Path Integrals over a Manifold
with Lars Andersson and Adrian Lim

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Canonical Quantization

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<td>STATE SPACE</td>
<td>$T^*\mathbb{R}^d \cong \mathbb{R}^d \times \mathbb{R}^d \ni (p, q)$</td>
<td>$K = PL^2(\mathbb{R}^d, dm)$ $\psi \in L^2(\mathbb{R}^d, dm) \ni |\psi|_K = 1$.</td>
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| OBSERVABLES      | Functions on $T^*\mathbb{R}^d$ | S.A. ops. on $K$
| Examples         | $p_k$ $q_k$ | $\hat{p}_k = \hbar \frac{\partial}{\partial q_k}$ $\hat{q}_k = M_{q_k}$ $H = -\frac{\hbar^2}{2m} \Delta + V(q)$ |
| DYNAMICS         | Newtons Equations of Motion | Schrödinger, Eq. $i\hbar \dot{\psi}(t) = \hat{H}\psi(t)$, $\psi(t) \in K$ |
| MEASUREMENTS     | Evaluation $f(q, p)$ | $(\psi, \theta \psi)$ - expected value. |

The Path Integral Prescription on $\mathbb{R}^d$

Notation 1. For $x \in \mathbb{R}^d$ and $T > 0$, let

$$W(\mathbb{R}^d; x, T) := \{ \omega \in C([0, T] \to \mathbb{R}^d) : \omega(0) = x \}$$

and let

$$H(\mathbb{R}^d; T) := \{ \omega \in W(\mathbb{R}^d; T) : \int_0^T |\dot{\omega}(s)|^2 ds < \infty \}.$$
Kac’s Formula (1949) (A Rigorous Interpretation)

Theorem 3 (Kac’s Formula).
\[
e^{-TH} f(x) = \int_{W(\mathbb{R}^d; T)} e^{-\int_0^T V(x + \omega(t)) dt} f(x + \omega(T)) d\mu(\omega)
\]

where \(\mu\) is Wiener measure (1923).

Informally,
\[
d\mu(\omega) = \frac{1}{Z} e^{-\frac{1}{2} \int_0^T |\omega'(s)|^2 ds} \, D\omega.
\]

Formally, \(\mu\) is the unique measure on \(W(\mathbb{R}^d; T)\) such that
\[
\int_{W(\mathbb{R}^d; T)} e^{i\varphi(\omega)} d\mu(\omega) = \exp \left( -\frac{1}{2} \langle \varphi, \varphi \rangle_{H(\mathbb{R}^d; T)} \right)
\]
for all \(\varphi \in W(\mathbb{R}^d; T)^*\).

Classical Mechanics on a Manifold

- Let \((M, g)\) be a Riemannian manifold.

\[
\begin{align*}
\frac{d\sigma(t)}{dt} &= -\nabla V(q(t)) \quad (2)
\end{align*}
\]

Classical Energy and Hamiltonian

- \(L(x, v) := \frac{1}{2} m |v|^2_g - V(x)\) is the Lagrangian.
- \(E(x, v) := \frac{1}{2} m |v|^2_g + V(x)\) is the energy.
- \(p = \frac{\partial L(x,v)}{\partial v} = mg(v, \cdot)\) is the conjugate momentum to \(v\).
- \(H(x, p) = \frac{1}{2m} |p|^2_g + V(x)\) is the Hamiltonian.
- \(H(x, p) = E(x, v) = p(v) - L(x, v)\) where \(p = \frac{\partial L(x,v)}{\partial v} = mg(v, \cdot)\).
“Canonical” Quantization

We now set \( m = 1, \sqrt{\mathcal{G}} = \sqrt{\det (g_{ij})} \), and \( d \text{Vol} := \sqrt{\mathcal{G}} dx^1 \ldots dx^d \).

- In local coordinates,
  \[
  H = \frac{1}{2} g^{ij} \langle \dot{q}_i \dot{q}_j + \mathcal{V}(q) \rangle = \frac{1}{2} g^{ij} \left( \sqrt{g} g^{ij} \langle \dot{q}_i \dot{q}_j + \mathcal{V}(q) \rangle \right).
  \]

- Quantize:
  \[
  p_i \to \hat{p}_i := \frac{1}{i} \frac{\partial}{\partial q^i} \quad \text{and} \quad q_i \to \hat{q}_i := Mq^i.
  \]

- Then \( H \to \hat{H} \) acting on \( L^2(M, d\text{Vol}) \) by
  \[
  \hat{H} = -\frac{1}{2} \frac{1}{\sqrt{\mathcal{G}}} \frac{\partial}{\partial q^i} \left( \sqrt{\mathcal{G}} g^{ij} \frac{\partial}{\partial q^j} \right) + \mathcal{V}(q) = -\frac{1}{2} \Delta_M + M \mathcal{V}.
  \]

\[ 
\]

A Motivation: Yang – Mills Equations

- The Yang – Mills equations are the Euler Lagrange equations for
  \[
  I(A) = \int_{\mathbb{R}^d} \langle F^A \rangle^2 dtdx.
  \]

- \( \mathfrak{g} = \text{Lie}(G) \) and \( G \) is a compact Lie group.

- \( A : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \otimes \mathfrak{g} \) is a connection one form.

- \( F^A = dA + A \wedge A \) is the curvature tensor.

- \( \langle \cdot \rangle^2 \) is a non-degenerate quadratic form determined by the Lorentzian metric on \( \mathbb{R}^{d+1} \) and an inner product on \( \mathfrak{g} \).

- Path integral quantization measure is
  \[
  d\mu(A) = \frac{1}{Z} \exp \left( -\frac{1}{2} \int_{\mathbb{R}^d} |F^A|^2 dtdx \right) dA.
  \]

- \( \mu \) is to be interpreted on \( M := \mathcal{M}/G \). (See http://www.claymath.org.)

- When \( d = 1 \) and \( \mathbb{R}^d = \mathbb{R}^1 \) is replace by \( S^1 \) the space \( \mathcal{M}/G \) simply becomes \( G \) itself and the path integral in (4) reduces to the one like that in Eq. (3) with \( M = G \) and \( \mathcal{V} = 0 \). See Driver and Hall [ Comm. Math. Phys. 201 (1999).]

Path Integral Quantization of \( \hat{H} \)

\[
\left( e^{-\hat{H}T} f \right)(x_0) = \frac{1}{Z_T} \int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t), \dot{\sigma}(t))dt} f(\sigma(T)) d\sigma
\]

where \( E(x, v) \) is the classical energy as above;

\[
E(x, v) := \frac{1}{2} g(v, v) + V(x)
\]

We now set \( T = 1 \).

Goal

Make sense out of the measure \( \nu \), "defined" by

\[
d\nu(\sigma) = \frac{1}{Z} e^{-\int_0^1 [2|\dot{\sigma}(t)|^2 + V(\sigma(t))] dt} d\sigma.
\]

Some Background

If \( \hat{H} \) is "defined" by

\[
e^{-T\hat{H}} f(x_0) = \frac{1}{Z_T} \int_{\sigma(0)=x_0} e^{-\int_0^T E(\sigma(t), \dot{\sigma}(t))dt} f(\sigma(T)) d\sigma
\]

then various rigorous and not so rigorous results indicate:

\[
\hat{H} = -\frac{1}{2} \Delta_M + \frac{1}{\kappa} S
\]

where

- \( S \) is the scalar curvature of \( M \), and

- \( \kappa \in \{6, 8, 12, \infty\} \).

- For example, see Cheng 72 with \( \kappa = 6 \). Um 73, Atsushi & Maeda 85, and Darling 85. Geo. Quant. gives \( \kappa = 12 \). Also see Kärki, Topi, Niemi, Antti J. Phys. Rev. D (3) 56 (1997) – quoted below.

Remark 4 (Scalar Curvature).

\[
\text{Vol}(B_e(m)) = |B_e(0)| \left( 1 - \frac{e^2}{6(d + 2)} S(m) + O(e^3) \right)
\]
Path Spaces

**Notation 5** (Path Spaces). Given a pointed Riemannian manifold \((M, g, o)\), let
\[ W(M) = \{ \sigma \in C([0, 1] \to M) \mid \sigma(0) = o \}. \]
For those \(\sigma \in W(M)\) which are absolutely continuous, let
\[ E_M(\sigma) := \int_0^1 |\sigma'(s)|^2 \, ds \]
denote the energy of \(\sigma\). The space of finite energy paths \(H(M)\) is given by
\[ H(M) := \{ \sigma \in W(M) \mid \text{\(\sigma\) is absolutely continuous and } E_M(\sigma) < \infty \}. \]

Wiener Measure on \(W(M)\)

**Notation 6.** Let \(M\) be a Riemannian manifold with base point \(o \in M\).

**Theorem 7** (Wiener measure). There exists a unique probability measure \(\nu_{W(M)}\) on \(W(M)\) such that
\[ \int_{W(M)} F(\sigma(s_1), \ldots, \sigma(s_n)) \, d\nu_{W(M)}(\sigma) = \int_M F(x_1, \ldots, x_n) \prod_{i=0}^{n-1} p_{\Delta_i}(x_i, x_{i+1}) \, dx_1 \cdots dx_n, \]
where, \(\Delta_i := s_i - s_{i-1}, x_0 = o, dx\) denotes the volume measure on \(M\), and \(p_t(x, y) = \ker e^{\Delta^2/2}(x, y)\).

**Example 1.** When \(M = \mathbb{R}^d\),
\[ p_t(x, y) = \left( \frac{1}{2\pi t} \right)^{d/2} \exp \left( -\frac{1}{2t} |x - y|^2 \right). \]
We call, \(\mu := \nu_{W(\mathbb{R}^d)},\) classical *Wiener Measure*.

Piecewise Geodesics

- \(P := \{ 0 = s_0 < s_1 < s_2 < \ldots < s_n = 1 \}\)
- \(\Delta_i := s_i - s_{i-1}\)
- Piecewise geodesics:

\[ H_P(M) = \{ \sigma \in H(M) \mid \nabla_\sigma'(s)/ds = 0 \text{ off } P \} \]

Tangent Spaces

\[ T_\sigma H(M) = \left\{ X : [0, 1] \to TM : X(s) \in T_{\sigma(s)}M, \ X(0) = 0, \ \int_0^1 \left| \frac{\nabla X(s)}{ds} \right|^2 \, ds < \infty \right\}. \]

\[ T_\sigma H_P(M) = \{ X \in T_\sigma H(M) : X \text{ satisfies } (\text{Jacobi}) \} \]

\[ \frac{\nabla^2 X(s)}{ds^2} = R(\sigma'(s), X(s)) \sigma'(s) \text{ for } s \notin P. \] (Jacobi)
Example: $M = \mathbb{R}^d$

$$H_P(\mathbb{R}^d) := \{ \omega \in H(\mathbb{R}^d) : \omega''(s) = 0 \text{ if } s \notin P \}.$$ 

A tangent vector, $X \in T_\sigma H_P(\mathbb{R}^d)$.

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**Metrics**

Let $\sigma \in H_P(M)$, and $X, Y \in T_\sigma H_P(M)$. Metrics:

- **$H^1$–Metric on $H(M)$**
  $$G^1(X, X) := \int_0^1 \left( \frac{\nabla X(s)}{ds}, \frac{\nabla X(s)}{ds} \right) ds,$$

- **$H^1$–Metric on $H_P(M)$ (Riemannian Sum Approximation)**
  $$G_P^1(X, Y) := \sum_{i=1}^n \left( \frac{\nabla X(s_{i-1}+)}{ds}, \frac{\nabla Y(s_{i-1}+)}{ds} \right) \Delta_i,$$

- **$H^0$–Metric on $H_P(M)$ (Riemannian Sum Approximation)**
  $$G_P^0(X, Y) := \sum_{i=1}^n \langle X(s_i), Y(s_i) \rangle \Delta_i,$$

- **$H^1$–Metric restricted to $H_P(M)$ – $G^1|_{TH_P(M)}$ (the hardest case).**

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**Approximating Measures**

**Definition 8** (Approximates to Wiener Measure to $\mu_W(M)$). For each partition $P = \{0 = s_0 < s_1 < s_2 < \cdots < s_n = 1\}$ of $[0, 1]$, let $\nu_0^p$ and $\nu_1^p$ denote measures on $H_P(M)$ defined by

$$d\nu_0^p := \frac{1}{Z_0^p} e^{-\frac{1}{2} E_M} \cdot d\text{Vol}_{G_P^0},$$

$$d\nu_1^p = \frac{1}{Z_1^p} e^{-\frac{1}{2} E_M} \cdot d\text{Vol}_{G_P^1},$$

and $d\nu_P := \frac{1}{Z_P} e^{-\frac{1}{2} E_M} \cdot d\text{Vol}_{G_P|_{TH_P(M)}}$

where $E_M : H(M) \to [0, \infty)$ is the energy functional

$$E_M(\sigma) := \int_0^1 |\sigma'(s)|^2 ds$$

and $Z_0^p$ and $Z_1^p$ are normalization constants given by

$$Z_0^p := \prod_{i=1}^n (\sqrt{2\pi} (s_i - s_{i-1}))^d$$

and

$$Z_1^p := (2\pi)^{dn/2}.$$  

(6)
Flat Case \((M = \mathbb{R}^d)\) Example

- \(H^1\) and \(H^0\) – Metrics on \(H(\mathbb{R}^d)\)
  \[ G^1(h, k) := \int_0^1 \langle h'(s), k'(s) \rangle ds \quad \text{and} \quad G^0(h, k) := \int_0^1 \langle h(s), k(s) \rangle ds \]

- \(H^1\)-Metric on \(H_P(\mathbb{R}^d)\)
  \[ G^1_P(h, k) := \sum_{i=1}^n \langle h'(s_{i-1}+), k'(s_{i-1}+) \rangle \Delta_i \]

- \(H^0\)-Metric on \(H_P(\mathbb{R}^d)\)
  \[ G^0_P(h, k) := \sum_{i=1}^n \langle k(s_i), h(s_i) \rangle \Delta_i \]

Limiting Measures for \(M = \mathbb{R}^d\)

**Theorem 9** (Wiener 1923). Let

\[ \mu^1_P = \frac{1}{Z^1_P} e^{-\frac{1}{4}E_{\mathbb{R}^d} \text{Vol}_{G^1_P}}, \quad \text{and} \quad \mu^0_P = \frac{1}{Z^0_P} e^{-\frac{1}{4}E_{\mathbb{R}^d} \text{Vol}_{G^0_P}}, \]

where \(Z^1_P\) and \(Z^0_P\) are normalization constants;

\[ Z^1_P := (2\pi)^{dn/2}, \quad Z^0_P := \prod_{i=1}^n (\sqrt{2}\pi \Delta_i)^d. \]

Then

\[ \mu = \lim_{|P| \to 0} \mu^1_P = \lim_{|P| \to 0} \mu^0_P, \]

where \(\mu\) is standard Wiener measure on \(W(\mathbb{R}^d)\).

**Proof**

Let \(* \in \{0, 1\}\). For \(\omega \in H_P(\mathbb{R}^d)\), let \(x_i := \omega(s_i)\). Then one shows;

\[ \int_{H_P(\mathbb{R}^d)} f(\omega) d\mu^*_P(\omega) = \int_{W(\mathbb{R}^d)} f(\omega_P) d\mu(\omega) \]

- Now suppose \(f\) is a bounded and continuous on \(W(\mathbb{R}^d)\).
- Apply the dominated convergence theorem and uniform continuity to show

\[ \lim_{|P| \to 0} \int_{H_P(\mathbb{R}^d)} f(\omega) d\mu^*_P(\omega) = \int_{W(\mathbb{R}^d)} f(\omega_P) d\mu(\omega) = \int_{W(\mathbb{R}^d)} f(\omega) d\mu(\omega). \]
Limits in the Manifold Case

Theorem 10 (Andersson and D. 1999.). Suppose that \( f : W(M) \rightarrow \mathbb{R} \) is a bounded and continuous, then

\[
\lim_{|P| \to 0} \int_{H_{\nu}(M)} f(\sigma) d\nu_{P}^{b}(\sigma) = \int_{W(M)} f(\sigma) d\nu_{W(M)}(\sigma)
\]

and

\[
\lim_{|P| \to 0} \int_{H_{\nu}(M)} f(\sigma) d\nu_{P}^{b}(\sigma) = \int_{W(M)} f(\sigma)e^{-k|s(\sigma)|} d\nu_{W(M)}(\sigma),
\]

where \( S \) is the scalar curvature of \((M, g)\).

There is a large literature pertaining to results of the type in Theorem 10, see for example Cheng72, Um74, Pinsky78, Fujimura80, Darling84, A. Inoue and Y. Maeda 85, W. Ichinose 97 and Jyh-Yang Wu 98. The version given here is contained in Andersson and Driver 98.

Notation 11. Let \( R_p \) be the curvature tensor at \( p \in M \) and \( \{e_i\}_{i=1,2,...,d} \) is any orthonormal basis in \( T_p(M) \).

Adrian Lim’s Theorem

Theorem 12 (Adrian Lim 2006). Let \((M^d, g)\) be a \( d \)-dimensional compact Riemannian manifold such that

\[ 0 \leq \text{Sectional-Curvatures} \leq \varepsilon (d) = \frac{3}{17d} \]

and \( P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\} \).

If \( f : W(M) \rightarrow \mathbb{R} \) is a bounded and continuous function, then

\[
\lim_{|P| \to 0} \int_{H_{\nu}(M)} f(\sigma) d\nu_{P}(\sigma) = \int_{W(M)} f(\sigma)e^{-\frac{k}{n}|s(\sigma)|} ds \sqrt{\det \left( I + \frac{1}{12} K_\sigma \right)} d\nu(\sigma).
\]

where, for \( \sigma \in H(M) \), \( K_\sigma \) is the integral operator acting on \( L^2([0,1]; \mathbb{R}^d) \) defined by

\[
(K_\sigma f)(s) = \int_0^1 (s \wedge t) \Gamma_\sigma(t)f(t) dt
\]

with

\[
\Gamma_\sigma = \sum_{i,j=1}^d \left( R_p(e_i, R_p(e_i, \cdot)e_j) e_j + R_p(e_i, R_p(e_j, \cdot)e_i) e_j \right).
\]

On the proofs.

Notation 13. To each \( \sigma \in H(M) \) and \( s \in [0,1] \) let

- Parallel translation: \( /s(\sigma) : T_\sigma M \rightarrow T_{\sigma(s)} M \)

\[
\frac{\nabla}{ds} /s(\sigma) = 0 \text{ with } /s(0) = I d_{T_\sigma M}.
\]

- Cartan’s rolling map: \( \psi : H(T_\sigma M) \rightarrow H(M) \) given by \( \sigma = \psi(\omega) \) where

\[
\sigma'(s) = /s(\sigma) \omega'(s) \text{ with } \sigma(0) = \omega.
\]

Proof of the \(G^1_P \) – Theorem

On \( H_{\nu}(M) \), let

\[
\nu_{P}^1 = \frac{1}{Z_P} \exp\left( \frac{1}{2} E_M \right) vol_{G_P}.
\]

Then \( \lim_{|P| \to 0} \nu_{P}^1 = \nu_{W(M)} \).

Proof Sketch: Although the rolling map \( \psi : H(\mathbb{R}^d) \rightarrow H(M) \) is not an isomorphism, we do have (with \( \psi_P := \psi|_{H_P(\mathbb{R}^d)} \)):

1. \( \det(D\psi_P) = \det(I + T_P)^2 = 1 \) because one shows that \( T_P \) is nilpotent.
2. Equivalently: \( \psi_P^* vol_{G_P} = vol_{G_P} \)
3. \( E_M(\omega) = E_M(\psi(\omega)) \) for \( \omega \in H(\mathbb{R}^d) \).
4. 2 & 3 imply that

\[
\psi_P^* \nu_P = \nu_P.
\]

5. Eeles & Elworthy (Gangolli) show

\[
\tilde{\psi}^* \mu = \nu,
\]

where \( \tilde{\psi} : W(\mathbb{R}^d) \rightarrow W(M) \) is the stochastic version of \( \psi \).
6. 4 & 5 along with Wong and Zakai approximation theorem shows \( \lim_{|P| \to 0} \nu_{P}^1 = \nu \).
Proof of the $G^0_P$ – Theorem

On $H_P(M)$, let

$$\nu^0_P = \frac{1}{Z_P} e^{-\frac{1}{2} \Delta \nu} \text{Vol}_{g^0_P}.$$  

Then

$$\lim_{|P| \to 0} \nu^0_P = \exp \left( -\frac{1}{6} \int_0^1 S(\sigma(s)) ds \right) \nu(\sigma)$$

where $S$ is the scalar curvature of $M$.

**Proof:** One shows that

$$\nu^0_P = \rho_P \nu_P$$

and that

$$\lim_{|P| \to 0} \rho_P = \exp \left( -\frac{1}{6} \int_0^1 S(\sigma(s)) ds \right)$$

See De Witt (57), Cheng (72), Um (73), Pinski(78), Darling (84), Atsushi(85), ...

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Proof of Adrian Lim’s Theorem

**Theorem 14** (Adrian Lim 2006).

$$\lim_{|P| \to 0} \int_{H_P(M)} f(\sigma) d\nu_P(\sigma) = \int_{W(M)} f(\sigma) e^{-\frac{1}{2} \int_0^1 S(\sigma(s)) ds} \sqrt{\det \left( I + \frac{1}{12} K_\sigma \right)} d\nu(\sigma).$$

where, for $\sigma \in H(M)$, $K_\sigma$ is the integral operator acting on $L^2([0, 1]; \mathbb{R}^d)$ defined by

$$(K_\sigma f)(s) = \int_0^1 (s \wedge t) \Gamma_\sigma(t) f(t) dt$$

with

$$\Gamma_p = \sum_{i,j=1}^d \left( R_p(e_i, R_p(e_i, e_j)) e_j + R_p(e_i, R_p(e_j, e_i)) e_j \right).$$

**Proposition 15** (Formula for $\rho_n$). Let $h_{i,a}(s)$ solve

$$\frac{d^2 h(s)}{ds^2} = \Omega_{i,a}(b', h(s)) b'_i$$

with

$$h_{i,a}(0) = 0,$$

and

$$h_{i,a}(s_{j-1}^+) = \delta_{ij} e_a \text{ for } j = 1, \ldots, n.$$  

Let $Q^n$ denote the $dn \times dn$ matrix which is given in $d \times d$ blocks, $Q^n = \{ (Q^n_m)_{m,k=1}^n \}$, with

$$(Q^n_{m,k} e_a, e_c) := \int_0^1 \langle h'_m(s), h'_k(s) \rangle ds \text{ for } a, c = 1, 2, \ldots, d.$$  

Then

$$\rho^2_P = \det(n Q^n).$$

**Proposition 16.** Suppose that $M$ is a symmetric positive definite $N \times N$ matrix and $\alpha \geq 1$. Then

$$\det(M) \leq \alpha^N e^{\text{tr}(M-I)} \leq \alpha^N e^{\text{tr}(M-I)}.$$  

• Now do 60+ pages of analysis!
Corollary 17. For \( \alpha \geq 1 \),
\[
\det(nQ^n) \leq a^{nd} \exp \left( \alpha^{-1} \text{tr} \left( nQ^n - I_{nd \times nd} \right) \right) = a^{nd} \exp \left( \alpha^{-1} \sum_{m=1}^{n} \text{tr} \left( nQ^n_{m,m} - I_{d \times d} \right) \right) \leq a^{nd} \exp \left( \alpha^{-1} d \sum_{m=1}^{n} \| nQ^n_{m,m} - I_{d \times d} \| \right).
\]

\[
Q^n_{mm} = \int_0^{1/n} S^r_m (b, s)^T S^r_m (b, s) \, ds + \sum_{j=m+1}^{n} V^T_{mj} \left[ \int_0^{1/n} C'_j(b, s)^T C'_j(b, s) \, ds \right] V_{mj},
\]
where
\[
V_{mj} := \prod_{k=m+1}^{j-1} C_k(b, \Delta k s) S_m(b, \Delta m s)
\]
and \( C_j \) and \( S_j \) are certain fundamental solutions to Jacobi's equation,
\[
\frac{d^2 h(s)}{ds^2} = \Omega_m(b', h(s)) b',
\]

Proof. This is a special case of the \( L^2 \) – limit theorem. The main points are:

- \( \nu^0_P \) is essentially product measure on \( M^n \).
- From this one shows that
\[
\left( Q^n_{s/n} F \right)(x) = \lim_{n \to \infty} \left( Q^n_{s/n} F \right)(x).
\]

See also Chorin, McCracken, Hughes, Marsden (78) and Wu (98).

Application

Corollary 18 (Trotter Product Formula for \( e^{\Delta/2} \)). For \( s > 0 \) let \( Q_s \) be the symmetric integral operator on \( L^2(M, dx) \) defined by the kernel
\[
Q_s(x, y) = (2\pi s)^{-d/2} \exp \left( -\frac{1}{2s} d^2(x, y) + \frac{s}{12} S(x) + \frac{s}{12} S(y) \right)
\]
for all \( x, y \in M \). Then for all continuous functions \( F : M \to \mathbb{R} \) and \( x \in M \),
\[
\left( e^{\Delta/2} F \right)(x) = \lim_{n \to \infty} \left( Q^n_{s/n} F \right)(x).
\]

Corollary 2: Integration by Parts for \( \nu \) on \( W(M) \)

See Bismut, Driver, Enchev, Elworthy, Hsu, Li, Lyons, Norris, Stroock, Taniguchi,

Let \( k \in PC^1 \), and \( z \) solve:
\[
z' + \frac{1}{2} \text{Ric}_{\gamma, x}(z) = k'(s), \quad z(0) = 0.
\]
and \( f \) be a cylinder function on \( W(M) \). Then
\[
\int_{W(M)} X_z f \, d\nu = \int_{W(M)} f \int_0^1 \langle k', d\bar{b} \rangle \, d\nu,
\]
where
\[
(X_z f)(\sigma) = \sum_{i=1}^{n} \langle \nabla_i f \rangle(\sigma), X^i(\sigma) \rangle
\]
and \( \langle \nabla_i f \rangle(\sigma) \) denotes the gradient \( F \) in the \( i^{th} \) variable evaluated at \( (\sigma(s_1), \sigma(s_2), \ldots, \sigma(s_n)) \).

Proof

Integrate by parts in on \( H_P (M) \) and then pass to the limit as \( |P| \to 0 \).
Quasi-Invariance Theorem for $\nu_{W(M)}$

**Theorem 19** (D. 92, Hsu 95). Let $h \in H(T_0 M)$ and $X^h$ be the $\nu_{W(M)}$ – a.e. well defined vector field on $W(M)$ given by

$$X^h_\sigma(s) = /s(\sigma)h(s) \text{ for } s \in [0, 1].$$  \hspace{1cm} (13)

Then $X^h$ admits a flow $e^{tX^h}$ on $W(M)$ and this flow leaves $\nu_{W(M)}$ quasi-invariant. *(Ref: D. 92, Hsu 95, Enchev-Strook 95, Lyons 96, Norris 95, ...)*