} (t^k/k!) |\alpha_k|_{q^2}^2 = M < \infty.$$

Then

$$|\alpha_{k+l}|^2_{q^2} \le M(k+l)!/t^{(k+l)}. \tag{8.8}$$

Combining Eq. (8.8) with Eq. (8.7) shows that

$$\begin{aligned}
|\alpha_k|^2_{q^1} &\le M \sum_{l=0}^{k} \binom{k}{l} \frac{2^l}{(l!)} \sum_{j=0}^{l} \binom{l}{j} (k+l)!/t^{(k+l)} \\
&= \frac{M}{t^k} \sum_{l=0}^{k} \frac{(k+l)!}{(l!)^2} \sum_{j=0}^{l} \frac{k!}{j!(l-j)!(k-l)!} \left(\frac{2}{t}\right)^l \\
&\le \frac{Mk!C_k}{t^k} \sum_{l=0}^{k} \sum_{j=0}^{l} \frac{k!}{j!(l-j)!(k-l)!} \left(\frac{2}{t}\right)^{l-j} \left(\frac{2}{t}\right)^j 1^{(k-l)} \\
&= \frac{Mk!C_k}{t^k} \left(\frac{2}{t} + \frac{2}{t} + 1\right)^k
\end{aligned}$$

wherein we have used the trinomial formula in the last line and the constant C_k is given by $C_k = \max\{(k+l)!/(k!(l!)^2) : 0 \le l \le k\}$.

In order to estimate the constant C_k we may write $(k+l)!/(k!(l!)^2)$ as a product of factors $(k+j)/(j(l+1-j)), j = 1, \ldots, l$. Let $f(x) = (k+x)/(x(l+1-x))$ for $x \in [1, l]$. Then $f'(x) = (x^2 + 2kx - k(l+1))/\{x(l+1-x)\}^2$. The quadratic formula shows that the numerator has only one zero on the positive x axis. Since $f'(l) > 0$, this zero lies to the left of $x = l$. Hence f takes its maximum on $[1, l]$ at one of the two endpoints. Since $f(l) \ge f(1)$ for $l \ge 1$ we see that

$$\frac{(k+l)!}{k!(l!)^2} \le \left(\frac{k+l}{l}\right)^l \le e^k \tag{8.9}$$

for $l \ge 1$. The overall inequality clearly holds for $l = 0$ also. Hence $C_k \le e^k$

Inserting this estimate for C_k into the previous inequalities we find that

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} |\alpha_k|^2_{q^1} \le \sum_{k=0}^{\infty} s^k e^k (4/t+1)^k \|\alpha\|^2_{q^2,t} = C(s,t) \, \|\alpha\|^2_{q^2,t}$$

where

$$C(s,t) = \sum_{k=0}^{\infty} (es/t)^k \left(\frac{4}{t} + 1\right)^k,$$

which is finite provided that $(es/t)\left(\frac{4}{t}+1\right) < 1$. Q.E.D.

9. The Fourier-Wigner transform and holomorphic functions.

We are going to show in this section how the harmonic oscillator Hamiltonian produces a natural source of holomorphic functions on the complex three dimensional Heisenberg group. There is a correspondence between analytic vectors for the quantum mechanical harmonic oscillator Hamiltonian and holomorphic functions on the complex Heisenberg group. The correspondence is induced by the Fourier-Wigner transform and also by the Wigner transform itself. The former seems easier to deal with. We will study only the Fourier-Wigner transform in this paper.

We are going to give functional analytic proofs in Section 9.1 and then reprove some parts of the main theorem in Section 9.2 by use of the known explicit kernel of the key integral operator.

9.1. **The main theorem.**

Notation 9.1. The complex Heisenberg group is $H_3^{\mathbb{C}} = \mathbb{C}^3$ with the group law

$$(z_1, z_2, z_3) \cdot (z_1', z_2', z_3') = (z_1 + z_1', z_2 + z_2', z_3 + z_3' + (1/2)(z_1 z_2' - z_2 z_1')) .$$

Let us observe here that if $z_j = x_j + iy_j$ then

$$z_1 z_2' - z_2 z_1' = [x_1 x_2' - x_2 x_1' - (y_1 y_2' - y_2 y_1')] + i[x_1 y_2' - y_2 x_1' + y_1 x_2' - x_2 y_1']$$

Define vector fields by

$$\begin{aligned} X_1 &= \partial/\partial x_1 - (x_2/2)\partial/\partial x_3 - (y_2/2)\partial/\partial y_3 \\ X_2 &= \partial/\partial x_2 + (x_1/2)\partial/\partial x_3 + (y_1/2)\partial/\partial y_3 \\ Y_1 &= \partial/\partial y_1 + (y_2/2)\partial/\partial x_3 - (x_2/2)\partial/\partial y_3 \\ Y_2 &= \partial/\partial y_2 - (y_1/2)\partial/\partial x_3 + (x_1/2)\partial/\partial y_3 \end{aligned}$$

These are the left invariant vector fields which reduce at the origin to $\partial/\partial x_1$, etc. We will use the sub-Laplacian given by

$$\Delta = X_1^2 + X_2^2 + Y_1^2 + Y_2^2 \tag{9.1}$$

Define the kernel ρ_t on $H_3^{\mathbb{C}}$ by the identity $e^{t\Delta/4} = \rho_t *$.

Notation 9.2. Let Q denote the operator of multiplication by x on $L^2(\mathbb{R})$ with its natural domain of self-adjointness and let P denote the operator $-id/dx$ with its natural domain of self-adjointness. Denote by H_0 the operator, $(1/2)\,(P^2 + Q^2)$ with its natural domain of self-adjointness.

We will show in Lemma 9.4 that for any real numbers u and v the closure of the operator $(uP + vQ)|\mathcal{S}(\mathbb{R})$ is self-adjoint. This is a very well known fact that goes back at least to J.M. Cook [5, Theorem 10]. See also [39, Theorem X.41]. These proofs show that these operators are essentially self-adjoint on any domain that contains the Hermite functions. Nevertheless we will give a short self contained proof in Lemma 9.4 because it comes right out of an identity that we will need anyway. We will always interpret the sum $uP + vQ$ as this self-adjoint operator.

The main theorem of this section is the following.

Theorem 9.3. *Let $s > 0$. Suppose that f is in the domain of e^{sH_0}. Then the Fourier-Wigner transform*

$$W(u, v, w) := e^{iw}\left(e^{i(uP+vQ)}f, f\right), \quad u, v, w \in \mathbb{R}, \tag{9.2}$$

has a unique analytic continuation to an entire function $\tilde{W}$ on $\mathbb{C}^3$. Moreover $\tilde{W}$ is in $\mathcal{H}L^2(H_3^{\mathbb{C}}, \rho_t)$ if

$$t < \frac{\tanh s}{2(1 + \tanh s)} = \frac{1}{4}(1 - e^{-2s}). \tag{9.3}$$

The proof depends on the following lemmas, of which the first is in part a precise restatement of a well known identity (cf. (9.5) below) expressing the evolution of the harmonic oscillator in the Heisenberg picture. It can be found in many elementary books on quantum mechanics. See e.g. [18, page 257].

Lemma 9.4 (Rotation in P, Q space)**.** *Let u and v be real, let $r = (u^2 + v^2)^{1/2}$ and let $s > 0$. Then $uP + vQ$ is essentially self-adjoint on $\mathcal{S}$ and*

$$\|e^{uP+vQ}e^{-sH_0}\| = \|e^{rQ}e^{-sH_0}\| \tag{9.4}$$

Proof: Since $\mathcal{S} = C^\infty(H_0)$ we have, for any real number θ, $e^{i\theta H_0}\mathcal{S} = \mathcal{S}$. Therefore, since $\mathcal{S}$ is a core for Q it is also a core for $Qe^{-i\theta H_0}$ and for $e^{i\theta H_0}Qe^{-i\theta H_0}$. Let $\epsilon > 0$ and note that range $e^{-\epsilon H_0} \subset \mathcal{S}$. Define an operator valued function of θ by

$$T(\theta) := \{e^{-i\theta H_0}e^{-\epsilon H_0}\}\{(P\sin\theta + Q\cos\theta)e^{-\epsilon H_0}\}\{e^{i\theta H_0}e^{-\epsilon H_0}\}.$$

Each of the three operators in braces is a bounded operator valued function of θ and each is differentiable with respect to θ with the operator norm on the range. Using the commutation relations $[iH_0, P] = -Q, [iH_0, Q] = P$ on $\mathcal{S}$ (and therefore on range $e^{-\epsilon H_0}$) it is straightforward to compute that $dT(\theta)/d\theta = 0$ by a computation which is easily justified, given the preceding information. Hence $T(\theta) = T(0)$ for all real θ. That is,

$$e^{-\epsilon H_0}e^{-i\theta H_0}\{P\sin\theta + Q\cos\theta\}e^{i\theta H_0}e^{-2\epsilon H_0} = e^{-\epsilon H_0}Qe^{-2\epsilon H_0}$$

We may cancel the injective operator $e^{-\epsilon H_0}$ on the left and then multiply by $e^{i\theta H_0}$ on the left and by $e^{-i\theta H_0}$ on the right to find

$$\{P\sin\theta + Q\cos\theta\}e^{-2\epsilon H_0}f = e^{i\theta H_0}Qe^{-i\theta H_0}e^{-2\epsilon H_0}f$$

for all $f \in L^2(\mathbb{R})$ and all $\epsilon > 0$. Let $g \in L^2(\mathbb{R})$ and insert $f := (H_0 + 1)^{-1}g$ into this equality. Shift the factors $(H_0+1)^{-1}$ to the left of the factors $e^{-2\epsilon H_0}$. We may then let $\epsilon \downarrow 0$ because the product to the left of the operator $e^{-2\epsilon H_0}$ on each side of the equation is a bounded operator. Since any function f in $\mathcal{S}$ may be written in the form $f = (H_0 + 1)^{-1}g$ with $g \in L^2(\mathbb{R})$, we have shown

$$\{P\sin\theta + Q\cos\theta\} = e^{i\theta H_0}Qe^{-i\theta H_0} \tag{9.5}$$

on $\mathcal{S}$. Since $\mathcal{S}$ is a core for the selfadjoint operator on the right, $\{P\sin\theta + Q\cos\theta\}$ is essentially self-adjoint on $\mathcal{S}$ and (9.5) holds on the full domain of the closure of $P\sin\theta + Q\cos\theta$. The functional calculus now shows that

$$\begin{aligned} e^{r(P\sin\theta+Q\cos\theta)}e^{-sH_0} &= e^{e^{i\theta H_0}rQe^{-i\theta H_0}}e^{-sH_0} \\ &= e^{i\theta H_0}e^{rQ}e^{-i\theta H_0}e^{-sH_0} \\ &= e^{i\theta H_0}e^{rQ}e^{-sH_0}e^{-i\theta H_0}, \end{aligned}$$

from which (9.4) follows. Q.E.D.

Notation 9.5. The ground state (lowest eigenfunction) for the operator H_0 is the function $\psi_0(x) = \pi^{-1/4}e^{-x^2/2}$. The associated ground state transformation, [32, page 71] and [6, page 458], is defined as follows. Define the ground state measure γ by $\gamma(dx) = \psi_0(x)^2dx = \pi^{-1/2}e^{-x^2}dx$. Under the unitary map $U : f \to f(x)/\psi_0(x)$ from $L^2(\mathbb{R}, dx)$ to $L^2(\mathbb{R}, \gamma)$ the Hamiltonian H_0 transforms to $UH_0U^{-1} = N + (1/2)$ where N is the Dirichlet form operator associated to the measure γ by the formula $(Nf, g)_{L^2(\gamma)} = (1/2)\int_{\mathbb{R}} f'(x)\overline{g}'(x)d\gamma(x)$. Under the unitary transform U the operator Q goes over to an operator $\hat{Q} := UQU^{-1}$, which again consists of multiplication by x (but on different functions).

Lemma 9.6 (Hypercontractive estimates)**.** *Let $s > 0$ and let r be real. Then*

$$\|e^{r\hat{Q}}e^{-sN}\|_{L^2(\gamma)\to L^2(\gamma)} \le e^{r^2/2\tanh s}. \tag{9.6}$$

Moreover

$$\|e^{rQ}e^{-sH_0}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le e^{r^2/2\tanh s}e^{-s/2} \tag{9.7}$$

and

$$\|e^{uP+vQ}e^{-sH_0}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \le e^{r^2/2\tanh s}e^{-s/2} \tag{9.8}$$

when $r = (u^2+v^2)^{1/2}$.

Proof: Let $f \in L^2(\gamma)$ and let $g = e^{-sN}f$. Define p by the equation $e^{2s} = 2p-1$. Then e^{-sN} is a contraction from $L^2(\gamma)$ to $L^{2p}(\gamma)$, [23]. [A change of variance and a change to $(1/2)\nabla^*\nabla$ cancel.] So $\|g\|_{2p} \le \|f\|_2$. Let $q = p/(p-1) = (\tanh s)^{-1}$. Then

$$\|e^{r\hat{Q}}g\|_2^2 = \int_{\mathbb{R}} e^{2rx}|g(x)|^2 d\gamma(x) \le \|e^{2rx}\|_q\|g\|_{2p}^2 \le \|e^{2rx}\|_q\|f\|_2^2.$$

But $\|e^{2rx}\|_q^q = \int_{\mathbb{R}} e^{2qrx}d\gamma(x) = e^{(2qr)^2/4}$. Hence $\|e^{2rx}\|_q = e^{qr^2}$. So

$$\|e^{r\hat{Q}}e^{-sN}f\|_2 = \|e^{r\hat{Q}}g\|_2 \le e^{qr^2/2}\|f\|_2.$$

Returning now to Lebesgue measure, the inequality (9.7) follows from (9.6) because $UH_0U^{-1} = N + (1/2)$, while $Ue^{rQ}U^{-1} = e^{r\hat{Q}}$. (9.8) now follows from (9.4) and (9.7). Q.E.D.

Lemma 9.7 (Taylor coefficient estimates)**.** *Suppose that* $n_1, \ldots, n_{2r}$ *are non-negative integers with* $n_1 + \cdots + n_{2r} = k$. *Then*

$$\|P^{n_1}Q^{n_2}\cdots Q^{n_{2r}}f\| \le 2^{k/2}\|(H_0+k)^{k/2}f\| \tag{9.9}$$

for all $f \in \mathcal{S}(\mathbb{R})$. *There is a constant* C *such that, for* $s > 0$,

$$\|P^{n_1}Q^{n_2}\cdots Q^{n_{2r}}e^{-sH_0}\| \le C\sqrt{k!}\Big(\frac{e^{2s}}{s}\Big)^{k/2}/k^{1/4}, \quad k \ge 1 \tag{9.10}$$

Proof: We are going to give a proof here for the reader's convenience. But we want to emphasize that the machinery we will use is quite well known in the literature of quantum field theory. See e.g. [39], Section X.6, Example 2 and also [39], Section X.7.

Let $a = (Q+iP)/\sqrt{2}$, interpreted as the closure of the actual sum. Then $a^* = (Q-iP)/\sqrt{2}$ (closure of sum). Moreover $\mathcal{S}(\mathbb{R})$ is a core for both operators and both leave $\mathcal{S}(\mathbb{R})$ invariant. Let $M = a^*a$. Then M is a non-negative self-adjoint operator with core $\mathcal{S}(\mathbb{R})$ and leaves $\mathcal{S}(\mathbb{R})$ invariant. One can easily verify on $\mathcal{S}(\mathbb{R})$ the identities $aa^* = a^*a+1$, $H_0 = M + (1/2)$, $Ma = a(M-1)$ and $Ma^* = a^*(M+1)$.

Since $Q = (a+a^*)/\sqrt{2}$ and $P = (a-a^*)/i\sqrt{2}$ the product $P^{n_1}Q^{n_2}\cdots Q^{n_{2r}}$ is a sum of products $A_1\cdots A_k$ with each $A_j = a$ or a^* and with an overall factor of $2^{-k/2}$ in magnitude. Hence the left side of (9.9) is at most $2^{-k/2}\sum\|A_1\cdots A_kf\|$ where the sum is over all possible choices, $A_j = a$ or a^*, for each $j \in \{1,\ldots,k\}$.

We may now use, on each of these 2^k terms, the inequality

$$\|A_1\cdots A_kf\| \le \|(M+k)^{k/2}f\|$$

stated in [40, Problem 36, page 178] and proved in [39], Section X.7. This proves (9.9).

In order to derive (9.10) note first that the range of $e^{-sH_0} \subset \mathcal{S}(\mathbb{R})$ because $\mathcal{S}(\mathbb{R})$ is exactly the set of C^∞ vectors for H_0. Taking $f = e^{-sH_0}g$ in (9.9) with $\|g\| = 1$, the inequality (9.10) can be deduced from (9.9) by observing that

$$\|P^{n_1}Q^{n_2}\cdots Q^{n_{2r}}f\| \le 2^{k/2}\|(H_0+k)^{k/2}e^{-sH_0}g\| \le 2^{k/2}\|(H_0+k)^{k/2}e^{-sH_0}\|,$$

while

$$\begin{aligned} 2^{k/2}\|(H_0+k)^{k/2}e^{-sH_0}\| &\le 2^{k/2}\sup_{u\ge 1/2}(u+k)^{k/2}e^{-s(u+k)}e^{sk} \\ &\le 2^{k/2}e^{sk}\sup_{v\ge 0} v^{k/2}e^{-sv} \\ &= 2^{k/2}e^{sk}(k/2s)^{k/2}e^{-k/2} \\ &= k^{k/2}\Big(\frac{e^{2s}}{es}\Big)^{k/2}, \end{aligned}$$

since $v^{k/2}e^{-sv}$ has a maximum on $[0,\infty)$ at $v = k/(2s)$. Stirling's formula, $k^{k/2} \sim (k!)^{1/2}e^{k/2}/(2\pi k)^{1/4}$, now shows that

$$2^{k/2}\|(H_0+k)^{k/2}e^{-sH_0}\| \sim (k!)^{1/2}(e^{2s}/s)^{k/2}/(2\pi k)^{1/4}$$

for large k. This proves (9.10). Q.E.D.

Lemma 9.8 (Convergence of power series)**.** *Let $s > 0$ and suppose that $f \in D(e^{sH_0})$. Then the power series expansion of $e^{z_1P+z_2Q}f$ in the two complex variables z_1, z_2 converges absolutely. Moreover if z_1 and z_2 are both real then the sum is $e^{z_1P+z_2Q}f$, where the exponential is defined by the spectral theorem for the self-adjoint operator z_1P+z_2Q. Similarly, if $z_1 = ib_1$ and $z_2 = ib_2$ are purely imaginary then the sum is $e^{i(b_1P+b_2Q)}f$ where the exponential is defined by the spectral theorem for the self-adjoint operator $b_1P + b_2Q$.*

Proof: We may assume that $f = e^{-sH_0}g$ with $\|g\| = 1$. Each term of the series

$$\sum_{k=0}^{\infty}\frac{(z_1P+z_2Q)^k}{k!}f$$

is well defined by Lemma 9.7 and has the form

$$(1/k!)\sum_{j=0}^{k} z_1^j z_2^{k-j}E_j f$$

where E_j is a sum of $\binom{k}{j}$ products of k factors of P and Q as in Lemma 9.7. In view of the estimate (9.10) we find

$$\begin{aligned} ||(1/k!)\sum_{j=0}^{k} z_1^j z_2^{k-j}E_j f\| &\le (1/k!)\sum_{j=0}^{k}|z_1|^j|z_2|^{k-j}\binom{k}{j}C(k!)^{1/2}\Big(\frac{e^{2s}}{s}\Big)^{k/2}/k^{1/4} \\ &\le (|z_1|+|z_2|)^k(k!)^{-1/2}C\Big(\frac{e^{2s}}{s}\Big)^{k/2}/k^{1/4}. \end{aligned}$$

Therefore the series converges absolutely.

In particular if z_1 and z_2 are real then f is an analytic vector for the self-adjoint operator $z_1P + z_2Q$. Hence the series converges to the exponential defined by the spectral theorem. A similar observation applies to $e^{i(b_1P+b_2Q)}f$. This proves Lemma 9.8. Q.E.D.

Lemma 9.9 (Power series vs. spectral theorem)**.** *Let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$. Then, for $s > 0$,*

$$e^{i(z_1P+z_2Q)}e^{-sH_0} = e^{(a_2b_1-a_1b_2)/2}e^{i(a_1P+a_2Q)}e^{-(b_1P+b_2Q)}e^{-sH_0} \qquad (9.11)$$

wherein the left side is defined as a power series as in Lemma 9.8 while the operators on the right side are all defined by the spectral theorem.

Proof: Let $f = e^{-sH_0}g$. If u_1, u_2, z_1, z_2 are all real then the Weyl form of the canonical commutation relations, cf. [11] [Equation (1.24)], imply

$$e^{i((u_1+z_1)P+(u_2+z_2)Q)}f = e^{i(u_1z_2-u_2z_1)/2}e^{i(u_1P+u_2Q)}e^{i(z_1P+z_2Q)}f \tag{9.12}$$

Since $e^{i(u_1P+u_2Q)}$ is unitary, both sides of (9.12) are analytic functions of $z_1, z_2 \in \mathbb{C}^2$. Hence (9.12) holds for all complex z_1, z_2. Choose $z_1 = iv_1$ and $z_2 = iv_2$ purely imaginary. Then (9.12) reduces to

$$e^{i((u_1+iv_1)P+(u_2+iv_2)Q)}f = e^{(u_2v_1-v_2u_1)/2}e^{i(u_1P+u_2Q)}e^{-(v_1P+v_2Q)}f$$

Q.E.D.

Lemma 9.10 (Operator bounds for complex exponents)**.** *For $s > 0$ and any two complex numbers $z_j = a_j + ib_j$ we have the operator bound*

$$\|e^{i(z_1P+z_2Q)}e^{-sH_0}\| \le e^{(a_2b_1-a_1b_2)/2}e^{(|z_1|^2+|z_2|^2)/2\tanh s}e^{-s/2} \tag{9.13}$$

Proof: If $z_j = a_j + ib_j$ for $j = 1, 2$ then, by (9.11) and (9.8),

$$\begin{aligned}\|(e^{i(z_1P+z_2Q)}e^{-sH_0}\| &= e^{(a_2b_1-a_1b_2)/2}\|e^{i(a_1P+a_2Q)}e^{-(b_1P+b_2Q)}e^{-sH_0}\| \\ &= e^{(a_2b_1-a_1b_2)/2}\|e^{-(b_1P+b_2Q)}e^{-sH_0}\| \\ &\le e^{(a_2b_1-a_1b_2)/2}e^{(b_1^2+b_2^2)/2\tanh s}e^{-s/2}.\end{aligned}$$

Q.E.D.

Lemma 9.11 (Form bounds for complex exponents)**.** *If $(z_1, z_2) \in \mathbb{C}^2$ and f is in the domain of e^{sH_0} then*

$$|\left(e^{i(z_1P+z_2Q)}f, f\right)| \le e^{m^2/(4\tanh s)}\|e^{sH_0}f\|^2e^{-s}$$

where $m^2 = |z_1|^2 + |z_2|^2$.

Proof: Observe first that $\left(e^{i(z_1P+z_2Q)}f, f\right) = \left(e^{(i/2)(z_1P+z_2Q)}f, e^{-(i/2)(\bar{z}_1P+\bar{z}_2Q)}f\right)$ when z_1 and z_2 are real. Since both sides of this equality are holomorphic in z_1 and z_2, the equality holds for all z_1 and z_2. Hence

$$\begin{aligned}|\left(e^{i(z_1P+z_2Q)}f, f\right)| &\le \|e^{(i/2)(z_1P+z_2Q)}f\|\|e^{-(i/2)(\bar{z}_1P+\bar{z}_2Q)}f\| \\ &\le \|e^{(i/2)(z_1P+z_2Q)}e^{-sH_0}\|\|e^{-(i/2)(\bar{z}_1P+\bar{z}_2Q)}e^{-sH_0}\|\|e^{sH_0}f\|^2 \\ &\le \{e^{(a_2b_1-a_1b_2)/8}e^{m^2/(8\tanh s)}e^{-s/2}\|e^{sH_0}f\|\}\cdot \\ &\qquad \{e^{-(a_2b_1-a_1b_2)/8}e^{m^2/(8\tanh s)}e^{-s/2}\|e^{sH_0}f\|\} \\ &\le e^{m^2/(4\tanh s)}e^{-s}\|e^{sH_0}f\|^2\end{aligned}$$

In the second from last line we have used (9.13) twice, but with opposite signs for the a_j. Q.E.D.

Proof: Consider the functions

$$\varphi(z) = k(|z_1|^2 + |z_2|^2)^2 + |z_3|^2, \qquad \psi(z) = \varphi(z)^{1/4}.$$

We will choose a number $k > 0$ later. Writing $m^2 = |z_1|^2 + |z_2|^2$, a computation shows that

$$X_1\varphi = 4kx_1m^2 - x_2x_3 - y_2y_3$$

$$X_2\varphi = 4kx_2m^2 + x_1x_3 + y_1y_3$$
$$Y_1\varphi = 4ky_1m^2 + y_2x_3 - x_2y_3$$
$$Y_2\varphi = 4ky_2m^2 - y_1x_3 + x_1y_3$$

Another computation then shows that

$$|\nabla\varphi(z)|^2 = m^2[16k^2m^4 + |z_3|^2].$$

If we choose $k = 1/16$ then we find

$$|\nabla\varphi(z)|^2 = m^2\varphi$$

and therefore

$$|\nabla\psi| = \frac{1}{4}\frac{|\nabla\varphi|}{\varphi^{3/4}} = \frac{m}{4\varphi^{1/4}} \le \frac{1}{2}.$$

The intrinsic distance d is defined by

$$d(x,y) = \sup\{f(x) - f(y) : f \in C^1(H_3^{\mathbb{C}}),\ |\nabla f| \le 1\}.$$

Thus the distance, $d(z)$, from the origin to (z_1, z_2, z_3) satisfies

$$d(z) \ge 2\psi = 2[2^{-4}(|z_1|^2 + |z_2|^2)^2 + |z_3|^2]^{1/4} = [(|z_1|^2 + |z_2|^2)^2 + 16|z_3|^2]^{1/4}.$$

It follows that $m^2 \le d^2(z)$ and $4|z_3| \le d^2(z)$.

Now by Lemma 9.11

$$\begin{aligned}|\tilde{W}(z_1,z_2,z_3)| &\le e^{|z_3|}e^{m^2/(4\tanh s)}\|e^{sH_0}f\|^2e^{-s}\\ &\le e^{d^2(z)\frac{1+\tanh s}{4\tanh s}}\|e^{sH_0}\|^2e^{-s}\end{aligned}$$

Hence

$$|\tilde{W}(z_1,z_2,z_3)|^2 \le e^{d^2(z)\frac{1+\tanh s}{2\tanh s}}\|e^{sH_0}f\|^4e^{-2s} \tag{9.14}$$

By [54, Theorem IV.4.2], for any $\varepsilon > 0$ there exists a finite constant C_ε such that

$$\rho_t(z) \le C_\varepsilon t^{-4}\exp\left(-\frac{d^2(z)}{4(1+\varepsilon)t}\right).$$

Here the 4 in t^{-4} is 8/2 where $8 = 4\times 1 + 2\times 2$ is the volume growth exponent of $H_3^{\mathbb{C}}$. Moreover, for any $\eta > 0$,

$$t^{-4}\int_{H_3^C} e^{-\eta d^2(z)}dz \le A_\eta < \infty.$$

Thus $\widetilde{W}$ belongs to $L^2(H_3^C, \rho_t)$ if $1/(4t) > \frac{1+\tanh s}{2\tanh s}$. That is, if (9.3) holds then $\tilde{W} \in L^2(H_3^{\mathbb{C}}, \rho_t)$. Q.E.D.

Remark 9.12. In applying Theorem IV.4.2 of [54] a reader might notice that the definition of the distance used there differs slightly from that given above. It is however a well known fact that these definitions coincide, [34].

Theorem 9.13 (Insertion of Planck's constant)**.** *Let h be a strictly positive real number. Define $P_h = -ihD$ and let*

$$H = \frac{1}{2}\left(P_h^2 + Q^2\right) \tag{9.15}$$

If f is in the domain of e^{sH} then the function

$$\tilde{W}_h(u,v,w) = e^{ihw}\left(e^{i(uP_h+vQ)}f, f\right) \tag{9.16}$$

on the real Heisenberg group H_3 *has a holomorphic extension to all of* H_3^C. *Moreover, if*

$$t < (1 - e^{-2sh})/(4h) \tag{9.17}$$

then the extension is in $\mathcal{H} \cap L^2(H_3^C, \rho_t)$

Proof: The scale transformation $(Sf)(x) = h^{-1/4} f(x/h^{1/2})$ is a unitary operator on $L^2(\mathbb{R})$ and one can compute easily that $S^{-1}P_h S = h^{1/2}P$ and $S^{-1}QS = h^{1/2}Q$. Consequently $S^{-1}(uP_h + vQ)S = h^{1/2}(uP + vQ)$ and $S^{-1}HS = hH_0$. Therefore

$$\begin{aligned}\|e^{(uP_h+vQ)}e^{-sH}\| &= \|e^{h^{1/2}(uP+vQ)}e^{-shH_0}\| \\ &\leq e^{h(|u|^2+|v|^2)/2\tanh sh}e^{-sh/2}\end{aligned}$$

The same argument leading to (9.14) now gives

$$|\tilde{W}_h(z_1, z_2, z_3)|^2 \leq e^{h(\frac{1+\tanh sh}{2\tanh sh})d^2(z)}\|e^{sH}f\|^4 e^{-2sh} \tag{9.18}$$

Consequently $\tilde{W}_h \in L^2(H_3^{\mathbb{C}}, \rho_t)$ if $1/(4t) > h(\frac{1+\tanh sh}{2\tanh sh})$. That is , if (9.17) holds. Q.E.D.

Remark 9.14. The artificial relation (9.3) between t and s should be attributed to the fact that we are analytically continuing the Fourier-Wigner transform in (9.2) rather than the Wigner transform itself, [11]. The Wigner transform will be studied from the point of view of coherent states elsewhere. We expect a more perspicuous relation between t and s in that case. An analytic continuation of the Wigner transform has already been discussed in [35] using a description of the range space which is not based on the heat kernel measure that we are using in this paper.

Remark 9.15. We might point out, however, that the condition (9.17) suggests some kind of "semiclassical limit": as $h \downarrow 0$ the relation (9.17) goes over to $t < s/2$. On the other hand, keeping h fixed and letting $s \to \infty$, the relation (9.17) goes over to $th < 1/4$. This limit can be loosely interpreted to suggest that even for the "most" regular functions f the Fourier-Wigner transform associated to Planck's constant h will be in $\mathcal{H} \cap L^2(H_3^{\mathbb{C}}, \rho_t)$ for only a bounded set of t, depending on h. In this sense Theorem 9.13 seems analogous to Theorem 4.6 in [25], according to which, the matrix elements of an irreducible unitary representation of a compact Lie group K have holomorphic extensions to the complexification of K lying in a certain L^2 space over the complexification if and only if the Casimir operator for the representation is appropriately related to the measure.

Remark 9.16. Extension of our results from the lowest dimensional Heisenberg group to higher dimensional Heisenberg groups is routine. The relation between analytic vectors for the harmonic oscillator Hamiltonian H and real analytic functions was discussed systematically in E. Nelson's paper [38]. Such a connection between the domain of e^{sH} and analytic functions was also discussed for Hamiltonians in infinitely many variables in [22].

9.2. **Alternative Proofs.** Suppose $f \in \mathcal{D}\left(e^{sH_0}\right)$ and let $F := e^{sH_0}f \in L^2(\mathbb{R})$. In this subsection we are going sketch an alternative proof to the fact that the function,

$$V(u, v) := \left(e^{i(uP+vQ)}f, f\right)$$

has an extension to an analytic function on $\mathbb{C}^2$ which satisfies the bounds in Lemma 9.11. We begin by using Mehler's formula, see for example [47, p. 38], which shows that $f = e^{-sH_0}F$ may be represented as

$$f(z) = \sqrt{\frac{1}{2\pi \sinh s}} \int_{\mathbb{R}} \exp\left\{-\frac{1}{2}\coth s \cdot (z^2 + w^2) + \frac{1}{\sinh s} zw\right\} F(w)\, dw. \tag{9.19}$$

It is now evident that f has an analytic continuation to the complex plane given by the the right hand side of Eq. (9.19). Moreover, an application of the Cauchy Schwarz inequality along with an explicit Gaussian integration shows,

$$|f(x+iy)| \le \left(\frac{1}{4\pi \sinh s \cdot \cosh s}\right)^{1/4} \|F\|_2 \exp\left(-\frac{x^2}{2}\tanh s + \frac{y^2}{2}\coth s\right). \tag{9.20}$$

According to Folland [11, p. 30],

$$V(u,v) = \int_{\mathbb{R}} e^{ivx} f(x+u/2)\, \overline{f(x-u/2)} dx.$$

Using this representation along with properties of f just described, it is easily seen that V also has an analytic continuation to $\mathbb{C}^2$ given by

$$V(z_1,z_2) = \int_{\mathbb{R}} e^{iz_2 x/2} f(x+z_1/2)\, e^{iz_2 x/2} \overline{f(x-\bar{z}_1/2)} dx.$$

Let $z_l = a_l + ib_l$. Using the Cauchy Schwarz inequality and the translation invariance of Lebesgue measure, we find

$$\begin{aligned} |V(z_1,z_2)| &\le \left\| e^{-b_2(\cdot)/2} f(\cdot + z_1/2)\right\|_2 \left\| e^{-b_2(\cdot)/2} f(\cdot - \bar{z}_1/2)\right\|_2 \\ &= \left\| e^{-b_2(\cdot)/2} f(\cdot + ib_1/2)\right\|_2^2 . \end{aligned} \tag{9.21}$$

Another application of the bound in Eq. (9.20) along with an explicit Gaussian integration, shows

$$\begin{aligned} |V(z_1,z_2)| &\le \left(\frac{1}{4\pi \sinh s \cdot \cosh s}\right)^{1/2} \|F\|_2^2\, e^{\frac{b_1^2}{4}\coth s} \int_{\mathbb{R}} e^{-x^2 \tanh s} e^{-b_2 x} dx \\ &= \frac{1}{2\sinh s} \|F\|_2^2\, e^{\frac{1}{4}\coth s \left(b_1^2 + b_2^2\right)}. \end{aligned} \tag{9.22}$$

This is the same bound appearing in Lemma 9.11 except that e^{-s} has been replaced by $(2\sinh s)^{-1} \ge e^{-s}$.

We can improve on the estimate (9.22) if we allow ourselves to use the hyper-contractivity estimate in Lemma 9.6. Indeed, it is simple to verify from Eq. (9.19) that

$$|f(x+iy)| \le e^{\coth s \cdot y^2/2} e^{-sH_0} |F|(x) \;\; \forall\; x,y \in \mathbb{R}.$$

Using this estimate in Eq. (9.21) along with the hyper-contractivity estimate in Eq. (9.7) then shows

$$\begin{aligned} |V(z_1,z_2)| &\le \left\| e^{-b_2(\cdot)/2} f(\cdot + ib_1/2)\right\|_2^2 \le e^{\frac{b_1^2}{4}\coth s} \left\| e^{-b_2(\cdot)/2} e^{-sH_0} |F|\right\|_2^2 \\ &\le e^{\frac{b_1^2}{4}\coth s} e^{\frac{b_2^2}{4}\coth s} e^{-s} \||F|\|_2^2 = e^{\coth s \cdot \left(b_1^2 + b_2^2\right)/4} e^{-s} \|F\|_2^2 . \end{aligned}$$

This is is precisely the estimate appearing in Lemma 9.11.

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