Equivalence of heat kernel measure and pinned Wiener measure on loop groups

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1 French Abridged Version

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We show that heat kernel measure and pinned Wiener measure on loop groups over simply connected compact Lie groups are equivalent.

Let $K$ be a connected compact Lie group, $e \in K$ be the identity, $\mathfrak{k} \equiv T_e K$ be the Lie algebra of $K$, and $\langle \cdot, \cdot \rangle$ be an $Ad_K$-invariant inner product on $\mathfrak{k}$. Also let $\mathcal{L}$ be the loop group in Eq. (2.1), $\mu_0^0$ be pinned Wiener measure on $\mathcal{L}$ with variance $t$ and $\nu_0^0$ be the time $t$ heat kernel measure on $\mathcal{L}$, so that $\nu_0^0$ is the law of the process $s \rightarrow \Sigma(t, s)$ described in Eq. (2.4). In [6], it was shown that the heat kernel measure $\nu_0^0$ is absolute continuous relative to the pinned Wiener measure $\mu_0^0$ and it was conjectured that these two measures are equivalent if $K$ is simply connected. In this note, we will prove in Theorem 3.2 that this conjecture is true. The idea of the proof is as follows.

Let $Z_t(\gamma) = \frac{d\nu_0^0}{d\mu_0^0}(\gamma)$, $N := \{ \gamma \in \mathcal{L} : Z_t(\gamma) = 0 \}$, and $H^1(K)$ denotes the subgroup of finite energy loops in $\mathcal{L}$. Using the quasi-invariance of the pinned measure $\mu_0^0$ proved in Malliavin and Malliavin [9] and of $\nu_0^0$ proved in Driver [4] under left translation by $H^1(K)$, one shows that $h \cdot N = N$ modulo sets of $\mu_0^0$ - measure zero for all $h \in H^1(K)$. Using Gross’ ergodicity theorem, see [8] and Aida [1], it follows that either $\mu_0^0(N) = 0$ or that $\mu_0^0(\mathcal{L} \setminus N) = 0$. Since $\nu_0^0$ is not the zero measure, we conclude that $\mu_0^0(N) = 0$, i.e. that $Z(\gamma) > 0$ on $\mathcal{L}$ for $\mu_0^0$ - a.e. $\gamma$. That is to say that $\mu_0^0$ and $\nu_0^0$ are equivalent. A generalization of this result to non-simply connected groups $K$ is given in Corollary 3.3.

2 Introduction

Let $K$ be a connected compact Lie group, $e \in K$ be the identity, $\mathfrak{k} \equiv T_e K$ be the Lie algebra of $K$ and $\langle \cdot, \cdot \rangle$ be an $Ad_K$-invariant inner product on $\mathfrak{k}$. (To simplify notation later we will, with
out loss of generality, assume that $K$ is a matrix group.) The loop group on $K$ is defined by

$$\mathcal{L} = \mathcal{L}(K) \equiv \{\gamma \in C([0,1], K) : \gamma(0) = \gamma(1) = e\}$$

(2.1)

and when $K$ is not simply connected, we let $\mathcal{L}_0$ denote those loops in $\mathcal{L}$ which are homotopic to the constant loop at $e \in K$. We view $\mathcal{L}$ as a topological space equipped with the topology of uniform convergence.

Let $\mu_t^0$ denote pinned Wiener measure with variance $t$. This measure may be described as the law of the process $\{g_s : 0 \leq s \leq 1\}$ pinned so that $g_1 = e$, where $g$ is the solution to the Stratonovich stochastic differential equation

$$dg_s = g_s \circ \sqrt{t} \, db_s \text{ with } g_0 = e.$$  

(2.2)

Here $b$ is a standard $\mathfrak{t}$-valued Brownian motion with covariance determined by $\langle \cdot, \cdot \rangle$. As introduced in [5] and [10], the heat kernel measure $\nu_t^0$ on $\mathcal{L}$ is the time $t$ distribution of a certain $\mathcal{L}$-valued Brownian motion which we now describe.

Let $\{\chi(t,s)\}_{0 \leq s \leq 1, 0 \leq t < \infty}$ be a $\mathfrak{t}$-valued Brownian bridge sheet, i.e. $\chi$ is a $\mathfrak{t}$-valued centered continuous Gaussian process such that

$$\mathbb{E}[\langle A, \chi(t,s) \rangle \langle B, \chi(t,s) \rangle] = \langle A, B \rangle \min(t,\sigma) \min(s,\sigma) - \sigma s$$

(2.3)

for all $s, \sigma \in [0,1], t, \tau \in [0, \infty)$ and $A,B \in \mathfrak{t}$. Let $\Sigma(t,s)$ for $0 \leq t < \infty$ and $0 \leq s \leq 1$ denote the solution to the Stratonovich stochastic differential equation in $t$ (with $s$ as a parameter),

$$\Sigma(dt,s) = \Sigma(t,s) \circ \chi(dt,s) \text{ with } \Sigma(0,0) = e \quad \forall s \in [0,1].$$

(2.4)

By Malliavin [10] (see also Baxendale [3] or Driver [4]), we may and do choose $\Sigma(t,s)$ to be jointly continuous in $(t,s)$. The heat kernel measure $\nu_t^0$ at time $t > 0$ is the law of the process $s \to \Sigma(t,s) \in K$. Notice by construction that $\nu_t^0(\mathcal{L}_0) = 1$. For further details and motivations for these constructions the reader is referred to [4, 5, 6].

In [6], it was shown that $\nu_t^0$ is absolute continuous relative to $\mu_t^0$ and it was conjectured that these two measures are equivalent if $K$ is simply connected. In this note, we will prove in Theorem 3.2 that this conjecture is true. The proof is based on an elementary fact (Lemma 3.1 below) on the equivalence of two quasi-invariant measures.

### 3 An elementary lemma and the equivalence of $\nu_t^0$ and $\mu_t^0$

Let $X$ be a topological group and $Y$ be a subgroup. For a finite Borel measure $\nu$ and $h \in Y$, define the new measure $\nu^h$ on $X$ by $\nu^h(A) = \nu(h \cdot A)$, where $A$ is a Borel measurable subset and $h \cdot A = \{hx \mid x \in A\}$.

We recall the basic notion of measures on groups. We say that $\nu$ is quasi-invariant relative to $Y$ if $\nu^h$ is equivalent to $\nu$ for all $h \in Y$. For a quasi-invariant measure $\nu$ relative to $Y$, $\nu$ is said to be ergodic relative to the left action of $Y$ if a Borel measurable subset $A$ satisfying $\nu((A \setminus h \cdot A) \cup (h \cdot A \setminus A)) = 0$ for any $h \in Y$, implies that either $\nu(X \setminus A) = 0$ or $\nu(A) = 0$.

**Lemma 3.1** Let $\nu_1$ and $\nu_2$ be quasi-invariant finite non-zero measures on $X$ relative to a subgroup $Y$. Let $\alpha_i^h(x) = \frac{d\nu_i^h}{d\nu_i}(x)$ $(i = 1, 2)$. Suppose that
(1) $\nu_1$ is ergodic relative to $Y$ and that

(2) $\nu_2$ is absolute continuous relative to $\nu_1$.

Then $\nu_1$ and $\nu_2$ are equivalent.

Proof. Let $Z(x) = \frac{d\nu_2}{d\nu_1}(x)$. We need only prove that $Z(x) > 0$ for $\nu_1$-a.e. $x$. We will do this by showing that $\nu_1(N) = \nu_1(\{x \in X \mid Z(x) = 0\}) = 0$.

By definition of $\alpha_i^h(x)$,

$$\int_X f(h^{-1}x)d\nu_2(x) = \int_X f(x)\alpha_i^h(x)d\nu_1(x) \quad (i = 1, 2) \tag{3.1}$$

for all bounded measurable functions $f$ on $X$ and $h \in Y$. Using Eq. (3.1) and the definition of $Z$ we have

$$\int_X f(h^{-1}x)Z(x)d\nu_1(x) = \int_X f(x)Z(hx)\alpha_i^h(x)d\nu_1(x) \tag{3.2}$$

and

$$\int_X f(h^{-1}x)d\nu_2(x) = \int_X f(x)\alpha_i^h(x)d\nu_2(x) = \int_X f(x)\alpha_i^h(x)Z(x)d\nu_1(x). \tag{3.3}$$

Since these equations hold for all bounded measurable functions, it follows

$$Z(hx)\alpha_i^h(x) = Z(x)\alpha_i^h(x) \quad \text{for } \nu_1 - \text{a.e. } x \tag{3.4}$$

for all $h \in Y$. Similar calculations show that $\alpha_i^{h^{-1}}(hx)\alpha_i^h(x) = 1$ for $\nu_i - \text{a.e. } x, i = 1, 2$ and $h \in Y$. In particular

$$\alpha_i^h(x) > 0 \quad \text{for } \nu_i - \text{a.e. } x. \tag{3.5}$$

Thus we may write Eq. (3.4) as

$$Z(hx) = Z(x)\frac{\alpha_i^h(x)}{\alpha_i^1(x)}. \tag{3.6}$$

By Eqs. (3.5) and (3.6) and the fact that $\nu_2(N) = 0$,

$$h \cdot N = N \quad \text{up to a } \nu_1 - \text{null set} \tag{3.7}$$

for all $h \in Y$. Hence by the ergodicity of $\nu_1$, either $\nu_1(X \setminus N) = 0$ or $\nu_1(N) = 0$. Since $\nu_2$ is not zero we know $\nu_1(X \setminus N) > 0$ and therefore, $\nu_1(N) = 0$ which completes the proof.

Now we are going to prove the equivalence of heat kernel measure and pinned Wiener measure. Let $\bar{\mu}_0^1$ be the restriction of the pinned Wiener measure $\mu_0^1$ to $L_0$. The absolute continuity of $\nu_0^1$ with $\mu_0^1$ was proved in [6]. Let $Z_i(\gamma) = \frac{d\nu_i^0}{d\mu_0^1}(\gamma)$.

**Theorem 3.2** The Radon–Nikodým derivative $Z_i(\gamma)$ is positive for $\bar{\mu}_0^1$-a.e. $\gamma$. Namely $\mu_0^1$ and $\nu_0^1$ are equivalent on $L_0$. 

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Proof. Let $H^1(K)$ and $H^1(K)_0$ denote the subgroups of finite energy loops in $\mathcal{L}$ and $\mathcal{L}_0$ respectively. The quasi-invariance of the un-pinned measure relative to $H^1(K)$ is standard (see [2], [7], [11]). However the quasi-invariance of pinned Wiener measure relative to $H^1(K)$ is much more delicate and is due to M.-P. Malliavin and P. Malliavin [9]. The quasi-invariance of heat kernel measure $\mu^0_t$ relative to $H^1(K)_0$ was proved in [4]. Furthermore the ergodicity of $\mu^0_t$ relative to $H^1(K)_0$ was proved by Gross [8], see also Aida [1] for this result and its generalizations. Hence the theorem is a consequence of Lemma 3.1 after setting $X = \mathcal{L}_0$, $Y = H^1(K)_0$, $\nu_1 = \mu^0_t$ and $\nu_2 = \nu^0_t$. 

Corollary 3.3 For each $h \in H^1(K)$, let $\nu^0_t(h, A) := \mu^0_t(h^{-1}A)$ - heat kernel measure on $\mathcal{L}$ starting at $h$. Let $\Pi \subset H^1(K)$ be chosen so that to each homotopy class in $\mathcal{L}$, there is a unique representative in $\Pi$. Then $\mu^0_t$ and $\sum_{h \in \Pi} \nu^0_t(h, \cdot)$ are equivalent measures on $\mathcal{L}$.

Proof. For $h \in H^1(K)$, let $\mathcal{L}_h$ denote those loops in $\mathcal{L}$ which are homotopic to $h$. By Theorem 3.2, $\nu^0_t(h, \cdot)$ is equivalent to $\mu^0_t(h^{-1}(\cdot))$ on $\mathcal{L}_{h^{-1}}$. By the quasi-invariance of $\mu^0_t$ relative to $H^1(K)$, it follows that $\mu^0_t(h^{-1}(\cdot))$ is equivalent to $\mu^0_t|_{\mathcal{L}_{h^{-1}}}$ on $\mathcal{L}_{h^{-1}}$. Therefore $\nu^0_t(h, \cdot)$ is equivalent to $\mu^0_t|_{\mathcal{L}_{h^{-1}}}$ on $\mathcal{L}_{h^{-1}}$ and hence $\sum_{h \in \Pi} \nu^0_t(h, \cdot)$ is equivalent to $\sum_{h \in \Pi} \mu^0_t|_{\mathcal{L}_{h^{-1}}} = \mu^0_t$. 

References


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