Path integrals on Riemannian Manifolds

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Newtonian Mechanics on $\mathbb{R}^d$

Given a potential energy function $V : \mathbb{R}^d \to \mathbb{R}$ we look to solve

$$m\ddot{q}(t) = -\nabla V(q(t)) \quad \text{for} \quad q(t) \in \mathbb{R}^d$$

that is

$$\text{Force} = \text{mass} \cdot \text{acceleration}$$

Recall that $p = m\dot{q}$ and

$$H(q, p) = \frac{1}{2m} p \cdot p + V(q)$$

$= \text{Conserved Energy}$

$$= E(q, \dot{q}) := \frac{1}{2m} |\dot{q}|^2 + V(q)$$
We want to find
\[ \psi (t, x) = \left( e^{\frac{t \hat{H}}{\hbar}} \psi_0 \right) (x) \]
i.e. solve the Schrödinger equation
\[ i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi (t) \text{ for } \psi (t) \in L^2 (\mathbb{R}^d) \]
with \( \psi (0, x) = \psi_0 (x) \)
where by “Canonical Quantization,”
\[ q \rightsquigarrow \hat{q} = M_q, \ p \rightsquigarrow \hat{p} = \frac{\hbar}{i} \nabla = \frac{\hbar}{i} \frac{\partial}{\partial q} \text{ and } \]
\[ H (q, p) \rightsquigarrow H (\hat{q}, \hat{p}) = -\frac{\hbar^2}{2m} \nabla^2 + M_{V(q)}. \]
Feynman explained that the solution to the Schrödinger equation should be given by

$$
\left(e^{\frac{T\hat{H}}{\hbar}}\psi_0\right)(x) = \frac{1}{Z(T)} \int_{W_{x,T}(\mathbb{R}^3)} e^{\frac{i}{\hbar} \int_0^T (K.E. - P.E.)(t) dt} \psi_0(\omega(T)) \, d\text{vol}(\omega)
$$

where $\psi_0(x)$ is the initial wave function,

$$(K.E. - P.E.) (t) = \frac{m}{2} |\dot{\omega}(t)|^2 - V(\omega(t)),$$

and

$$Z(T) = \int_{W_{x_0,T}(\mathbb{R}^3)} e^{\frac{i}{\hbar} \int_0^T (K.E.)(t) dt} \, d\text{vol}(\omega).$$

Figure 1: $W_{x,T}(\mathbb{R}^d) = \text{the paths in } \mathbb{R}^d \text{ starting at } x \text{ which are parametrized by } [0, T]$. 
The Path Integral Prescription on $\mathbb{R}^d$

**Theorem 1** (Meta-Theorem – Feynman (Kac) Quantization). Let $V : \mathbb{R}^d \to \mathbb{R}$ be a nice function and

$$W \left( \mathbb{R}^d; x, T \right) := \{ \omega \in C \left( [0, T] \to \mathbb{R}^d \right) : \omega (0) = x \}.$$  

Then

$$\left( e^{-T \hat{H}} f \right) (x) = \frac{1}{Z_T} \int_{W(\mathbb{R}^d;x,T)} e^{-\int_0^T E(\omega(t),\dot{\omega}(t)) dt} f(\omega(T)) d\omega,$$

where $E(x,v) = \frac{1}{2}m|v|^2 + V(x)$ is the classical energy and

$$Z_T := \int_{W(\mathbb{R}^d;x,T)} e^{-\frac{1}{2} \int_0^T |\dot{\omega}(t)|^2 dt} d\omega.$$
Proof of the Path Integral Prescription

**Theorem 2** (Trotter Product Formula). Let $A$ and $B$ be $n \times n$ matrices. Then

$$e^{(A+B)} = \lim_{n \to \infty} \left( e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n.$$  

**Proof:** Since

$$\frac{d}{d\varepsilon} \log(e^{\varepsilon A} e^{\varepsilon B}) = A + B,$$

$$\log(e^{\varepsilon A} e^{\varepsilon B}) = \varepsilon (A + B) + O(\varepsilon^2),$$

i.e.

$$e^{\varepsilon A} e^{\varepsilon B} = e^{\varepsilon (A+B) + O(\varepsilon^2)}$$

and therefore

$$(e^{n^{-1} A} e^{n^{-1} B})^n = \left[ e^{n^{-1} A + n^{-1} B + O(n^{-2})} \right]^n = e^{A+B+O(n^{-1})} \to e^{(A+B)} \text{ as } n \to \infty.$$
• Let $A := \frac{1}{2} \Delta$;

$$
(e^{t\Delta/2} f)(x) = \int_{\mathbb{R}^d} p_t(x, y) f(y) dy
$$

where

$$
p_t(x, y) = \left( \frac{1}{2\pi t} \right)^{d/2} \exp \left( \frac{1}{2t} |x - y|^2 \right)
$$

• Let $B = -MV$ – multiplication by $V$; $e^{-tMV} = M e^{-iV}$

• By Trotter ($x_0 := x$),

$$
\left( \left( e^{\frac{T}{n} \Delta/2} e^{-\frac{T}{n} V} \right)^n f \right)(x) = \int_{(\mathbb{R}^d)^n} p_{\frac{T}{n}}(x_0, x_1) e^{-\frac{T}{n} V(x_1)} \ldots p_{\frac{T}{n}}(x_{n-1}, x_n) e^{-\frac{T}{n} V(x_n)} f(x_n) dx_1 \ldots dx_n
$$

$$
= \frac{1}{Z_n(T)} \int_{(\mathbb{R}^d)^n} e^{-\frac{n}{2T} \sum_{i=1}^{n} |x_i - x_{i-1}|^2 - \frac{T}{n} \sum_{i=1}^{n} V(x_i)} f(x_n) dx_1 \ldots dx_n
$$

$$
= \frac{1}{Z_n(T)} \int_{H_n} e^{-\int_0^T \left[ \frac{1}{2} |\omega'(s)|^2 + V(\omega(s)) \right] ds} f(\omega(T)) d\mu_{H_n}(\omega)
$$

(3)
where \( Z_n(T) := (2\pi T/n)^{d n/2} \), \( \mathcal{P}_n = \left\{ \frac{k T}{n} \right\}_{k=0}^{n} \), and

\[
H_n = \left\{ \omega \in W\left( \mathbb{R}^d; x, T \right) : \omega''(s) = 0 \text{ for } s \notin \mathcal{P}_n \right\}.
\]

Q.E.D.
Euclidean Path Integral Quantization on $\mathbb{R}^d$

**Theorem 3** (Meta-Theorem – Path integral quantization). *We can define $\hat{H}$ by;*

$$
\left(e^{-T\hat{H}}\psi_0\right)(x)^{\prime} = \frac{1}{Z_T} \int_{\omega(0)=x} e^{-\int_0^T E(\omega(t),\dot{\omega}(t))dt}\psi_0(\omega(T))D\omega
$$

where

"$Z_T := \int_{\omega(0)=0} e^{-\frac{1}{2} \int_0^T |\dot{\omega}(t)|^2 dt}D\omega".$

and

$D\omega = "\text{Infinite Dimensional Lebesgue Measure}."

**Question:** what does this formula really mean?

1. Problems, $Z_T = \lim_{n \to \infty} Z_n(T) = 0$.

2. There is not Lebesgue measure in infinite dimensions.

3. The paths $\omega$ appearing in Eq. (4) are very rough and in fact nowhere differentiable.
Summary of Flat Results

• Let $\mathcal{P} := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0, T]$.

• Let $H_{\mathcal{P}} (\mathbb{R}^d) := \{\omega : [0, T] \to \mathbb{R}^d : \omega (0) = 0 \text{ and } \dot{\omega} (t) = 0 \forall t \notin \mathcal{P}\}$

• $\lambda_{\mathcal{P}}$ be Lebesgue measure on $H_{\mathcal{P}} (\mathbb{R}^d)$

• $Z_{\mathcal{P}} := \int_{H_{\mathcal{P}} (\mathbb{R}^d)} \exp \left(-\frac{1}{2} \int_0^T |\dot{\omega} (t)|^2 \, dt\right) \, d\lambda_{\mathcal{P}} (\omega)$

• $d\mu_{\mathcal{P}} := \frac{1}{Z_{\mathcal{P}}} \exp \left(-\frac{1}{2} \int_0^T |\dot{\omega} (t)|^2 \, dt\right) \, d\lambda_{\mathcal{P}} (\omega)$

**Theorem 4** (Wiener 1923). There exist a measure $\mu$ on $W \left([0, T], \mathbb{R}^d\right)$ such that $\mu_{\mathcal{P}} \Longrightarrow \mu$ as $|\mathcal{P}| \to 0$. 
Theorem 5 (Feynman Kac). If $E(x, v) = \frac{1}{2} |v|^2 + V(x)$ where $V$ is a nice potential, then

$$\frac{1}{Z_P} \exp \left( - \int_0^T E(\omega(t), \dot{\omega}(t)) \, dt \right) \, d\lambda_P(\omega) \implies e^{-\int_0^T V(\omega(s)) \, ds} \, d\mu(\omega)$$

and moreover,

$$\left( e^{-t\hat{H}} f \right)(0) = \lim_{|P| \to 0} \frac{1}{Z_P} \int_{H_P(\mathbb{R}^d)} \exp \left( - \int_0^T E(\omega(t), \dot{\omega}(t)) \, dt \right) f(\omega(T)) \, d\lambda_P(\omega)$$

$$= \int_{W([0,T],[\mathbb{R}^d])} e^{-\int_0^T V(\omega(s)) \, ds} f(\omega(T)) \, d\mu(\omega).$$
Norbert Wiener

Figure 2: Norbert Wiener (November 26, 1894 – March 18, 1964). Graduated High School at 11, BA at Tufts College at the age of 14, and got his Ph.D. from Harvard at 18.
Classical Mechanics on a Manifold

• Let \((M, g)\) be a Riemannian manifold.

• Newton’s Equations of motion

\[ m \frac{\nabla \dot{\sigma}(t)}{dt} = -\nabla V(q(t)), \]  

i.e.

Force = mass \cdot \text{tangential acceleration}

• In local coordinates \((q^1, \ldots, q^d)\);

\[ H(q, p) = \frac{1}{2m} g^{ij}(q) p_i p_j + V(q) \text{ where} \]

\[ ds^2 = g_{ij}(q) dq^i dq^j \]
(Not) Canonical Quantization on $M$

\[ H(q, p) = \frac{1}{2} g^{ij}(q)p_i p_j + V(q) \]
\[ = \frac{1}{2} \frac{1}{\sqrt{g}} p_i \sqrt{g} g^{ij}(q) p_j + V(q). \]

- To quantize $H(q, p)$, let
  \[ q_i \mapsto \hat{q}_i := M_{q_i}, \quad p_i \mapsto \hat{p}_i := \frac{1}{i} \frac{\partial}{\partial q^i}, \text{ and } H(q, p) \mapsto H(\hat{q}, \hat{p}). \]

- Is
  \[ \hat{H} = -\frac{1}{2} g^{ij}(q) \frac{\partial^2}{\partial q^i \partial q^j} + V(q) \]
  or is it
  \[ \hat{H} = -\frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \sqrt{g} g^{ij}(q) \frac{\partial}{\partial q^j} + V(q) = -\frac{1}{2} \Delta_M + M_V, \]
  or something else?
Path Integral Quantization of $\hat{H}$

The previous formulas on $\mathbb{R}^d$ suggest we can **define** $\hat{H}$ in the manifold setting by:

$$
\left(e^{-T\hat{H}}\psi_0\right)(x_0) = \frac{1}{Z_T} \int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt} \psi_0(\sigma(T)) \mathcal{D}\sigma
$$

where

$$
E(x, v) = \frac{1}{2} g(v, v) + V(x)
$$

is the classical energy.

- Formally, there no longer seems to be any ambiguity as there was with canonical quantization.

- On the other hand what does Eq. (6) actually mean?
Recall we now wish to mathematically interpret the expression:

\[
d\nu(\sigma) = \frac{1}{Z(T)} e^{-\int_0^T [\frac{1}{2} |\dot{\sigma}(t)|^2 + V(\sigma(t))] dt} D\sigma.
\]

To simplify life (and w.o.l.o.g.) set \( V = 0, T = 1 \) so that we will now consider,

\[
\frac{1}{Z} \int_{W_o(M)} e^{-\frac{1}{2} \int_0^1 |\dot{\sigma}(t)|^2 dt} \psi_0(\sigma(1)) D\sigma.
\]

We need introduce (recall) six geometric ingredients.
I. Geometric Wiener Measure \((\nu)\) over \(M\)

**Fact** (Cartan’s Rolling Map). Relying on Itô to handle the technical (non-differentiability) difficulties, we may transfer Wiener’s measure, \(\mu\), on \(W_{0,T}(\mathbb{R}^d)\) to a measure, \(\nu\), on \(W_{0,T}(M)\).

Figure 4: Cartan’s rolling map gives a one to one correspondance between, \(W_{0,T}(\mathbb{R}^d)\) and \(W_{0,T}(M)\).
II. Riemannian Volume Measures

- On any finite dimensional Riemannian manifold \((M, g)\) there is an associated volume measure,

\[
d\text{Vol}_g = \sqrt{\det \left( g \left( \frac{\partial \Sigma}{\partial t_i}, \frac{\partial \Sigma}{\partial t_j} \right) \right)} \, dt_1 \ldots dt_n
\]  

(7)

where \(\mathbb{R}^n \ni (t_1, \ldots, t_n) \rightarrow \Sigma(t_1, \ldots, t_n) \in M\) is a (local) parametrization of \(M\).

**Example 1.** Suppose \(M\) is 2 dimensional surface, then we teach,

\[
dS = \| \partial_{t_1} \Sigma(t_1, t_2) \times \partial_{t_2} \Sigma(t_1, t_2) \| \, dt_1 dt_2.
\]  

(8)

Combining this with the identity,

\[
\| a \times b \|^2 = \| a \|^2 \| b \|^2 - (a \cdot b)^2
\]

\[
= \det \begin{bmatrix}
a \cdot a & a \cdot b \\
a \cdot b & b \cdot b
\end{bmatrix}
\]

shows,

\[
dS = \sqrt{\det \left[ \partial_{t_1} \Sigma \cdot \partial_{t_1} \Sigma \quad \partial_{t_1} \Sigma \cdot \partial_{t_2} \Sigma \\
\partial_{t_1} \Sigma \cdot \partial_{t_2} \Sigma \quad \partial_{t_2} \Sigma \cdot \partial_{t_2} \Sigma \right]} \, dt_1 dt_2
\]

that is Eq. (7) reduces to Eq. (8) for surfaces in \(\mathbb{R}^3\).
III. Scalar Curvature

- On any finite dimensional Riemannian manifold $(M, g)$ there is an associated function called scalar curvature,

$$\text{Scal} : M \rightarrow \mathbb{R}$$

such that at a point $m \in M$,

$$\text{Vol}_g(B_\varepsilon(m)) = \left| B_{\varepsilon}^{\mathbb{R}^d}(0) \right| \left( 1 - \frac{\varepsilon^2}{6(d + 2)} \text{Scal}(m) + O(\varepsilon^3) \right) \text{ for } \varepsilon \sim 0,$$

where $\left| B_{\varepsilon}^{\mathbb{R}^d}(0) \right|$ is the volume of a $\varepsilon$ – Euclidean ball in $\mathbb{R}^d$. 
IV. Tangent Vectors in Path Spaces

• The space

\[ H(M) = \left\{ \sigma \in W_o(M) : E(\sigma) := \int_0^1 |\dot{\sigma}(t)|^2 \, dt < \infty \right\} \]

is an infinite dimensional Hilbert manifold.

• The tangent space to \( \sigma \in H(M) \) is

\[ T_\sigma H(M) = \left\{ X : [0, 1] \to TM : X(t) \in T_{\sigma(t)}M \text{ and } G^1(X, X) := \int_0^1 g\left( \frac{\nabla X(t)}{dt}, \frac{\nabla X(t)}{dt} \right) \, dt < \infty \right\}. \]
Figure 5: A tangent vector at $\sigma \in H(M)$. 
V. Piecewise Geodesics Approximations

- Given a partition $\mathcal{P}$ of $[0, 1]$ the space

$$ H_\mathcal{P}(M) = \left\{ \sigma \in W_o(M) : \frac{\nabla}{dt} \dot{\sigma}(t) = 0 \text{ for } t \notin \mathcal{P} \right\} $$

is a smooth finite dimensional embedded sub-manifold of $H(M)$. 
VI. Four Riemannian Metrics on $H_P(M)$

Let $\sigma \in H_P(M)$, and $X, Y \in T_\sigma H_P(M)$. **Metrics:**

- **$H^0$–Metric on $H(M)$**
  \[ G^0(X, X) := \int_0^1 \langle X(s), X(s) \rangle \, ds, \]

- **$H^1$–Metric on $H(M)$**
  \[ G^1(X, X) := \int_0^1 \left\langle \frac{\nabla X(s)}{ds}, \frac{\nabla X(s)}{ds} \right\rangle \, ds, \]

- **$H^1$–Metric on $H_P(M)$ (Riemannian Sum Approximation)**
  \[ G^1_P(X, Y) := \sum_{i=1}^n \left\langle \frac{\nabla X(s_{i-1}^+)}{ds}, \frac{\nabla Y(s_{i-1}^+)}{ds} \right\rangle \Delta_i s, \]

- **$H^0$–“Metric” on $H_P(M)$ (Riemannian Sum Approximation)**
  \[ G^0_P(X, Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i) \rangle \Delta_i s. \]
Riemann Sum Metric Results

**Theorem 6** (Andersson and D. JFA 1999.). *Suppose that $f : W(M) \to \mathbb{R}$ is a bounded and continuous and*

$$
d\nu_P^*(\sigma) = \frac{1}{Z_P}e^{-\frac{1}{2} \int_0^1 |\dot{\sigma}(t)|^2 dt} d\text{vol}_{G_P^*}(\sigma) \text{ for } * \in \{0, 1\}.
$$

*Then*

$$
\lim_{|P| \to 0} \int_{H_P(M)} f(\sigma) d\nu_1^P(\sigma) = \int_{W(M)} f(\sigma) d\nu(\sigma)
$$

$$
\implies \hat{H} = -\frac{1}{2} \Delta_M = -\frac{1}{2} \Delta_M + \frac{1}{\infty} \text{Scal}.
$$

*and*

$$
\lim_{|P| \to 0} \int_{H_P(M)} f(\sigma) d\nu_0^P(\sigma)
$$

$$
= \int_{W(M)} f(\sigma) e^{-\frac{1}{6} \int_0^1 \text{Scal}(\sigma(s)) ds} d\nu(\sigma)
$$

$$
\implies \hat{H} = -\frac{1}{2} \Delta_M + \frac{1}{6} \text{Scal}.
$$
Some Other (Markovian) Results

If $\hat{H}$ is “defined” by

$$e^{-T\hat{H}}f(x_0) = \frac{1}{Z_T} \int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t),\dot{\sigma}(t))dt} f(\sigma(T)) D\sigma$$

then

$$\hat{H} = -\frac{1}{2}\Delta + \frac{1}{\kappa}S$$

where

- $S$ is the scalar curvature of $M$, and
- $\kappa \in \{6, 8, 12, \infty\}$.
- $\kappa = 6$ Cheng 72.
- $\kappa = 12$, De Witt 1957, Um 73, Atsuchi & Maeda 85, and Darling 85. Geometric Quantization. (AIDA says to check these names: Atsuchi & Maeda as at least one is a given name rather than the family name.)

• Semi-group proofs and extensions of AD1999;
  - Butko (2006)
  - Bär and Frank Pfäffle, Crelle 2008.


• In the real Feynman case see for example S. Albeverio and R. Hoegh-Krohn (1976), Lapidus and Johnson, etc. etc.
Continuum $H^1$ – Metric Result

Now let

$$d\nu^1_P(\sigma) = \frac{1}{Z_P} e^{-\frac{1}{2} \int_0^1 |\dot{\sigma}(t)|^2 dt} d\text{vol}_{G^1|_{H^1_P(M)}}(\sigma).$$

**Theorem 7** (Adrian Lim 2006). (Reviews in Mathematical Physics 19 (2007), no. 9, 967–1044.) Assume $(M,g)$ satisfies,

$$0 \leq \text{Sectional-Curvatures} \leq \frac{1}{2d}.$$

If $f : W(M) \to \mathbb{R}$ is a bounded and continuous function, then

$$\lim_{|P| \to 0} \int_{H^1_P(M)} f(\sigma) \, d\nu^1_P(\sigma)$$

$$= \int_{W(M)} f(\sigma) e^{-\frac{1}{6} \int_0^1 \text{Scal}(\sigma(s)) \, ds} \sqrt{\det \left( I + \frac{1}{12} K_\sigma \right)} \, d\nu(\sigma).$$

where, for $\sigma \in H(M)$, $K_\sigma$ is a certain integral operator acting on $L^2([0, 1]; \mathbb{R}^d)$. 

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• $K_{\sigma}$ is defined by

\[
(K_{\sigma}f)(s) = \int_0^1 (s \wedge t) \Gamma_{\sigma(t)} f(t) \, dt
\]

where

\[
\Gamma_m = \sum_{i,j=1}^d \left( R_m (e_i, R_m(e_i, \cdot) e_j) e_j + R_m (e_i, R_m(e_j, \cdot) e_i) e_j \right) + R_m (e_i, R_m(e_j, \cdot) e_j) e_i
\]

Here $R_m$ is the curvature tensor at $m \in M$ and $\{e_i\}_{i=1,2,\ldots,d}$ is any orthonormal basis in $T_m(M)$.

• Adrian Lim’s limiting measure has lost the Markov property and no nice $\hat{H}$ in this case. See “Fredholm Determinant of an Integral Operator driven by a Diffusion Process,” Journal of Applied Mathematics and Stochastic Analysis, Vol. 2008, Article ID 130940.
Continuum $H^0$ – Metric Result

**Theorem 8** (Tom Laetsch: JFA 2013). If

$$d\nu_0^0(\sigma) = \frac{1}{Z_\mathcal{P}} e^{-\frac{1}{2} \int^1_0 |\dot{\sigma}(t)|^2 dt} d\text{vol}|_{H^0(\mathcal{M})}(\sigma),$$

then

$$\lim_{|\mathcal{P}| \to 0} \int_{H^0(\mathcal{M})} f(\sigma) \, d\nu_0^0(\sigma) = \int_{W(\mathcal{M})} f(\sigma) e^{-\frac{2 + \sqrt{3}}{20\sqrt{3}} \int^1_0 \text{Scal}(\sigma(s)) \, ds} \, d\nu(\sigma).$$

- The quantization implication of this result is that we should take

$$\hat{H} = -\frac{1}{2} \Delta_M + \frac{2 + \sqrt{3}}{20\sqrt{3}} \text{Scal}.$$
Summary: Quantization of Free Hamiltonian

\[ \hat{H} = -\frac{1}{2}\Delta_M + \frac{1}{\kappa}\text{Scal}. \]

- \( \kappa \in \{8, 12\} \cup \{\infty, 6, \emptyset, 10\} \).

Non Intrinsic Considerations

- Sidorova, Smolyanov, Weizsäcker, and Olaf Wittich, JFA2004, consider squeezing a ambient Brownian motion onto an embedded submanifold. This then result in

\[ \hat{H} = -\frac{1}{2}\Delta_M - \frac{1}{4}S + V_{SF} \]

where \( V_{SF} \) is a potential depending on the embedding through the second fundamental form.
Applications

**Corollary 9** (Trotter Product Formula for $e^{t\Delta/2}$). For $s > 0$ let $Q_s$ be the symmetric integral operator on $L^2(M, dx)$ defined by the kernel

$$Q_s(x, y) = (2\pi s)^{-d/2} \exp \left( -\frac{1}{2s}d^2(x, y) + \frac{s}{12}S(x) + \frac{s}{12}S(y) \right)$$

for all $x, y \in M$. Then for all continuous functions $F : M \to \mathbb{R}$ and $x \in M$,

$$(e^{s\Delta/2}F)(x) = \lim_{n \to \infty} (Q_s^{n}F)(x).$$

See also Chorin, McCracken, Huges, Marsden (78) and Wu (98).

**Proof.** This is a special case of the $L^2$ – limit theorem. The main points are:

- $\nu_0^P$ is essentially product measure on $M^n$.

- From this one shows that

$$(Q_s^{n}F)(x) \cong \int_{H^P(M)} e^s \int_0^1 S(\sigma(s)) ds F(\sigma(s)) d\nu_0^P(\sigma)$$
Corollary 2: Integration by Parts for \( \nu \) on \( W(M) \)

See Bismut, Driver, Enchev, Elworthy, Hsu, Li, Lyons, Norris, Stroock, Taniguchi,

Let \( k \in PC^1 \), and \( z \) solve:

\[
z'(s) + \frac{1}{2} \text{Ric}_{\tilde{g}(s)} z(s) = k'(s), \quad z(0) = 0.
\]

and \( f \) be a cylinder function on \( W(M) \). Then

\[
\int_{W(M)} X^z f \, d\nu = \int_{W(M)} f \int_0^1 \langle k', d\tilde{b} \rangle \, d\nu,
\]

where

\[
(X^z f)(\sigma) = \sum_{i=1}^n \langle \nabla_if(\sigma), X^z_{s_i}(\sigma) \rangle
\]

\[
= \sum_{i=1}^n \langle \nabla_if(\sigma), /s_i(\sigma) z(s_i, \sigma) \rangle
\]

and \( (\nabla_if)(\sigma) \) denotes the gradient \( F \) in the \( i^{th} \) variable evaluated at \( (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n)) \). \textbf{Proof.} Integrate by parts on \( H_P(M) \) and then pass to the limit as \( |\mathcal{P}| \rightarrow 0 \).
More Detailed Proof

Proof. Given \( k \in C^1 \cap H(T_o M) \), let \( X^P(\sigma) \in T_\sigma H_P(M) \) such that

\[
\frac{\nabla X^P_s(\sigma)}{ds}
\bigg|_{s=s_i+} = /\!/s_i(\sigma)k'(s_i+).
\]

1. \( X^P(\sigma) \) is a certain projection of \( /\!/.(\sigma)k(\cdot) \) into \( T_\sigma H_P(M) \).

2.

\[
dE(X^P) = 2 \int_0^1 \langle \sigma'(s), \frac{\nabla X^P_s}{ds} \rangle ds
= 2 \sum_{i=1}^n \langle \Delta_i b, k'(s_{i-1}+) \rangle
\]

3. \( L_{X^P} \text{Vol}_{G^1_P} = 0 \).

4. 1 & 2 imply that

\[
L_{X^P} \nu^1_P = - \sum_{i=1}^n \langle \Delta_i b, k'(s_{i-1}+) \rangle \nu^1_P.
\]
Equivalently:

$$\int_{H_P(M)} (X^{k_P} f) \nu_P^1 = \int_{H_P(M)} \sum_{i=1}^n \langle k'(s_{i-1}+), \Delta_i b \rangle f \nu_P^1.$$ 

5. After some work one shows

$$\lim_{|P| \to 0} \int_{H_P(M)} (X^{k_P} f) \nu_P^1 = \int_{W(M)} X^z f \nu$$

and

6.

$$\lim_{|P| \to 0} \int_{H_P(M)} \sum_{i=1}^n \langle k'(s_{i-1}+), \Delta_i b \rangle f \nu_P^1 = \int_{W(M)} X^z f \nu$$

7. The previous three equations and the limit theorem imply the IBP result.
Quasi-Invariance Theorem for $\nu_{W(M)}$

**Theorem 10** (D. 92, Hsu 95). Let $h \in H(T_0M)$ and $X^h$ be the $\nu_{W(M)}$ – a.e. well defined vector field on $W(M)$ given by

$$X^h_s(\sigma) = \parallel_s(\sigma)h(s) \quad \text{for} \quad s \in [0, 1].$$

(10)

Then $X^h$ admits a flow $e^{tX^h}$ on $W(M)$ and this flow leaves $\nu_{W(M)}$ quasi-invariant. (**Ref:** D. 92, Hsu 95, Enchev-Strook 95, Lyons 96, Norris 95, ...)

Bruce Driver 35
A word from our sponsor: 
Quantized Yang-Mills Fields

- A $1,000,000 question, [http://www.claymath.org/millennium-problems](http://www.claymath.org/millennium-problems)

- “... Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. ...”

- Roughly speaking one needs to make sense out of the path integral expressions above when $[0, T]$ is replaced by $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$:

\[
\begin{align*}
d\mu(A) &= \frac{1}{Z} \exp \left( -\frac{1}{2} \int_{\mathbb{R} \times \mathbb{R}^3} |F^A|^2 \, dt \, dx \right) \mathcal{D}A,
\end{align*}
\]
More Motivation: Physics proof of the Atiyah–Singer Index Theorem

Physics proof of the Atiyah–Singer Index Theorem (Alvarez-Gaumé, Friedan & Windey, Witten)

\[
\text{index}(D) = \lim_{T \to 0} \int_{L(M)} e^{-\int_0^T \left[ |\sigma'(s)|^2 - \psi(s) \cdot \nabla \psi(s) \right]} ds D\sigma D\psi
\]

\[
= C^{2n} \int_M \hat{A}(R).
\]

- Toy Model for Constructive Field Theory,
- Intuitive understanding of smoothness properties of \( \nu \).
- Heuristic path integral methods have lead to many interesting conjectures and theorems.

End