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180B Lecture Notes, W2011

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Basic Probability Facts / Conditional Expectations

0.1 Course Notation

1. (Ω, P) will denote a probability spaces and S will denote a set which is called **state space**.
2. If S is a discrete set, i.e. finite or countable and $X : \Omega \to S$ we let
$$\rho_X(s) := P(X = s).$$
More generally if $X_i : \Omega \to S_i$ for $1 \le i \le n$ we let
$$\rho_{X_1,\dots,X_n}(\mathbf{s}) := P(X_1 = s_1, \dots, X_n = s_n)$$
for all $\mathbf{s} = (s_1, \dots, s_n) \in S_1 \times \dots \times S_n$.
3. If S is $\mathbb{R}$ or $\mathbb{R}^n$ and $X : \Omega \to S$ is a continuous random variable, we let $\rho_X(x)$ be the operability density function of X, namely,
$$\mathbb{E}[f(X)] = \int_S f(x)\,\rho_X(x)\,dx.$$
4. Given random variables X and Y we let;
   a) $\mu_X := \mathbb{E}X$ be the mean of X.
   b) $\mathrm{Var}(X) := \mathbb{E}\left[(X-\mu_X)^2\right] = \mathbb{E}X^2 - \mu_X^2$ be the variance of X.
   c) $\sigma_X = \sigma(X) := \sqrt{\mathrm{Var}(X)}$ be the standard deviation of X.
   d) $\mathrm{Cov}(X,Y) := \mathbb{E}[(X-\mu_X)(Y-\mu_Y)] = \mathbb{E}[XY] - \mu_X\mu_Y$ be the covariance of X and Y.
   e) $Corr(X,Y) := \mathrm{Cov}(X,Y)/(\sigma_X\sigma_Y)$ be the **correlation** of X and Y.

0.2 Some Discrete Distributions

Definition 0.1 (Generating Function). *Suppose that $N : \Omega \to \mathbb{N}_0$ is an integer valued random variable on a probability space, $(\Omega, \mathcal{B}, P)$. The generating function associated to N is defined by*
$$G_N(z) := \mathbb{E}\left[z^N\right] = \sum_{n=0}^{\infty} P(N=n)\,z^n \text{ for } |z| \le 1. \tag{0.1}$$

By Corollary **??**, it follows that $P(N=n) = \frac{1}{n!}G_N^{(n)}(0)$ so that G_N can be used to completely recover the distribution of N.

Proposition 0.2 (Generating Functions). *The generating function satisfies,*
$$G_N^{(k)}(z) = \mathbb{E}\left[N(N-1)\dots(N-k+1)\,z^{N-k}\right] \text{ for } |z| < 1$$
and
$$G^{(k)}(1) = \lim_{z\uparrow 1} G^{(k)}(z) = \mathbb{E}[N(N-1)\dots(N-k+1)],$$
where it is possible that one and hence both sides of this equation are infinite. In particular, $G'(1) := \lim_{z\uparrow 1} G'(z) = \mathbb{E}N$ and if $\mathbb{E}N^2 < \infty$,
$$\mathrm{Var}(N) = G''(1) + G'(1) - [G'(1)]^2. \tag{0.2}$$

Proof. By Corollary **??** for $|z| < 1$,
$$\begin{aligned} G_N^{(k)}(z) &= \sum_{n=0}^{\infty} P(N=n)\cdot n(n-1)\dots(n-k+1)\,z^{n-k} \\ &= \mathbb{E}\left[N(N-1)\dots(N-k+1)\,z^{N-k}\right]. \end{aligned} \tag{0.3}$$
Since, for $z \in (0,1)$,
$$0 \le N(N-1)\dots(N-k+1)\,z^{N-k} \uparrow N(N-1)\dots(N-k+1) \text{ as } z \uparrow 1,$$
we may apply the MCT to pass to the limit as $z \uparrow 1$ in Eq. (0.3) to find,
$$G^{(k)}(1) = \lim_{z\uparrow 1} G^{(k)}(z) = \mathbb{E}[N(N-1)\dots(N-k+1)].$$
■

Exercise 0.1 (Some Discrete Distributions). Let $p \in (0,1]$ and $\lambda > 0$. In the four parts below, the distribution of N will be described. You should work out the generating function, $G_N(z)$, in each case and use it to verify the given formulas for $\mathbb{E}N$ and $\mathrm{Var}(N)$.

1. Bernoulli(p) : $P(N=1) = p$ and $P(N=0) = 1-p$. You should find $\mathbb{E}N = p$ and $\mathrm{Var}(N) = p - p^2$.

2. Binomial(n,p) : $P(N=k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0,1,\dots,n$. ($P(N=k)$ is the probability of k successes in a sequence of n independent yes/no experiments with probability of success being p.) You should find $\mathbb{E}N = np$ and $\operatorname{Var}(N) = n\left(p-p^2\right)$.
3. Geometric(p) : $P(N=k) = p(1-p)^{k-1}$ for $k \in \mathbb{N}$. ($P(N=k)$ is the probability that the k^{th} – trial is the first time of success out a sequence of independent trials with probability of success being p.) You should find $\mathbb{E}N = 1/p$ and $\operatorname{Var}(N) = \frac{1-p}{p^2}$.
4. Poisson(λ) : $P(N=k) = \frac{\lambda^k}{k!}e^{-\lambda}$ for all $k \in \mathbb{N}_0$. You should find $\mathbb{E}N = \lambda = \operatorname{Var}(N)$.

Solution to Exercise (0.1).

1. $G_N(z) = pz^1 + (1-p)z^0 = pz + 1 - p$. Therefore, $G_N'(z) = p$ and $G_N''(z) = 0$ so that $\mathbb{E}N = p$ and $\operatorname{Var}(N) = 0 + p - p^2$.
2. $G_N(z) = \sum_{k=0}^n z^k \binom{n}{k} p^k (1-p)^{n-k} = (pz + (1-p))^n$. Therefore,

$$\begin{aligned} G_N'(z) &= n(pz + (1-p))^{n-1} p, \\ G_N''(z) &= n(n-1)(pz + (1-p))^{n-2} p^2 \end{aligned}$$

and

$$\mathbb{E}N = np \text{ and } \operatorname{Var}(N) = n(n-1)p^2 + np - (np)^2 = n\left(p - p^2\right).$$

3. For the geometric distribution,

$$G_N(z) = \mathbb{E}\left[z^N\right] = \sum_{k=1}^\infty z^k p (1-p)^{k-1} = \frac{zp}{1 - z(1-p)} \text{ for } |z| < (1-p)^{-1}.$$

Differentiating this equation in z implies,

$$\begin{aligned} \mathbb{E}\left[N z^{N-1}\right] = G_N'(z) &= \frac{p\left[1 - z(1-p)\right] + (1-p)pz}{(1 - z(1-p))^2} \\ &= \frac{p}{(1 - z(1-p))^2} \text{ and} \\ \mathbb{E}\left[N(N-1)z^{N-2}\right] = G_N''(z) &= \frac{2(1-p)p}{(1-z(1-p))^3}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}N &= G_N'(1) = 1/p, \\ \mathbb{E}\left[N(N-1)\right] &= \frac{2(1-p)p}{p^3} = \frac{2(1-p)p}{p^2}, \end{aligned}$$

and

$$\operatorname{Var}(N) = 2\frac{1-p}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}.$$

Alternative method. Starting with $\sum_{n=0}^\infty z^n = \frac{1}{1-z}$ for $|z| < 1$ we learn that

$$\begin{aligned} \frac{1}{(1-z)^2} &= \frac{d}{dz}\frac{1}{1-z} = \sum_{n=0}^\infty n z^{n-1} = \sum_{n=1}^\infty n z^{n-1} \text{ and} \\ \sum_{n=0}^\infty n^2 z^{n-1} &= \frac{d}{dz}\frac{z}{(1-z)^2} = \frac{(1-z)^2 + 2z(1-z)}{(1-z)^4} = \frac{1+z}{(1-z)^3}. \end{aligned}$$

Taking $z = 1-p$ in these formulas shows,

$$\mathbb{E}N = p\sum_{n=1}^\infty n(1-p)^{n-1} = p\frac{1}{p^2} = \frac{1}{p}$$

and

$$\mathbb{E}N^2 = p\sum_{n=1}^\infty n^2(1-p)^{n-1} = p \cdot \frac{2-p}{p^3} = \frac{2-p}{p^2}$$

and therefore,

$$\operatorname{Var}(N) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

4. In the Poisson case,

$$G_N(z) = \mathbb{E}\left[z^N\right] = \sum_{k=0}^\infty z^k \frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}e^{\lambda z} = e^{\lambda(z-1)}$$

and $G_N^{(k)}(z) = \lambda^k e^{\lambda(z-1)}$. Therefore, $\mathbb{E}N = \lambda$ and $\mathbb{E}\left[N \cdot (N-1)\right] = \lambda^2$ so that $\operatorname{Var}(N) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

Remark 0.3 (Memoryless property of the geometric distribution). Suppose that $\{X_i\}$ are i.i.d. Bernoulli random variables with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$ and $N = \inf\{i \geq 1 : X_i = 1\}$. Then $P(N = k) = P(X_1 = 0, \dots, X_{k-1} = 0, X_k = 1) = (1-p)^{k-1}p$, so that N is geometric with parameter p. Using this representation we easily and intuitively see that

$$\begin{aligned} P(N = n+k | N > n) &= \frac{P(X_1 = 0, \dots, X_{n+k-1} = 0, X_{n+k} = 1)}{P(X_1 = 0, \dots, X_n = 0)} \\ &= P(X_{n+1} = 0, \dots, X_{n+k-1} = 0, X_{n+k} = 1) \\ &= P(X_1 = 0, \dots, X_{k-1} = 0, X_k = 1) = P(N = k). \end{aligned}$$

This can be verified by first principles as well;

$$\begin{aligned}
P(N=n+k|N>n) &= \frac{P(N=n+k)}{P(N>n)} = \frac{p(1-p)^{n+k-1}}{\sum_{k>n} p(1-p)^{k-1}} \\
&= \frac{p(1-p)^{n+k-1}}{\sum_{j=0}^{\infty} p(1-p)^{n+j}} = \frac{(1-p)^{n+k-1}}{(1-p)^n \sum_{j=0}^{\infty} (1-p)^j} \\
&= \frac{(1-p)^{k-1}}{\frac{1}{1-(1-p)}} = p(1-p)^{k-1} = P(N=k).
\end{aligned}$$

Exercise 0.2. Let $S_{n,p} \overset{d}{=} \text{Binomial}(n,p)$, $k \in \mathbb{N}$, $p_n = \lambda_n/n$ where $\lambda_n \to \lambda > 0$ as $n \to \infty$. Show that

$$\lim_{n\to\infty} P(S_{n,p_n} = k) = \frac{\lambda^k}{k!} e^{-\lambda} = P(\text{Poisson}(\lambda) = k).$$

Thus we see that for $p = O(1/n)$ and k not too large relative to n that for large n,

$$P(\text{Binomial}(n,p) = k) \cong P(\text{Poisson}(pn) = k) = \frac{(pn)^k}{k!} e^{-pn}.$$

(We will come back to the Poisson distribution and the related Poisson process later on.)

Solution to Exercise (0.2). We have,

$$\begin{aligned}
P(S_{n,p_n} = k) &= \binom{n}{k} (\lambda_n/n)^k (1-\lambda_n/n)^{n-k} \\
&= \frac{\lambda_n^k}{k!} \frac{n(n-1)\dots(n-k+1)}{n^k} (1-\lambda_n/n)^{n-k}.
\end{aligned}$$

The result now follows since,

$$\lim_{n\to\infty} \frac{n(n-1)\dots(n-k+1)}{n^k} = 1$$

and

$$\begin{aligned}
\lim_{n\to\infty} \ln(1-\lambda_n/n)^{n-k} &= \lim_{n\to\infty} (n-k) \ln(1-\lambda_n/n) \\
&= -\lim_{n\to\infty} [(n-k)\lambda_n/n] = -\lambda.
\end{aligned}$$

1

Course Overview and Plan

This course is an introduction to some basic topics in the theory of stochastic processes. After finishing the discussion of multivariate distributions and conditional probabilities initiated in Math 180A, we will study Markov chains in discrete time. We then begin our investigation of stochastic processes in continuous time with a detailed discussion of the Poisson process. These two topics will be combined in Math 180C when we study Markov chains in continuous time and renewal processes.

In the next two quarters we will study some aspects of Stochastic Processes. Stochastic (from the Greek $\sigma\tau\acute{o}\chi o\xi$ for aim or guess) means random. A stochastic process is one whose behavior is non-deterministic, in that a system's subsequent state is determined both by the process's predictable actions and by a random element. However, according to M. Kac[1] and E. Nelson[2], any kind of time development (be it deterministic or essentially probabilistic) which is analyzable in terms of probability deserves the name of stochastic process.

Mathematically we will be interested in collection of random variables or vectors $\{X_t\}_{t\in T}$ with $X_t : \Omega \to S$ (S is the **state space**) on some probability space, (Ω, P). Here T is typically in $\mathbb{R}_+$ or $\mathbb{Z}_+$ but not always.

Example 1.1. 1. X_t is the value of a spinner at times $t \in \mathbb{Z}_+$.
2. X_t denotes the prices of a stock (or stocks) on the stock market.
3. X_t denotes the value of your portfolio at time t.
4. X_t is the position of a dust particle like in Brownian motion.
5. X_A is the number of stars in a region A contained in space or the number of raisins in a region of a cake, etc.
6. $X_n \in S = Perm(\{1, \dots, 52\})$ is the ordering of cards in a deck of cards after the n^{th} shuffle.

Our goal in this course is to introduce and analyze models for such random objects. This is clearly going to require that we make assumptions on $\{X_t\}$ which will typically be some sort of dependency structures. This is where we will begin our study – namely heading towards conditional expectations and related topics.

1.1 180B Course Topics:

1. Review the linear algebra of orthogonal projections in the context of least squares approximations in the context of Probability Theory.
2. Use the least squares theory to interpret covariance and correlations.
3. Review of conditional probabilities for discrete random variables.
4. Introduce conditional expectations as least square approximations.
5. Develop conditional expectation relative to discrete random variables.
6. Give a short introduction to martingale theory.
7. Study in some detail discrete time Markov chains.
8. Review of conditional probability densities for continuous random variables.
9. Develop conditional expectations relative to continuous random variables.
10. Begin our study of the Poisson process.

The bulk of this quarter will involve the study of Markov chains and processes. These are processes for which the past and future are independent given the present. This is a typical example of a dependency structure that we will consider in this course. For an example of such a process, let $S = \mathbb{Z}$ and place a coin at each site of S (perhaps the coins are biased with different probabilities of heads at each site of S.) Let $X_0 = s_0$ be some point in S be fixed and then flip the coin at s_0 and move to the right on step if the result is heads and to left one step if the result is tails. Repeat this process to determine the position X_{n+1} from the position X_n along with a flip of the coin at X_n. This is a typical example of a Markov process.

Before going into these and other processes in more detail we are going to develop the extremely important concept of **conditional expectation.** The idea is as follows. Suppose that X and Y are two random variables with $\mathbb{E}|Y|^2 < \infty$. We wish to find the function h such that $h(X)$ is the minimizer of $\mathbb{E}(Y - f(X))^2$ over all functions f such that $\mathbb{E}\left[f(X)^2\right] < \infty$, that is $h(X)$ is a least squares approximation to Y among random variables of the form $f(X)$, i.e.

$$\mathbb{E}(Y - h(X))^2 = \min_f \mathbb{E}(Y - f(X))^2 . \tag{1.1}$$

Fact: a minimizing function h always exist and is "essentially unique." We denote $h(X)$ as $\mathbb{E}[Y|X]$ and call it the **conditional expectation of Y given**

[1] M. Kac & J. Logan, in Fluctuation Phenomena, eds. E.W. Montroll & J.L. Lebowitz, North-Holland, Amsterdam, 1976.

[2] E. Nelson, Quantum Fluctuations, Princeton University Press, Princeton, 1985.

X. We are going to spend a fair amount of time filling in the details of this construction and becoming familiar with this concept.

As a warm up to conditional expectation, we are going to consider a simpler problem of best linear approximations. The goal now is to find $a_0, b_0 \in \mathbb{R}$ such that

$$\mathbb{E}\left(Y - a_0 X + b_0\right)^2 = \min_{a,b\in\mathbb{R}} \mathbb{E}\left(Y - aX + b\right)^2 . \tag{1.2}$$

This is the same sort of problem as finding conditional expectations except we now only allow consider functions of the form $f(x) = ax + b$. (You should be able to find a_0 and b_0 using the first derivative test from calculus! We will carry this out using linear algebra ideas below.) It turns out the answer to finding (a_0, b_0) solving Eq. (1.2) only requires knowing the first and second moments of X and Y and $\mathbb{E}[XY]$. On the other hand finding $h(X)$ solving Eq. (1.1) require full knowledge of the joint distribution of (X, Y).

By the way, you are asked to show on your first homework that $\min_{c\in\mathbb{R}} \mathbb{E}(Y-c)^2 = \operatorname{Var}(Y)$ which occurs for $c = \mathbb{E}Y$. Thus $\mathbb{E}Y$ is the least squares approximation to Y by a constant function and $\operatorname{Var}(Y)$ is the least square error associated with this problem.

2

Covariance and Correlation

Suppose that (Ω, P) is a probability space. We say that $X : \Omega \to \mathbb{R}$ is **integrable** if $\mathbb{E}\left|X\right| < \infty$ and X is **square integrable** if $\mathbb{E}\left|X\right|^2 < \infty$. We denote the set of integrable random variables by $L^1(P)$ and the square integrable random variables by $L^2(P)$. When X is integrable we let $\mu_X := \mathbb{E}X$ be the **mean** of X. If Ω is a finite set, then

$$\mathbb{E}\left[|X|^p\right] = \sum_{\omega \in \Omega} |X(\omega)|^p P(\{\omega\})) < \infty$$

for any $0 < p < \infty$. So when the sample space is finite requiring integrability or square integrability is no restriction at all. On the other hand when Ω is infinite life can become a little more complicated.

Example 2.1. Suppose that N is a geometric with parameter p so that $P(N = k) = p(1-p)^{k-1}$ for $k \in \mathbb{N} = \{1, 2, 3, \dots\}$. If $X = f(N)$ for some function $f : \mathbb{N} \to \mathbb{R}$, then

$$\mathbb{E}\left[f(N)\right] = \sum_{k=1}^{\infty} p(1-p)^{k-1} f(k)$$

when the sum makes sense. So if $X_\lambda = \lambda^N$ for some $\lambda > 0$ we have

$$\mathbb{E}\left[X_\lambda^2\right] = \sum_{k=1}^{\infty} p(1-p)^{k-1}\lambda^{2k} = p\lambda^2 \sum_{k=1}^{\infty}\left[(1-p)\lambda^2\right]^{k-1} < \infty$$

iff $(1-p)\lambda^2 < 1$, i.e. $\lambda < 1/\sqrt{1-p}$. Thus we see that $X_\lambda \in L^2(P)$ iff $\lambda < 1/\sqrt{1-p}$.

Lemma 2.2. *$L^2(P)$ is a subspace of the vector space of random variables on (Ω, P). Moreover if $X, Y \in L^2(P)$, then $XY \in L^1(P)$ and in particular (take $Y = 1$) it follows that $L^2(P) \subset L^1(P)$.*

Proof. If $X, Y \in L^2(P)$ and $c \in \mathbb{R}$ then $\mathbb{E}|cX|^2 = c^2\mathbb{E}|X|^2 < \infty$ so that $cX \in L^2(P)$. Since

$$0 \le (|X| - |Y|)^2 = |X|^2 + |Y|^2 - 2|X||Y|,$$

it follows that

$$|XY| \le \frac{1}{2}|X|^2 + \frac{1}{2}|Y|^2 \in L^1(P).$$

Moreover,

$$(X+Y)^2 = X^2 + Y^2 + 2XY \le X^2 + Y^2 + 2|XY| \le 2\left(X^2 + Y^2\right)$$

from which it follows that $\mathbb{E}(X+Y)^2 < \infty$, i.e. $X + Y \in L^2(P)$. ■

Definition 2.3. *The **covariance,** $\operatorname{Cov}(X, Y)$, of two square integrable random variables, X and Y, is defined by*

$$\operatorname{Cov}(X, Y) = \mathbb{E}\left[(X - \mu_X)(Y - \mu_Y)\right] = \mathbb{E}\left[XY\right] - \mathbb{E}X \cdot \mathbb{E}Y$$

*where $\mu_X := \mathbb{E}X$ and $\mu_Y := \mathbb{E}Y$. The **variance** of X,*

$$\operatorname{Var}(X) = \operatorname{Cov}(X, X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}X)^2 \tag{2.1}$$

$$= \mathbb{E}\left[(X - \mu_X)^2\right] \tag{2.2}$$

*We say that X and Y are **uncorrelated** if $\operatorname{Cov}(X, Y) = 0$, i.e. $\mathbb{E}[XY] = \mathbb{E}X \cdot \mathbb{E}Y$. More generally we say $\{X_k\}_{k=1}^n \subset L^2(P)$ are **uncorrelated** iff $\operatorname{Cov}(X_i, X_j) = 0$ for all $i \neq j$.*

Definition 2.4 (Correlation). *Given two non-constant random variables we define $Corr(X, Y) := \frac{\operatorname{Cov}(X,Y)}{\sigma(X)\cdot\sigma(Y)}$ to be the **correlation** of X and Y.*

It follows from Eqs. (2.1) and (2.2) that

$$0 \le \operatorname{Var}(X) \le \mathbb{E}\left[X^2\right] \text{ for all } X \in L^2(P). \tag{2.3}$$

Exercise 2.1. Let X, Y be two random variables on $(\Omega, \mathcal{B}, P)$;

1. Show that X and Y are independent iff $\operatorname{Cov}(f(X), g(Y)) = 0$ (i.e. $f(X)$ and $g(Y)$ are **uncorrelated**) for bounded measurable functions, $f, g : \mathbb{R} \to \mathbb{R}$. (In this setting X and Y may take values in some arbitrary state space, S.)
2. If $X, Y \in L^2(P)$ and X and Y are independent, then $\operatorname{Cov}(X, Y) = 0$.
   Note well: we will see in examples below that $\operatorname{Cov}(X, Y) = 0$ does **not** necessarily imply that X and Y are independent.

Solution to Exercise (2.1). (Only roughly sketched the proof of this in class.)

1. Since

$$\operatorname{Cov}(f(X),g(Y))=\mathbb{E}[f(X)g(Y)]-\mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]$$

it follows that $\operatorname{Cov}(f(X),g(Y))=0$ iff

$$\mathbb{E}[f(X)g(Y)]=\mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]$$

from which item 1. easily follows.

2. Let $f_M(x)=x1_{|x|\le M}$, then by independence,

$$\mathbb{E}[f_M(X)g_M(Y)]=\mathbb{E}[f_M(X)]\,\mathbb{E}[g_M(Y)]. \tag{2.4}$$

Since

$$\begin{aligned}|f_M(X)g_M(Y)|&\le|XY|\le\frac{1}{2}\left(X^2+Y^2\right)\in L^1(P),\\ |f_M(X)|&\le|X|\le\frac{1}{2}\left(1+X^2\right)\in L^1(P),\ \text{and}\\ |g_M(Y)|&\le|Y|\le\frac{1}{2}\left(1+Y^2\right)\in L^1(P),\end{aligned}$$

we may use the DCT three times to pass to the limit as $M\to\infty$ in Eq. (2.4) to learn that $\mathbb{E}[XY]=\mathbb{E}[X]\,\mathbb{E}[Y]$, i.e. $\operatorname{Cov}(X,Y)=0$. (These technical details were omitted in class.)

End of 1/3/2011 Lecture.

Example 2.5. Suppose that $P(X\in dx,Y\in dy)=e^{-y}1_{0<x<y}dxdy$. Recall that

$$\int_0^\infty y^k e^{-\lambda y}dy=\left(-\frac{d}{d\lambda}\right)^k\int_0^\infty e^{-\lambda y}dy=\left(-\frac{d}{d\lambda}\right)^k\frac{1}{\lambda}=k!\frac{1}{\lambda^{k+1}}.$$

Therefore,

$$\begin{aligned}\mathbb{E}Y&=\int\int ye^{-y}1_{0<x<y}dxdy=\int_0^\infty y^2e^{-y}dy=2,\\ \mathbb{E}Y^2&=\int\int y^2e^{-y}1_{0<x<y}dxdy=\int_0^\infty y^3e^{-y}dy=3!=6\end{aligned}$$

$$\begin{aligned}\mathbb{E}X&=\int\int xe^{-y}1_{0<x<y}dxdy=\frac{1}{2}\int_0^\infty y^2e^{-y}dy=1,\\ \mathbb{E}X^2&=\int\int x^2e^{-y}1_{0<x<y}dxdy=\frac{1}{3}\int_0^\infty y^3e^{-y}dy=\frac{1}{3}3!=2\end{aligned}$$

and

$$\mathbb{E}[XY]=\int\int xye^{-y}1_{0<x<y}dxdy=\frac{1}{2}\int_0^\infty y^3e^{-y}dy=\frac{3!}{2}=3.$$

Therefore $\operatorname{Cov}(X,Y)=3-2\cdot1=1$, $\sigma^2(X)=2-1^2=1$, $\sigma^2(Y)=6-2^2=2$,

$$Corr(X,Y)=\frac{1}{\sqrt{2}}.$$

Lemma 2.6. *The covariance function,* $\operatorname{Cov}(X,Y)$ *is bilinear in* X *and* Y *and* $\operatorname{Cov}(X,Y)=0$ *if either* X *or* Y *is constant. For any constant* k, $\operatorname{Var}(X+k)=\operatorname{Var}(X)$ *and* $\operatorname{Var}(kX)=k^2\operatorname{Var}(X)$. *If* $\{X_k\}_{k=1}^n$ *are uncorrelated* $L^2(P)$ *– random variables, then*

$$\operatorname{Var}(S_n)=\sum_{k=1}^n\operatorname{Var}(X_k).$$

Proof. We leave most of this simple proof to the reader. As an example of the type of argument involved, let us prove $\operatorname{Var}(X+k)=\operatorname{Var}(X)$;

$$\begin{aligned}\operatorname{Var}(X+k)&=\operatorname{Cov}(X+k,X+k)=\operatorname{Cov}(X+k,X)+\operatorname{Cov}(X+k,k)\\ &=\operatorname{Cov}(X+k,X)=\operatorname{Cov}(X,X)+\operatorname{Cov}(k,X)\\ &=\operatorname{Cov}(X,X)=\operatorname{Var}(X),\end{aligned}$$

wherein we have used the bilinearity of $\operatorname{Cov}(\cdot,\cdot)$ and the property that $\operatorname{Cov}(Y,k)=0$ whenever k is a constant. ■

Example 2.7. Suppose that X and Y are distributed as follows;

	ρ_Y	$1/4$	$\frac{1}{2}$	$1/4$
ρ_X	$X \backslash Y$	-1	0	1
$1/4$	1	0	$1/4$	0
$3/4$	0	$1/4$	$1/4$	$1/4$

so that $\rho_{X,Y}(1,-1) = P(X=1, Y=-1) = 0$, $\rho_{X,Y}(1,0) = P(X=1,Y=0) = 1/4$, etc. In this case $XY = 0$ a.s. so that $\mathbb{E}[XY] = 0$ while

$$\begin{aligned}\mathbb{E}[X] &= 1 \cdot \frac{1}{4} + 0 \cdot \frac{3}{4} = \frac{1}{4}, \text{ and} \\ \mathbb{E}Y &= (-1)\,1/4 + 0\frac{1}{2} + 1\frac{1}{4} = 0\end{aligned}$$

so that $\operatorname{Cov}(X,Y) = 0 - \frac{1}{4} \cdot 0 = 0$. Again X and Y are not independent since $\rho_{X,Y}(x,y) \neq \rho_X(x)\,\rho_Y(y)$.

Example 2.8. Let X have an even distribution and let $Y = X^2$, then

$$\operatorname{Cov}(X,Y) = \mathbb{E}\left[X^3\right] - \mathbb{E}\left[X^2\right] \cdot \mathbb{E}X = 0$$

since,

$$\mathbb{E}\left[X^{2k+1}\right] = \int_{-\infty}^{\infty} x^{2k+1} \rho(x)\,dx = 0 \text{ for all } k \in \mathbb{N}.$$

On the other hand $\operatorname{Cov}\left(Y, X^2\right) = \operatorname{Cov}(Y,Y) = \operatorname{Var}(Y) \neq 0$ in general so that Y is not independent of X.

Example 2.9 (Not done in class.). Let X and Z be independent with $P(Z = \pm 1) = \frac{1}{2}$ and take $Y = XZ$. Then $\mathbb{E}Z = 0$ and

$$\begin{aligned}\operatorname{Cov}(X,Y) &= \mathbb{E}\left[X^2 Z\right] - \mathbb{E}[X]\,\mathbb{E}[XZ] \\ &= \mathbb{E}\left[X^2\right] \cdot \mathbb{E}Z - \mathbb{E}[X]\,\mathbb{E}[X]\,\mathbb{E}Z = 0.\end{aligned}$$

On the other hand it should be intuitively clear that X and Y are not independent since knowledge of X typically will give some information about Y. To verify this assertion let us suppose that X is a discrete random variable with $P(X=0) = 0$. Then

$$P(X = x, Y = y) = P(X = x, xZ = y) = P(X = x) \cdot P(X = y/x)$$

while

$$P(X = x)\,P(Y = y) = P(X = x) \cdot P(XZ = y).$$

Thus for X and Y to be independent we would have to have,

$$P(xX = y) = P(XZ = y) \text{ for all } x, y.$$

This is clearly not going to be true in general. For example, suppose that $P(X=1) = \frac{1}{2} = P(X=0)$. Taking $x = y = 1$ in the previously displayed equation would imply

$$\frac{1}{2} = P(X=1) = P(XZ=1) = P(X=1, Z=1) = P(X=1)\,P(Z=1) = \frac{1}{4}$$

which is false.

Presumably you saw the following exercise in Math 180A.

Exercise 2.2 (A Weak Law of Large Numbers). Assume $\{X_n\}_{n=1}^{\infty}$ is a sequence if uncorrelated square integrable random variables which are identically distributed, i.e. $X_n \overset{d}{=} X_m$ for all $m, n \in \mathbb{N}$. Let $S_n := \sum_{k=1}^{n} X_k$, $\mu := \mathbb{E}X_k$ and $\sigma^2 := \operatorname{Var}(X_k)$ (these are independent of k). Show;

$$\begin{aligned}\mathbb{E}\left[\frac{S_n}{n}\right] &= \mu, \\ \mathbb{E}\left(\frac{S_n}{n} - \mu\right)^2 &= \operatorname{Var}\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}, \text{ and} \\ P\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) &\leq \frac{\sigma^2}{n\varepsilon^2}\end{aligned}$$

for all $\varepsilon > 0$ and $n \in \mathbb{N}$.

3

Geometric aspects of $L^2(P)$

Definition 3.1 (Inner Product). *For $X, Y \in L^2(P)$, let $(X,Y) := \mathbb{E}[XY]$ and $\|X\| := \sqrt{(X,X)} = \sqrt{\mathbb{E}|X^2|}$.*

Example 3.2 (This was already mentioned in Lecture 1 with $N = 4$.). Suppose that $\Omega = \{1, \dots, N\}$ and $P(\{i\}) = \frac{1}{N}$ for $1 \le i \le N$. Then

$$(X,Y) = \mathbb{E}[XY] = \frac{1}{N}\sum_{i=1}^{N} X(i)\,Y(i) = \frac{1}{N}\mathbf{X}\cdot\mathbf{Y}$$

where

$$\mathbf{X} := \begin{bmatrix} X(1) \\ X(2) \\ \vdots \\ X(N) \end{bmatrix} \text{ and } \mathbf{Y} := \begin{bmatrix} Y(1) \\ Y(2) \\ \vdots \\ Y(N) \end{bmatrix}.$$

Thus the inner product we have defined in this case is essentially the dot product that you studied in math 20F.

Remark 3.3. The inner product on $H := L^2(P)$ satisfies,

1. $(aX + bY, Z) = a(X,Z) + b(Y,Z)$ i.e. $X \to (X,Z)$ is linear.
2. $(X,Y) = (Y,X)$ (symmetry).
3. $\|X\|^2 := (X,X) \ge 0$ with $\|X\|^2 = 0$ iff $X = 0$.

Notice that combining properties (1) and (2) that $X \to (Z,X)$ is linear for fixed $Z \in H$, i.e.

$$(Z, aX + bY) = a(Z,X) + b(Z,Y).$$

The following identity will be used frequently in the sequel without further mention,

$$\begin{aligned} \|X+Y\|^2 &= (X+Y, X+Y) = \|X\|^2 + \|Y\|^2 + (X,Y) + (Y,X) \\ &= \|X\|^2 + \|Y\|^2 + 2(X,Y). \end{aligned} \tag{3.1}$$

Theorem 3.4 (Schwarz Inequality). *Let $(H, (\cdot,\cdot))$ be an inner product space, then for all $X, Y \in H$*

$$|(X,Y)| \le \|X\|\|Y\|$$

and equality holds iff X and Y are linearly dependent. Applying this result to $|X|$ and $|Y|$ shows,

$$\mathbb{E}[|XY|] \le \|X\| \cdot \|Y\|.$$

Proof. If $Y = 0$, the result holds trivially. So assume that $Y \neq 0$ and observe; if $X = \alpha Y$ for some $\alpha \in \mathbb{C}$, then $(X,Y) = \alpha\|Y\|^2$ and hence

$$|(X,Y)| = |\alpha|\,\|Y\|^2 = \|X\|\|Y\|.$$

Now suppose that $X \in H$ is arbitrary, let $Z := X - \|Y\|^{-2}(X,Y)Y$. (So $\|Y\|^{-2}(X,Y)Y$ is the "orthogonal projection" of X along Y, see Figure 3.1.)

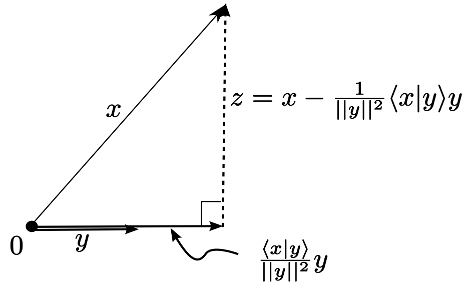

x
$z = x - \frac{1}{\|y\|^2}\langle x|y\rangle y$
0
y
$\frac{\langle x|y\rangle}{\|y\|^2}y$

Fig. 3.1. The picture behind the proof of the Schwarz inequality.

Then

$$\begin{aligned} 0 \le \|Z\|^2 = \left\| X - \frac{(X,Y)}{\|Y\|^2}Y \right\|^2 &= \|X\|^2 + \frac{|(X|Y)|^2}{\|Y\|^4}\|Y\|^2 - 2(X|\frac{(X|Y)}{\|Y\|^2}Y) \\ &= \|X\|^2 - \frac{|(X|Y)|^2}{\|Y\|^2} \end{aligned}$$

from which it follows that $0 \le \|Y\|^2\|X\|^2 - |(X|Y)|^2$ with equality iff $Z = 0$ or equivalently iff $X = \|Y\|^{-2}(X|Y)Y$.

Alternative argument: Let $c \in \mathbb{R}$ and $Z := X - cY$, then

$$0 \le \|Z\|^2 = \|X - cY\|^2 = \|X\|^2 - 2c(X,Y) + c^2\|Y\|^2.$$

The right side of this equation is minimized at $c = (X,Y)/\|Y\|^2$ and for this valued of c we find,

$$0 \le \|X - cY\|^2 = \|X\|^2 - (X,Y)^2/\|Y\|^2$$

with equality iff $X = cY$. Solving this last inequality for $|(X,Y)|$ gives the result. ■

Corollary 3.5. *The norm, $\|\cdot\|$, satisfies the triangle inequality and $(\cdot,\cdot)$ is continuous on $H \times H$.*

Proof. If $X, Y \in H$, then, using Schwarz's inequality,

$$\begin{aligned}\|X+Y\|^2 &= \|X\|^2 + \|Y\|^2 + 2(X,Y)\\ &\le \|X\|^2 + \|Y\|^2 + 2\|X\|\|Y\| = (\|X\| + \|Y\|)^2.\end{aligned}$$

Taking the square root of this inequality shows $\|\cdot\|$ satisfies the triangle inequality. (The rest of this proof may be skipped.)

Checking that $\|\cdot\|$ satisfies the remaining axioms of a norm is now routine and will be left to the reader. If $X, Y, \Delta X, \Delta Y \in H$, then

$$\begin{aligned}|(X+\Delta X, Y+\Delta Y) - (X,Y)| &= |(X,\Delta Y) + (\Delta X, Y) + (\Delta X, \Delta Y)|\\ &\le \|X\|\|\Delta Y\| + \|Y\|\|\Delta X\| + \|\Delta X\|\|\Delta Y\|\\ &\to 0 \text{ as } \Delta X, \Delta Y \to 0,\end{aligned}$$

from which it follows that $(\cdot,\cdot)$ is continuous. ■

Definition 3.6. *Let $(H,(\cdot,\cdot))$ be an inner product space, we say $X, Y \in H$ are* ***orthogonal*** *and write $X \perp Y$ iff $(X,Y) = 0$. More generally if $A \subset H$ is a set, $X \in H$ is* ***orthogonal to*** *A (write $X \perp A$) iff $(X,Y) = 0$ for all $Y \in A$. Let $A^\perp = \{X \in H : X \perp A\}$ be the set of vectors orthogonal to A. A subset $S \subset H$ is an* ***orthogonal set*** *if $X \perp Y$ for all distinct elements $X, Y \in S$. If S further satisfies, $\|X\| = 1$ for all $X \in S$, then S is said to be an* ***orthonormal set.***

Proposition 3.7. *Let $(H,(\cdot,\cdot))$ be an inner product space then*

1. ***(Pythagorean Theorem)*** *If $S \subset\subset H$ is a finite orthogonal set, then*

$$\left\|\sum_{X\in S} X\right\|^2 = \sum_{X\in S} \|X\|^2. \tag{3.2}$$

2. ***(Parallelogram Law)*** *(Skip this one.) For all $X, Y \in H$,*

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2 \tag{3.3}$$

Proof. Items 1. and 2. are proved by the following elementary computations;and

$$\begin{aligned}\left\|\sum_{X\in S} X\right\|^2 &= (\sum_{X\in S} X, \sum_{Y\in S} Y) = \sum_{X,Y\in S} (X,Y)\\ &= \sum_{X\in S} (X,X) = \sum_{X\in S} \|X\|^2\end{aligned}$$

and

$$\begin{aligned}&\|X+Y\|^2 + \|X-Y\|^2\\ &\quad= \|X\|^2 + \|Y\|^2 + 2(X,Y) + \|X\|^2 + \|Y\|^2 - 2(X,Y)\\ &\quad= 2\|X\|^2 + 2\|Y\|^2.\end{aligned}$$

■

Theorem 3.8 (Least Squares Approximation Theorem). *Suppose that V is a subspace of $H := L^2(P)$, $X \in V$, and $Y \in L^2(P)$. Then the following are equivalent;*

1. *$\|Y - X\| \ge \|Y - Z\|$ for all $Z \in V$ (i.e. X is a least squares approximation to Y by an element from V) and*
2. *$(Y - X) \perp V$.*

Moreover there is "essentially" at most one $X \in V$ satisfying 1. or equivalently 2. We denote random variable by $Q_V Y$ and call it ***orthogonal projection of*** *Y* ***along*** *V.*

Proof. 1 $\implies$ 2. If 1. holds then $f(t) := \|Y - (X + tZ)\|^2$ has a minimum at $t = 0$ and therefore $\dot f(0) = 0$. Since

$$f(t) := \|Y - X - tZ\|^2 = \|Y - X\|^2 + t^2\|Z\|^2 - 2t(Y - X, Z),$$

we may conclude that

$$0 = \dot f(0) = -2(Y - X, Z).$$

As $Z \in V$ was arbitrary we may conclude that $(Y - X) \perp V$.

2 $\implies$ 1. Now suppose that $(Y - X) \perp V$ and $Z \in V$, then $(Y - X) \perp (X - Z)$ and so

$$\|Y - Z\|^2 = \|Y - X + X - Z\|^2 = \|Y - X\|^2 + \|X - Z\|^2 \ge \|Y - X\|^2. \tag{3.4}$$

Moreover if Z is another best approximation to Y then $\|Y - Z\|^2 = \|Y - X\|^2$ which happens according to Eq. (3.4) iff

$$\|X - Z\|^2 = \mathbb{E}(X - Z)^2 = 0,$$

i.e. iff $X = Z$ a.s. ■

End of Lecture 3: 1/07/2011 (Given by Tom Laetsch)

Corollary 3.9 (Orthogonal Projection Formula). *Suppose that V is a subspace of $H := L^2(P)$ and $\{X_i\}_{i=1}^N$ is an orthogonal basis for V. Then*

$$Q_V Y = \sum_{i=1}^N \frac{(Y, X_i)}{\|X_i\|^2} X_i \text{ for all } Y \in H.$$

Proof. The best approximation $X \in V$ to Y is of the form $X = \sum_{i=1}^N c_i X_i$ where $c_i \in \mathbb{R}$ need to be chosen so that $(Y - X) \perp V$. Equivalently put we must have

$$0 = (Y - X, X_j) = (Y, X_j) - (X, X_j) \text{ for } 1 \leq j \leq N.$$

Since

$$(X, X_j) = \sum_{i=1}^N c_i (X_i, X_j) = c_j \|X_j\|^2,$$

we see that $c_j = (Y, X_j) / \|X_j\|^2$, i.e.

$$Q_V Y = X = \sum_{i=1}^N \frac{(Y, X_i)}{\|X_i\|^2} X_i.$$

■

Example 3.10. Given $Y \in L^2(P)$ the best approximation to Y by a constant function c is given by

$$c = \frac{\mathbb{E}[Y1]}{\mathbb{E}1^2} 1 = \mathbb{E}Y.$$

You already proved this on your first homework by a direct calculus exercise.

4

Linear prediction and a canonical form

Corollary 4.1 (Correlation Bounds). *For all square integrable random variables, X and Y,*

$$|\mathrm{Cov}\left(X,Y\right)| \leq \sigma\left(X\right)\cdot\sigma\left(Y\right)$$

or equivalently,

$$|Corr\left(X,Y\right)| \leq 1.$$

.

Proof. This is a simply application of Schwarz's inequality (Theorem 3.4);

$$|\mathrm{Cov}\left(X,Y\right)| = |\mathbb{E}\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right]| \leq \|X-\mu_X\|\cdot\|Y-\mu_Y\| = \sigma\left(X\right)\cdot\sigma\left(Y\right).$$

■

Since $Corr\left(X,Y\right)>0$ iff $\mathrm{Cov}\left(X,Y\right)>0$ iff $\mathbb{E}\left[\left(X-\mu_X\right)\left(Y-\mu_Y\right)\right]>0$, we see that X and Y are positively correlated iff $X-\mu_X$ and $Y-\mu_Y$ tend to have the same sign more often than not. While X and Y are negatively correlated iff $X-\mu_X$ and $Y-\mu_Y$ tend to have opposite signs more often than not. This description is of course rather crude given that it ignores size of $X-\mu_X$ and $Y-\mu_Y$ but should however give the reader a little intuition into the meaning of correlation. (See Corollary 4.4 below for the special case where $Corr\left(X,Y\right)=1$ or $Corr\left(X,Y\right)=-1$.)

Theorem 4.2 (Linear Prediction Theorem). *Let X and Y be two square integrable random variables, then*

$$\sigma\left(Y\right)\sqrt{1-Corr^2\left(X,Y\right)} = \min_{a,b\in\mathbb{R}}\|Y-\left(aX+b\right)\| = \|Y-W\| \tag{4.1}$$

where

$$W = \mu_Y + \frac{\mathrm{Cov}\left(X,Y\right)}{\mathrm{Var}\left(X\right)}\left(X-\mu_X\right) = \frac{\mathrm{Cov}\left(X,Y\right)}{\mathrm{Var}\left(X\right)}X + \left(\mathbb{E}Y - \mu_X\frac{\mathrm{Cov}\left(X,Y\right)}{\mathrm{Var}\left(X\right)}\right).$$

Proof. Let $\mu=\mathbb{E}X$ and $\bar{X}=X-\mu$. Then $\left\{1,\bar{X}\right\}$ is an orthogonal set and $V:=\mathrm{span}\left\{1,X\right\}=\mathrm{span}\left\{1,\bar{X}\right\}$. Thus best approximation of Y by random variable of the form $aX+b$ is given by

$$W = \left(Y,1\right)1 + \frac{\left(Y,\bar{X}\right)}{\|\bar{X}\|^2}\bar{X} = \mathbb{E}Y + \frac{\mathrm{Cov}\left(X,Y\right)}{\mathrm{Var}\left(X\right)}\left(X-\mu_X\right).$$

The **root mean square error** of this approximation is

$$\begin{aligned}\|Y-W\|^2 &= \left\|\bar{Y} - \frac{\mathrm{Cov}\left(X,Y\right)}{\mathrm{Var}\left(X\right)}\bar{X}\right\|^2 = \sigma^2\left(Y\right) - \frac{\mathrm{Cov}^2\left(X,Y\right)}{\sigma^2\left(X\right)}\\ &= \sigma^2\left(Y\right)\left(1-Corr^2\left(X,Y\right)\right),\end{aligned}$$

so that

$$\|Y-W\| = \sigma\left(Y\right)\sqrt{1-Corr^2\left(X,Y\right)}.$$

■

Example 4.3. Suppose that $P\left(X\in dx, Y\in dy\right) = e^{-y}1_{0<x<y}dxdy$. Recall from Example 2.5 that

$$\begin{aligned}\mathbb{E}X &= 1, & \mathbb{E}Y &= 2,\\ \mathbb{E}X^2 &= 2, & \mathbb{E}Y^2 &= 6\\ \sigma\left(X\right) &= 1, & \sigma\left(Y\right) &= \sqrt{2},\end{aligned}$$
$$\mathrm{Cov}\left(X,Y\right)=1,\text{ and } Corr\left(X,Y\right)=\frac{1}{\sqrt{2}}.$$

So in this case

$$W = 2 + \frac{1}{1}\left(X-1\right) = X+1$$

is the best linear predictor of Y and the root mean square error in this prediction is

$$\|Y-W\| = \sqrt{2}\sqrt{1-\frac{1}{2}} = 1.$$

Corollary 4.4. *If $Corr\left(X,Y\right)=\pm1$, then*

$$Y = \mu_Y \pm \frac{\sigma\left(Y\right)}{\sigma\left(X\right)}\left(X-\mu_X\right),$$

i.e. $Y-\mu_Y$ is a positive (negative) multiple of $X-\mu_X$ if $Corr\left(X,Y\right)=1$ $\left(Corr\left(X,Y\right)=-1\right)$.

Proof. According to Eq. (4.1) of Theorem 4.2, if $Corr\,(X,Y) = \pm 1$ then

$$\begin{aligned} Y &= \mu_Y + \frac{\operatorname{Cov}(X,Y)}{\operatorname{Var}(X)}(X - \mu_X) \\ &= \mu_Y \pm \frac{\sigma_X \sigma_Y}{\sigma_X^2}(X - \mu_X) = \mu_Y \pm \frac{\sigma_Y}{\sigma_X}(X - \mu_X), \end{aligned}$$

wherein we have used $\operatorname{Cov}(X,Y) = \operatorname{Cov}(X,Y)\,\sigma_X \sigma_Y = \pm 1 \sigma_X \sigma_Y$. ■

Theorem 4.5 (Canonical form). *If $X, Y \in L^2(P)$, then there are two mean zero uncorrelated Random variables $\{Z_1, Z_2\}$ such that $\|Z_1\| = \|Z_2\| = 1$ and*

$$\begin{aligned} X &= \mu_X + \sigma(X)\,Z_1, \text{ and} \\ Y &= \mu_Y + \sigma(Y)\left[\cos\theta \cdot Z_1 + \sin\theta \cdot Z_2\right], \end{aligned}$$

where $0 \le \theta \le \pi$ is chosen such that $\cos\theta := Corr\,(X,Y)$.

Proof. (Just sketch the main ideal in class!). The proof amounts to applying the Gram-Schmidt procedure to $\left\{\bar{X} := X - \mu_X, \bar{Y} := Y - \mu_Y\right\}$ to find Z_1 and Z_2 followed by expressing X and Y in uniquely in terms of the linearly independent set, $\{1, Z_1, Z_2\}$. The details follow.

Performing Gram-Schmidt on $\left\{\bar{X}, \bar{Y}\right\}$ gives $Z_1 = \bar{X}/\sigma(X)$ and

$$\tilde{Z}_2 = \bar{Y} - \frac{\left(\bar{Y}, \bar{X}\right)}{\sigma(X)^2}\bar{X}.$$

To get Z_2 we need to normalize $\tilde{Z}_2$ using;

$$\begin{aligned} \mathbb{E}\tilde{Z}_2^2 &= \sigma(Y)^2 - 2\frac{\left(\bar{Y}, \bar{X}\right)}{\sigma(X)^2}\left(\bar{X}, \bar{Y}\right) + \frac{\left(\bar{Y}, \bar{X}\right)^2}{\sigma(X)^4}\sigma(X)^2 \\ &= \sigma(Y)^2 - \frac{\left(\bar{X}, \bar{Y}\right)^2}{\sigma(X)^2} = \sigma(Y)^2\left(1 - Corr^2(X,Y)\right) \\ &= \sigma(Y)^2 \sin^2\theta. \end{aligned}$$

Therefore $Z_1 = \bar{X}/\sigma(X)$ and

$$\begin{aligned} Z_2 := \frac{\tilde{Z}_2}{\left\|\tilde{Z}_2\right\|} &= \frac{\bar{Y} - \frac{\left(\bar{Y}, \bar{X}\right)}{\sigma(X)^2}\bar{X}}{\sigma(Y)\sin\theta} = \frac{\bar{Y} - \frac{\sigma(X)\sigma(Y)Corr(X,Y)}{\sigma(X)^2}\bar{X}}{\sigma(Y)\sin\theta} \\ &= \frac{\bar{Y} - \frac{\sigma(Y)}{\sigma(X)}\cos\theta \cdot \bar{X}}{\sigma(Y)\sin\theta} = \frac{\bar{Y} - \sigma(Y)\cos\theta \cdot Z_1}{\sigma(Y)\sin\theta} \end{aligned}$$

Solving for $\bar{X}$ and $\bar{Y}$ shows,

$$\bar{X} = \sigma(X)\,Z_1 \text{ and } \bar{Y} = \sigma(Y)\left[\sin\theta \cdot Z_2 + \cos\theta \cdot Z_1\right]$$

which is equivalent to the desired result. ■

Remark 4.6. It is easy to give a second proof of Corollary 4.4 based on Theorem 4.5. Indeed, if $Corr\,(X,Y) = 1$, then $\theta = 0$ and $\bar{Y} = \sigma(Y)\,Z_1 = \frac{\sigma(Y)}{\sigma(X)}\bar{X}$ while if $Corr\,(X,Y) = -1$, then $\theta = \pi$ and therefore $\bar{Y} = -\sigma(Y)\,Z_1 = -\frac{\sigma(Y)}{\sigma(X)}\bar{X}$.

Exercise 4.1 (A correlation inequality). Suppose that X is a random variable and $f, g : \mathbb{R} \to \mathbb{R}$ are two increasing functions such that both $f(X)$ and $g(X)$ are square integrable, i.e. $\mathbb{E}|f(X)|^2 + \mathbb{E}|g(X)|^2 < \infty$. Show $\operatorname{Cov}(f(X), g(X)) \ge 0$. **Hint:** let Y be another random variable which has the same law as X and is independent of X. Then consider

$$\mathbb{E}\left[(f(Y) - f(X)) \cdot (g(Y) - g(X))\right].$$

5

Conditional Expectation

Notation 5.1 (Conditional Expectation 1) *Given $Y \in L^1(P)$ and $A \subset \Omega$ let*

$$\mathbb{E}[Y : A] := \mathbb{E}[1_A Y]$$

and

$$\mathbb{E}[Y|A] = \begin{cases} \mathbb{E}[Y : A]/P(A) & \text{if } P(A) > 0 \\ 0 & \text{if } P(A) = 0 \end{cases}. \tag{5.1}$$

(In point of fact, when $P(A) = 0$ we could set $\mathbb{E}[Y|A]$ to be any real number. We choose 0 for definiteness and so that $Y \to \mathbb{E}[Y|A]$ is always linear.)

Example 5.2 (Conditioning for the uniform distribution). Suppose that Ω is a finite set and P is the uniform distribution on P so that $P(\{\omega\}) = \frac{1}{\#(\Omega)}$ for all $\omega \in W$. Then for non-empty any subset $A \subset \Omega$ and $Y : \Omega \to \mathbb{R}$ we have $\mathbb{E}[Y|A]$ is the expectation of Y restricted to A under the uniform distribution on A. Indeed,

$$\begin{aligned} \mathbb{E}[Y|A] &= \frac{1}{P(A)}\mathbb{E}[Y : A] = \frac{1}{P(A)}\sum_{\omega \in A} Y(\omega) P(\{\omega\}) \\ &= \frac{1}{\#(A)/\#(\Omega)}\sum_{\omega \in A} Y(\omega)\frac{1}{\#(\Omega)} = \frac{1}{\#(A)}\sum_{\omega \in A} Y(\omega). \end{aligned}$$

Lemma 5.3. *If $P(A) > 0$ then $\mathbb{E}[Y|A] = \mathbb{E}_{P(\cdot|A)}Y$ for all $Y \in L^1(P)$.*

Proof. I will only prove this lemma when Y is a discrete random variable, although the result does hold in general. So suppose that $Y : \Omega \to S$ where S is a finite or countable subset of $\mathbb{R}$. Then taking expectation relative to $P(\cdot|A)$ of the identity, $Y = \sum_{y \in S} y 1_{Y=y}$, gives

$$\begin{aligned} \mathbb{E}_{P(\cdot|A)}Y &= \mathbb{E}_{P(\cdot|A)}\sum_{y \in S} y 1_{Y=y} = \sum_{y \in S} y \mathbb{E}_{P(\cdot|A)} 1_{Y=y} = \sum_{y \in S} y P(Y = y|A) \\ &= \sum_{y \in S} y P(Y = y|A) = \sum_{y \in S} y \frac{P(Y = y, A)}{P(A)} = \frac{1}{P(A)}\sum_{y \in S} y \mathbb{E}[1_A 1_{Y=y}] \\ &= \frac{1}{P(A)}\mathbb{E}\left[1_A \sum_{y \in S} y 1_{Y=y}\right] = \frac{1}{P(A)}\mathbb{E}[1_A Y] = \mathbb{E}[Y|A]. \end{aligned}$$

■

Lemma 5.4. *No matter whether $P(A) > 0$ or $P(A) = 0$ we always have,*

$$|\mathbb{E}[Y|A]| \le \mathbb{E}[|Y|\,|A] \le \sqrt{\mathbb{E}\left[|Y|^2|A\right]}. \tag{5.2}$$

Proof. If $P(A) = 0$ then all terms in Eq. (5.2) are zero and so the inequalities hold. For $P(A) > 0$ we have, using the Schwarz inequality in Theorem 3.4), that

$$|\mathbb{E}[Y|A]| = \left|\mathbb{E}_{P(\cdot|A)}Y\right| \le \mathbb{E}_{P(\cdot|A)}|Y| \le \sqrt{\mathbb{E}_{P(\cdot|A)}|Y|^2 \cdot \mathbb{E}_{P(\cdot|A)}1} = \sqrt{\mathbb{E}_{P(\cdot|A)}|Y|^2}.$$

This completes that proof as $\mathbb{E}_{P(\cdot|A)}|Y| = \mathbb{E}[|Y|\,|A]$ and $\mathbb{E}_{P(\cdot|A)}|Y|^2 = \mathbb{E}\left[|Y|^2|A\right]$. ■

Notation 5.5 *Let S be a set (often $S = \mathbb{R}$ or $S = \mathbb{R}^N$) and suppose that $X : \Omega \to S$ is a function. (So X is a random variable if $S = \mathbb{R}$ and a random vector when $S = \mathbb{R}^N$.) Further let V_X denote those random variables $Z \in L^2(P)$ which may be written as $Z = f(X)$ for some function $f : S \to \mathbb{R}$. (This is a subspace of $L^2(P)$ and we let $\mathcal{F}_X := \left\{f : S \to \mathbb{R} : f(X) \in L^2(P)\right\}$.)*

Definition 5.6 (Conditional Expectation 2). *Given a function $X : \Omega \to S$ and $Y \in L^2(P)$, we define $\mathbb{E}[Y|X] := Q_{V_X}Y$ where Q_{V_X} is orthogonal projection onto V_X. (**Fact:** $Q_{V_X}Y$ always exists. The proof requires technical details beyond the scope of this course.)*

Remark 5.7. By definition, $\mathbb{E}[Y|X] = h(X)$ where $h \in \mathcal{F}_X$ is chosen so that $[Y - h(X)] \perp V_X$, i.e. $\mathbb{E}[Y|X] = h(X)$ iff $(Y - h(X), f(X)) = 0$ for all $f \in \mathcal{F}_X$. So in summary, $\mathbb{E}[Y|X] = h(X)$ iff

$$\mathbb{E}[Y f(X)] = \mathbb{E}[h(X) f(X)] \text{ for all } f \in \mathcal{F}_X. \tag{5.3}$$

Corollary 5.8 (Law of total expectation). *For all random variables $Y \in L^2(P)$, we have $\mathbb{E}Y = \mathbb{E}(\mathbb{E}(Y|X))$.*

Proof. Take $f = 1$ in Eq. (5.3). ■

This notion of conditional expectation is rather abstract. It is now time to see how to explicitly compute conditional expectations. (In general this can be quite tricky to carry out in concrete examples!)

5.1 Conditional Expectation for Discrete Random Variables

Recall that if A and B are events with $P(A) > 0$, then we define $P(B|A) := \frac{P(B\cap A)}{P(A)}$. By convention we will set $P(B|A) = 0$ if $P(A) = 0$.

Example 5.9. If Ω is a finite set with N elements, P is the uniform distribution on Ω, and A is a non-empty subset of Ω, then $P(\cdot|A)$ restricted to events contained in A is the uniform distribution on A. Indeed, $a = \#(A)$ and $B \subset A$, we have

$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\#(B)/N}{\#(A)/N} = \frac{\#(B)}{\#(A)} = \frac{\#(B)}{a}.$$

Theorem 5.10. *Suppose that S is a finite or countable set and $X : \Omega \to S$, then $\mathbb{E}[Y|X] = h(X)$ where $h(s) := \mathbb{E}[Y|X = s]$ for all $s \in S$.*

Proof. First Proof. Our goal is to find $h(s)$ such that

$$\mathbb{E}[Yf(X)] = \mathbb{E}[h(X)f(X)] \text{ for all bounded } f.$$

Let $S' = \{s \in S : P(X = s) > 0\}$, then

$$\begin{aligned}\mathbb{E}[Yf(X)] &= \sum_{s\in S}\mathbb{E}[Yf(X) : X = s] = \sum_{s\in S'}\mathbb{E}[Yf(X) : X = s]\\ &= \sum_{s\in S'} f(s)\,\mathbb{E}[Y|X = s]\cdot P(X = s)\\ &= \sum_{s\in S'} f(s)\,h(s)\cdot P(X = s)\\ &= \sum_{s\in S} f(s)\,h(s)\cdot P(X = s) = \mathbb{E}[h(X)f(X)]\end{aligned}$$

where $h(s) := \mathbb{E}[Y|X = s]$.

Second Proof. If S is a finite set, such that $P(X = s) > 0$ for all $s \in S$. Then

$$f(X) = \sum_{s\in S} f(s)\,1_{X=s}$$

which shows that $V_X = \operatorname{span}\{1_{X=s} : s \in S\}$. As $\{1_{X=s}\}_{s\in S}$ is an orthogonal set, we may compute

$$\begin{aligned}\mathbb{E}[Y|X] &= \sum_{s\in S}\frac{(Y, 1_{X=s})}{\|1_{X=s}\|^2}1_{X=s} = \sum_{s\in S}\frac{\mathbb{E}[Y : X = s]}{P(X = s)}1_{X=s}\\ &= \sum_{s\in S}\mathbb{E}[Y|X = s]\cdot 1_{X=s} = h(X).\end{aligned}$$

■

Example 5.11. Suppose that X and Y are discrete random variables with joint distribution given as;

$$\begin{array}{ccccc} & \rho_Y & 1/4 & \frac{1}{2} & 1/4\\ \rho_X & X\backslash Y & -1 & 0 & 1\\ 1/4 & 1 & 0 & 1/4 & 0\\ 3/4 & 0 & 1/4 & 1/4 & 1/4\end{array}.$$

We then have

$$\begin{aligned}\mathbb{E}[Y|X = 1] &= \frac{1}{1/4}\left(-1\cdot 0 + 0\cdot\frac{1}{4} + 1\cdot 0\right) = 0 \text{ and}\\ \mathbb{E}[Y|X = 0] &= \frac{1}{3/4}\left(-1\cdot 1/4 + 0\cdot\frac{1}{4} + 1\cdot 1/4\right) = 0\end{aligned}$$

and therefore $\mathbb{E}[Y|X] = 0$. On the other hand,

$$\begin{aligned}\mathbb{E}[X|Y = -1] &= \frac{1}{1/4}\left(1\cdot 0 + 0\cdot\frac{1}{4}\right) = 0,\\ \mathbb{E}[X|Y = 0] &= \frac{1}{1/2}\left(1\cdot 1/4 + 0\cdot\frac{1}{4}\right) = \frac{1}{2}, \text{ and}\\ \mathbb{E}[X|Y = 1] &= \frac{1}{1/4}\left(1\cdot 0 + 0\cdot\frac{1}{4}\right) = 0.\end{aligned}$$

Therefore

$$\mathbb{E}[X|Y] = \frac{1}{2}1_{Y=0}.$$

Example 5.12. Let X and Y be discrete random variables with values in $\{1, 2, 3\}$ whose joint distribution and marginals are given by

$$\begin{array}{ccccc} & \rho_X & .3 & .35 & .35\\ \rho_Y & Y\backslash X & 1 & 2 & 3\\ .6 & 1 & .1 & .2 & .3\\ .3 & 2 & .15 & .15 & 0\\ .1 & 3 & .05 & 0 & .05\end{array}.$$

Then

$$\begin{aligned}\rho_{X|Y}(1,3) &= P(X = 1|Y = 3) = \frac{.05}{.1} = \frac{1}{2},\\ \rho_{X|Y}(2,3) &= P(X = 2|Y = 3) = \frac{0}{.1} = 0, \text{ and}\\ \rho_{X|Y}(3,3) &= P(X = 3|Y = 3) = \frac{.05}{.1} = \frac{1}{2}.\end{aligned}$$

Therefore,

$$\mathbb{E}\left[X|Y=3\right] = 1\cdot\frac{1}{2} + 2\cdot 0 + 3\cdot\frac{1}{2} = 2$$

or

$$h\left(3\right) := \mathbb{E}\left[X|Y=3\right] = \frac{1}{.1}\left(1\cdot .05 + 2\cdot 0 + 3\cdot .05\right) = 2$$

Similarly,

$$h\left(1\right) := \mathbb{E}\left[X|Y=1\right] = \frac{1}{.6}\left(1\cdot .1 + 2\cdot .2 + 3\cdot .3\right) = 2\frac{1}{3},$$
$$h\left(2\right) := \mathbb{E}\left[X|Y=2\right] = \frac{1}{.3}\left(1\cdot .15 + 2\cdot .15 + 3\cdot 0\right) = 1.5$$

and so

$$\mathbb{E}\left[X|Y\right] = h\left(Y\right) = 2\frac{1}{3}\cdot 1_{Y=1} + 1.5\cdot 1_{Y=2} + 2\cdot 1_{Y=3}.$$

Example 5.13 (Number of girls in a family). Suppose the number of children in a family is a random variable X with mean μ, and given $X = n$ for $n \geq 1$, each of the n children in the family is a girl with probability p and a boy with probability $1-p$. Problem. What is the expected number of girls in a family?

Solution. Intuitively, the answer should be $p\mu$. To show this is correct let G be the random number of girls in a family. Then,

$$\mathbb{E}\left[G|X=n\right] = p\cdot n$$

as $G = 1_{A_1} + \cdots + 1_{A_n}$ on $X = n$ where A_i is the event the i^{th} – child is a girl. We are given $P\left(A_i|X=n\right) = p$ so that $\mathbb{E}\left[1_{A_i}|X=n\right] = p$ and so $\mathbb{E}\left[G|X=n\right] = p\cdot n$. Therefore, $\mathbb{E}\left[G|X\right] = p\cdot X$ and

$$\mathbb{E}\left[G\right] = \mathbb{E}\mathbb{E}\left[G|X\right] = \mathbb{E}\left[p\cdot X\right] = p\mu.$$

Example 5.14. Suppose that X and Y are i.i.d. random variables with the geometric distribution,

$$P\left(X=k\right) = P\left(Y=k\right) = \left(1-p\right)^{k-1}p \text{ for } k\in\mathbb{N}.$$

We compute, for $n > m$,

$$\begin{aligned} P\left(X=m|X+Y=n\right) &= \frac{P\left(X=m, X+Y=n\right)}{P\left(X+Y=n\right)} \\ &= \frac{P\left(X=m, Y=n-m\right)}{\sum_{k+l=n}P\left(X=k, Y=l\right)} \end{aligned}$$

where

$$\begin{aligned} P\left(X=m, Y=n-m\right) &= p^2\left(1-p\right)^{m-1}\left(1-p\right)^{n-m-1} \\ &= p^2\left(1-p\right)^{n-2} \end{aligned}$$

and

$$\begin{aligned} \sum_{k+l=n}P\left(X=k, Y=l\right) &= \sum_{k+l=n}\left(1-p\right)^{k-1}p\left(1-p\right)^{l-1}p \\ &= \sum_{k+l=n}p^2\left(1-p\right)^{n-2} = p^2\left(1-p\right)^{n-2}\sum_{k=1}^{n-1}1. \end{aligned}$$

Thus we have shown,

$$P\left(X=m|X+Y=n\right) = \frac{1}{n-1} \text{ for } 1\leq m<n.$$

From this it follows that

$$\mathbb{E}\left[f\left(X\right)|X+Y=n\right] = \frac{1}{n-1}\sum_{m=1}^{n-1}f\left(m\right)$$

and so

$$\mathbb{E}\left[f\left(X\right)|X+Y\right] = \frac{1}{X+Y-1}\sum_{m=1}^{X+Y-1}f\left(m\right).$$

As a check if $f\left(m\right) = m$ we have

$$\begin{aligned} \mathbb{E}\left[X|X+Y\right] &= \frac{1}{X+Y-1}\sum_{m=1}^{X+Y-1}m \\ &= \frac{1}{X+Y-1}\frac{1}{2}\left(X+Y-1\right)\left(X+Y-1+1\right) \\ &= \frac{1}{2}\left(X+Y\right) \end{aligned}$$

as we will see hold in fair generality, see Example 5.24 below.

Example 5.15 (Durrett Example 4.6.2, p. 205). Suppose we want to determine the expected value of

$$Y = \#\text{ of rolls to complete one game of craps.}$$

Let X be the sum we obtain on the first roll. In this game, if;

$$\begin{aligned} &X\in\{2,3,12\} =: L \implies \text{game ends and you loose,} \\ &X\in\{7,11\} =: W \implies \text{game ends and you win,and} \\ &X\in\{4,5,6,8,9,10\} =: P \implies X \text{ is your “point.”} \end{aligned}$$

In the last case, you roll your dice again and again until you either throw until you get X (your point) or 7. (If you hit X before the 7 then you win.) We are going to compute $\mathbb{E}Y$ as $\mathbb{E}\left[\mathbb{E}\left[Y|X\right]\right].$

Clearly if $x \in L \cup W$ then $\mathbb{E}\left[Y|X=x\right] = 1$ while if $x \in P$, then $\mathbb{E}\left[Y|X=x\right] = 1 + \mathbb{E}N_x$ where N_x is the number of rolls need to hit either x or 7. This is a geometric random variable with parameter p_x (probability of rolling an x or a 7) and so $\mathbb{E}N_x = \frac{1}{p_x}$. For example if $x = 4$, then $p_x = \frac{3+6}{36} = \frac{9}{36}$ (3 is the number of ways to roll a 4 and 6 is the number of ways to roll as 7) and hence $1 + \mathbb{E}N_x = 1 + 4 = 5$. Similar calculations gives us the following table;

$x \in$	$\{2,3,7,11,12\}$	$\{4,10\}$	$\{5,9\}$	$\{6,8\}$
$\mathbb{E}\left[Y\|X=x\right]$	1	$\frac{45}{9}$	$\frac{46}{10}$	$\frac{47}{11}$
$P\left(\text{set}\right)$	$\frac{12}{36}$	$\frac{6}{36}$	$\frac{8}{36}$	$\frac{10}{36}$

(For example, there are 5 ways to get a 6 and 6 ways to get a 7 so when $x = 6$ we are waiting for an event with probability $11/36$ and the mean of this geometric random variables is $36/11$ and adding the first roll to this implies, $\mathbb{E}\left[Y|X=6\right] = 47/11$. Similarly for $x = 8$ and $P\left(X = 6 \text{ or } 8\right) = \left(5+5\right)/36.$) Putting the pieces together and using the law of total expectation gives,

$$\begin{aligned}\mathbb{E}Y = \mathbb{E}\left[\mathbb{E}\left[Y|X\right]\right] &= 1 \cdot \frac{12}{36} + \frac{45}{9} \cdot \frac{6}{36} + \frac{46}{10} \cdot \frac{8}{36} + \frac{47}{11} \cdot \frac{10}{36} \\ &= \frac{557}{165} \cong 3.376 \text{ rolls.}\end{aligned}$$

The following two facts are often helpful when computing conditional expectations.

Proposition 5.16 (Bayes formula). *Suppose that $A \subset \Omega$ and $\{A_i\}$ is a partition of A, then*

$$\mathbb{E}\left[Y|A\right] = \frac{1}{P\left(A\right)} \sum_i \mathbb{E}\left[Y|A_i\right] P\left(A_i\right) = \frac{\sum_i \mathbb{E}\left[Y|A_i\right] P\left(A_i\right)}{\sum_i P\left(A_i\right)}.$$

If we further assume that $\mathbb{E}\left[Y|A_i\right] = c$ independent of i, then $\mathbb{E}\left[Y|A\right] = c$.

The proof of this proposition is straight forward and is left to the reader.

Proposition 5.17. *Suppose that $X_i : \Omega \to S_i$ for $1 \le i \le n$ are independent random functions with each S_i being discrete. Then for any $T_i \subset S_i$ we have,*

$$\mathbb{E}\left[u\left(X_1, \dots, X_n\right)|X_1 \in T_1, \dots, X_n \in T_n\right] = \mathbb{E}\left[u\left(Y_1, \dots, Y_n\right)\right]$$

where $Y_i : \Omega \to T_i$ for $1 \le i \le n$ are independent random functions such that $P\left(Y_i = t\right) = P\left(X_i = t|X_i \in T_i\right)$ for all $t \in T_i$.

Proof. The proof is contained in the following computation,

$$\begin{aligned}&\mathbb{E}\left[u\left(X_1, \dots, X_n\right)|X_1 \in T_1, \dots, X_n \in T_n\right] \\ &= \frac{\mathbb{E}\left[u\left(X_1, \dots, X_n\right) : X_1 \in T_1, \dots, X_n \in T_n\right]}{P\left(X_1 \in T_1, \dots, X_n \in T_n\right)} \\ &= \frac{1}{P\left(X_1 \in T_1, \dots, X_n \in T_n\right)} \sum_{t_i \in T_i} u\left(t_1, \dots, t_n\right) P\left(X_1 = t_1, \dots, X_n = t_n\right) \\ &= \frac{1}{\prod_i P\left(X_i \in T_i\right)} \sum_{(t_1, \dots, t_n) \in T_1 \times \dots \times T_n} u\left(t_1, \dots, t_n\right) \prod_i P\left(X_i \in t_i\right) \\ &= \sum_{(t_1, \dots, t_n) \in T_1 \times \dots \times T_n} u\left(t_1, \dots, t_n\right) \prod_i \frac{P\left(X_i \in t_i\right)}{P\left(X_i \in T_i\right)} \\ &= \sum_{(t_1, \dots, t_n) \in T_1 \times \dots \times T_n} u\left(t_1, \dots, t_n\right) \prod_i P\left(X_i = t|X_i \in T_i\right) \\ &= \sum_{(t_1, \dots, t_n) \in T_1 \times \dots \times T_n} u\left(t_1, \dots, t_n\right) P\left(Y_1 = t_1, \dots, Y_n = t_n\right) \\ &= \mathbb{E}\left[u\left(Y_1, \dots, Y_n\right)\right].\end{aligned}$$

■

Here is an example of how to use these two propositions.

Example 5.18. Suppose we roll a die $n-$times with results $\{X_i\}_{i=1}^n$ where $X_i \in \{1,2,3,4,5,6\}$ for each i. Let

$$\begin{aligned}Y &= \sum_{i=1}^n 1_{\{1,3,5\}}\left(X_i\right) = \text{ number of odd rolls and} \\ Z &= \sum_{i=1}^n 1_{\{3,4,6\}}\left(X_i\right) \\ &= \text{ number of times 3, 4, or 6 are rolled.}\end{aligned}$$

We wish to compute $\mathbb{E}\left[Z|Y\right].$ So let $0 \le y \le n$ be given and let A be the event where X_i is odd for $1 \le i \le y$ and X_i is even for $y < i \le n$. Then

$$\mathbb{E}\left[Z|A\right] = y\frac{1}{3} + \left(n-y\right) \cdot \frac{2}{3}$$

where $\frac{1}{3} = P\left(X_1 \in \{3,4,6\}|X_1 \text{ is odd}\right)$ and $\frac{2}{3} = P\left(X_1 \in \{3,4,6\}|X_1 \text{ is even}\right).$ Now it is clear that $\{Y = y\}$ can be partitioned into events like the one above being labeled by which of the $y-$slots are even and the results are the same for all such choices by symmetry, therefore by Proposition 5.16 we may conclude

$$\mathbb{E}\left[Z|Y=y\right] = y\frac{1}{3} + (n-y)\cdot\frac{2}{3}$$

and therefore,

$$\mathbb{E}\left[Z|Y\right] = Y\frac{1}{3} + (n-Y)\cdot\frac{2}{3}.$$

As a check notice that

$$\begin{aligned}\mathbb{E}\mathbb{E}\left[Z|Y\right] &= \mathbb{E}Y\frac{1}{3} + (n-\mathbb{E}Y)\cdot\frac{2}{3} = \frac{n}{2}\frac{1}{3} + \left(n-\frac{n}{2}\right)\cdot\frac{2}{3}\\ &= \frac{n}{6}+\frac{n}{3} = \frac{1}{2}n = \mathbb{E}Z.\end{aligned}$$

The next lemma generalizes this result.

Lemma 5.19. *Suppose that $X_i : \Omega \to S$ for $1 \le i \le n$ are i.i.d. random functions into a discrete set S. Given a subset $A \subset S$ let*

$$Z_A := \sum_{i=1}^{n} 1_A\left(X_i\right) = \#\left(\{i : X_i \in A\}\right).$$

If B is another subset of S, then

$$\mathbb{E}\left[Z_A|Z_B\right] = Z_B \cdot P\left(X_1 \in A|X_1 \in B\right) + \left(n - Z_B\right)\cdot P\left(X_1 \in A|X_1 \notin B\right). \quad (5.4)$$

Proof. Intuitively, for a typical trial there are Z_B of the X_i in B and for these i we have $\mathbb{E}\left[1_A\left(X_i\right)|X_i \in B\right] = P\left(X_1 \in A|X_1 \in B\right)$. Likewise there are $n - Z_B$ of the X_i in $S \setminus B$ and for these i we have $\mathbb{E}\left[1_A\left(X_i\right)|X_i \notin B\right] = P\left(X_1 \in A|X_1 \notin B\right)$. On these grounds we are quickly lead to Eq. (5.4).

To prove Eq. (5.4) rigorously we will compute $\mathbb{E}\left[Z_A|Z_B = m\right]$ by partitioning $\{Z_B = m\}$ as $\cup Q_\Lambda$ where Λ runs through subsets of k elements of S and

$$Q_\Lambda = \left(\cap_{i\in\Lambda}\left\{X_i \in B\right\}\right) \cap \left(\cap_{i\in\Lambda^c}\left\{X_i \notin B\right\}\right).$$

Then according to Proposition 5.17,

$$\mathbb{E}\left[Z_A|Q_\Lambda\right] = \mathbb{E}\left[\sum_{i=1}^{n} 1_A\left(Y_i\right)\right]$$

where $\{Y_i\}$ are independent and

$$P\left(Y_i = s\right) = P\left(X_i = s|X_i \in B\right) = P\left(X_1 = s|X_1 \in B\right) \text{ for } i \in \Lambda$$

and

$$P\left(Y_i = s\right) = P\left(X_i = s|X_i \notin B\right) = P\left(X_1 = s|X_1 \notin B\right) \text{ for } i \notin \Lambda.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[Z_A|Q_\Lambda\right] &= \mathbb{E}\left[\sum_{i=1}^{n} 1_A\left(Y_i\right)\right] = \sum_{i=1}^{n}\mathbb{E}1_A\left(Y_i\right)\\ &= \sum_{i\in\Lambda} P\left(X_1 \in A|X_1 \in B\right) + \sum_{i\notin\Lambda} P\left(X_1 \in A|X_1 \notin B\right)\\ &= m\cdot P\left(X_1 \in A|X_1 \in B\right) + (n-m)\cdot P\left(X_1 \in A|X_1 \notin B\right).\end{aligned}$$

As the result is independent of the choice of Λ with $\#(\Lambda) = m$ we may use Proposition 5.16 to conclude that

$$\mathbb{E}\left[Z_A|Z_B = m\right] = m\cdot P\left(X_1 \in A|X_1 \in B\right) + (n-m)\cdot P\left(X_1 \in A|X_1 \notin B\right).$$

As $0 \le m \le n$ is arbitrary Eq. (5.4) follows.

As a check notice that $\mathbb{E}Z_A = n\cdot P\left(X_1 \in A\right)$ while

$$\begin{aligned}\mathbb{E}\mathbb{E}\left[Z_A|Z_B\right] =& \mathbb{E}Z_B\cdot P\left(X_1 \in A|X_1 \in B\right) + \mathbb{E}\left(n - Z_B\right)\cdot P\left(X_1 \in A|X_1 \notin B\right)\\ =& n\cdot P\left(X_1 \in B\right)\cdot P\left(X_1 \in A|X_1 \in B\right)\\ &+ \left(n - n\cdot P\left(X_1 \in B\right)\right)\cdot P\left(X_1 \in A|X_1 \notin B\right)\\ =& n\cdot\left[\begin{array}{c} P\left(X_1 \in B\right)\cdot P\left(X_1 \in A|X_1 \in B\right)\\ +\left(1 - P\left(X_1 \in B\right)\right)\cdot P\left(X_1 \in A|X_1 \notin B\right)\end{array}\right]\\ =& n\cdot\left[P\left(X_1 \in A|X_1 \in B\right)P\left(X_1 \in B\right) + P\left(X_1 \in A|X_1 \notin B\right)P\left(X_1 \notin B\right)\right]\\ =& n\cdot\left[P\left(X_1 \in A, X_1 \in B\right) + P\left(X_1 \in A, X_1 \notin B\right)\right]\\ =& n\cdot P\left(X_1 \in A\right) = \mathbb{E}Z_A.\end{aligned}$$

■

5.2 General Properties of Conditional Expectation

Let us pause for a moment to record a few basic general properties of conditional expectations.

Proposition 5.20 (Contraction Property). *For all $Y \in L^2(P)$, we have $\mathbb{E}\left|\mathbb{E}\left[Y|X\right]\right| \le \mathbb{E}\left|Y\right|$. Moreover if $Y \ge 0$ then $\mathbb{E}\left[Y|X\right] \ge 0$ (a.s.).*

Proof. Let $\mathbb{E}\left[Y|X\right] = h(X)$ (with $h : S \to \mathbb{R}$) and then define

$$f(x) = \begin{cases} 1 & \text{if } h(x) \ge 0\\ -1 & \text{if } h(x) < 0\end{cases}.$$

Since $h(x) f(x) = \left|h(x)\right|$, it follows from Eq. (5.3) that

$$\mathbb{E}\left[|h\left(X\right)|\right]=\mathbb{E}\left[Yf\left(X\right)\right]=\left|\mathbb{E}\left[Yf\left(X\right)\right]\right|\leq\mathbb{E}\left[|Yf\left(X\right)|\right]=\mathbb{E}\left|Y\right|.$$

For the second assertion take $f\left(x\right)=1_{h(x)<0}$ in Eq. (5.3) in order to learn

$$\mathbb{E}\left[h\left(X\right)1_{h(X)<0}\right]=\mathbb{E}\left[Y1_{h(X)<0}\right]\geq0.$$

As $h\left(X\right)1_{h(X)<0}\leq0$ we may conclude that $h\left(X\right)1_{h(X)<0}=0$ a.s. ■

Because of this proposition we may extend the notion of conditional expectation to $Y\in L^{1}\left(P\right)$ as stated in the following theorem which we do not bother to prove here.

Theorem 5.21. *Given $X:\Omega\rightarrow S$ and $Y\in L^{1}\left(P\right)$, there exists an "essentially unique" function $h:S\rightarrow\mathbb{R}$ such that Eq. (5.3) holds for all bounded functions, $f:S\rightarrow\mathbb{R}$. (As above we write $\mathbb{E}\left[Y|X\right]$ for $h\left(X\right)$.) Moreover the contraction property, $\mathbb{E}\left|\mathbb{E}\left[Y|X\right]\right|\leq\mathbb{E}\left|Y\right|$, still holds.*

Theorem 5.22 (Basic properties). *Let Y, Y_1, and Y_2 be integrable random variables and $X:\Omega\rightarrow S$ be given. Then:*

1. *$\mathbb{E}(Y_1+Y_2|X)=\mathbb{E}(Y_1|X)+\mathbb{E}(Y_2|X)$.*
2. *$\mathbb{E}(aY|X)=a\mathbb{E}(Y|X)$ for all constants a.*
3. *$\mathbb{E}(g(X)Y|X)=g(X)\mathbb{E}(Y|X)$ for all bounded functions g.*
4. *$\mathbb{E}(\mathbb{E}(Y|X))=\mathbb{E}Y$. (**Law of total expectation.**)*
5. *If Y and X are independent then $\mathbb{E}(Y|X)=\mathbb{E}Y$.*

Proof. 1. Let $h_i\left(X\right)=\mathbb{E}\left[Y_i|X\right]$, then for all bounded f,

$$\begin{aligned}\mathbb{E}\left[Y_1f\left(X\right)\right]&=\mathbb{E}\left[h_1\left(X\right)f\left(X\right)\right]\text{ and}\\ \mathbb{E}\left[Y_2f\left(X\right)\right]&=\mathbb{E}\left[h_2\left(X\right)f\left(X\right)\right]\end{aligned}$$

and therefore adding these two equations together implies

$$\begin{aligned}\mathbb{E}\left[\left(Y_1+Y_2\right)f\left(X\right)\right]&=\mathbb{E}\left[\left(h_1\left(X\right)+h_2\left(X\right)\right)f\left(X\right)\right]\\ &=\mathbb{E}\left[\left(h_1+h_2\right)\left(X\right)f\left(X\right)\right]\\ \mathbb{E}\left[Y_2f\left(X\right)\right]&=\mathbb{E}\left[h_2\left(X\right)f\left(X\right)\right]\end{aligned}$$

for all bounded f. Therefore we may conclude that

$$\mathbb{E}(Y_1+Y_2|X)=\left(h_1+h_2\right)\left(X\right)=h_1\left(X\right)+h_2\left(X\right)=\mathbb{E}(Y_1|X)+\mathbb{E}(Y_2|X).$$

2. The proof is similar to 1 but easier and so is omitted.

3. Let $h\left(X\right)=\mathbb{E}\left[Y|X\right]$, then $\mathbb{E}\left[Yf\left(X\right)\right]=\mathbb{E}\left[h\left(X\right)f\left(X\right)\right]$ for all bounded functions f. Replacing f by $g\cdot f$ implies

$$\mathbb{E}\left[Yg\left(X\right)f\left(X\right)\right]=\mathbb{E}\left[h\left(X\right)g\left(X\right)f\left(X\right)\right]=\mathbb{E}\left[\left(h\cdot g\right)\left(X\right)f\left(X\right)\right]$$

for all bounded functions f. Therefore we may conclude that

$$\mathbb{E}\left[Yg\left(X\right)|X\right]=\left(h\cdot g\right)\left(X\right)=h\left(X\right)g\left(X\right)=g\left(X\right)\mathbb{E}(Y|X).$$

4. Take $f\equiv1$ in Eq. (5.3).
5. If X and Y are independent and $\mu:=\mathbb{E}\left[Y\right]$, then

$$\mathbb{E}\left[Yf\left(X\right)\right]=\mathbb{E}\left[Y\right]\mathbb{E}\left[f\left(X\right)\right]=\mu\mathbb{E}\left[f\left(X\right)\right]=\mathbb{E}\left[\mu f\left(X\right)\right]$$

from which it follows that $\mathbb{E}\left[Y|X\right]=\mu$ as desired. ■

The next theorem says that conditional expectations essentially only depends on the distribution of $\left(X,Y\right)$ and nothing else.

Theorem 5.23. *Suppose that $\left(X,Y\right)$ and $\left(\tilde{X},\tilde{Y}\right)$ are random vectors such that $\left(X,Y\right)\overset{d}{=}\left(\tilde{X},\tilde{Y}\right)$, i.e. $\mathbb{E}\left[f\left(X,Y\right)\right]=\mathbb{E}\left[f\left(\tilde{X},\tilde{Y}\right)\right]$ for all bounded (or non-negative) functions f. If $h\left(X\right)=\mathbb{E}\left[u\left(X,Y\right)|X\right]$, then $\mathbb{E}\left[u\left(\tilde{X},\tilde{Y}\right)|\tilde{X}\right]=h\left(\tilde{X}\right)$.*

Proof. By assumption we know that

$$\mathbb{E}\left[u\left(X,Y\right)f\left(X\right)\right]=\mathbb{E}\left[h\left(X\right)f\left(X\right)\right]\text{ for all bounded }f.$$

Since $\left(X,Y\right)\overset{d}{=}\left(\tilde{X},\tilde{Y}\right)$, this is equivalent to

$$\mathbb{E}\left[u\left(\tilde{X},\tilde{Y}\right)f\left(\tilde{X}\right)\right]=\mathbb{E}\left[h\left(\tilde{X}\right)f\left(\tilde{X}\right)\right]\text{ for all bounded }f$$

which is equivalent to $\mathbb{E}\left[u\left(\tilde{X},\tilde{Y}\right)|\tilde{X}\right]=h\left(\tilde{X}\right)$. ■

Example 5.24. Let $\left\{X_i\right\}_{i=1}^{\infty}$ be i.i.d. random variables with $\mathbb{E}\left|X_i\right|<\infty$ for all i and let $S_m:=X_1+\dots+X_m$ for $m=1,2,\dots$. We wish to show,

$$\mathbb{E}\left[S_m|S_n\right]=\frac{m}{n}S_n\text{ for all }m\leq n.$$

for all $m\leq n$. To prove this first observe by symmetry[1] that

$$\mathbb{E}\left(X_i|S_n\right)=h\left(S_n\right)\text{ independent of }i.$$

Therefore

$$S_n=\mathbb{E}\left(S_n|S_n\right)=\sum_{i=1}^{n}\mathbb{E}\left(X_i|S_n\right)=\sum_{i=1}^{n}h\left(S_n\right)=n\cdot h\left(S_n\right).$$

[1] Apply Theorem 5.23 using $\left(X_{\ 1},S_n\right)\overset{d}{=}\left(X_i,S_n\right)$ for $1\leq i\leq n$.

Thus we see that

$$\mathbb{E}\left(X_i|S_n\right) = \frac{1}{n}S_n$$

and therefore

$$\mathbb{E}\left(S_m|S_n\right) = \sum_{i=1}^{m}\mathbb{E}\left(X_i|S_n\right) = \sum_{i=1}^{m}\frac{1}{n}S_n = \frac{m}{n}S_n.$$

If $m > n$, then $S_m = S_n + X_{n+1} + \cdots + X_m$. Since X_i is independent of S_n for $i > n$, it follows that

$$\begin{aligned}\mathbb{E}\left(S_m|S_n\right) &= \mathbb{E}\left(S_n + X_{n+1} + \cdots + X_m|S_n\right)\\ &= \mathbb{E}\left(S_n|S_n\right) + \mathbb{E}\left(X_{n+1}|S_n\right) + \cdots + \mathbb{E}\left(X_m|S_n\right)\\ &= S_n + (m-n)\,\mu \text{ if } m \geq n\end{aligned}$$

where $\mu = \mathbb{E}X_i$.

Example 5.25 (See Durrett, #8, p. 213). Suppose that X and Y are two integrable random variables such that

$$\mathbb{E}\left[X|Y\right] = 18 - \frac{3}{5}Y \text{ and } \mathbb{E}\left[Y|X\right] = 10 - \frac{1}{3}X.$$

We would like to find $\mathbb{E}X$ and $\mathbb{E}Y$. To do this we use the law of total expectation to find,

$$\begin{aligned}\mathbb{E}X &= \mathbb{E}\mathbb{E}\left[X|Y\right] = \mathbb{E}\left(18 - \frac{3}{5}Y\right) = 18 - \frac{3}{5}\mathbb{E}Y \text{ and}\\ \mathbb{E}Y &= \mathbb{E}\mathbb{E}\left[Y|X\right] = \mathbb{E}\left(10 - \frac{1}{3}X\right) = 10 - \frac{1}{3}\mathbb{E}X.\end{aligned}$$

Solving this pair of linear equations shows $\mathbb{E}X = 15$ and $\mathbb{E}Y = 5$.

5.3 Conditional Expectation for Continuous Random Variables

(This section will be covered later in the course when first needed.)

Suppose that Y and X are continuous random variables which have a joint density, $\rho_{(Y,X)}\left(y,x\right)$. Then by definition of $\rho_{(Y,X)}$, we have, for all bounded or non-negative, U, that

$$\mathbb{E}\left[U\left(Y,X\right)\right] = \int\int U\left(y,x\right)\rho_{(Y,X)}\left(y,x\right)dydx. \tag{5.5}$$

The marginal density associated to X is then given by

$$\rho_X\left(x\right) := \int \rho_{(Y,X)}\left(y,x\right)dy \tag{5.6}$$

and recall from Math 180A that the conditional density $\rho_{(Y|X)}\left(y,x\right)$ is defined by

$$\rho_{(Y|X)}\left(y,x\right) = \begin{cases}\frac{\rho_{(Y,X)}(y,x)}{\rho_X(x)} & \text{if } \rho_X\left(x\right) > 0\\ 0 & \text{if } \rho_X\left(x\right) = 0\end{cases}. \tag{5.7}$$

Observe that if $\rho_{(Y,X)}\left(y,x\right)$ is continuous, then

$$\rho_{(Y,X)}\left(y,x\right) = \rho_{(Y|X)}\left(y,x\right)\rho_X\left(x\right) \text{ for all } \left(x,y\right). \tag{5.8}$$

Indeed, if $\rho_X\left(x\right) = 0$, then

$$0 = \rho_X\left(x\right) = \int \rho_{(Y,X)}\left(y,x\right)dy$$

from which it follows that $\rho_{(Y,X)}\left(y,x\right) = 0$ for all y. If $\rho_{(Y,X)}$ is not continuous, Eq. (5.8) still holds for "a.e." $\left(x,y\right)$ which is good enough.

Lemma 5.26. *In the notation above,*

$$\rho\left(x,y\right) = \rho_{(Y|X)}\left(y,x\right)\rho_X\left(x\right) \text{ for a.e. } \left(x,y\right). \tag{5.9}$$

Proof. By definition Eq. (5.9) holds when $\rho_X\left(x\right) > 0$ and $\rho\left(x,y\right) \geq \rho_{(Y|X)}\left(y,x\right)\rho_X\left(x\right)$ for all $\left(x,y\right)$. Moreover,

$$\begin{aligned}\int\int \rho_{(Y|X)}\left(y,x\right)\rho_X\left(x\right)dxdy &= \int\int \rho_{(Y|X)}\left(y,x\right)\rho_X\left(x\right)1_{\rho_X(x)>0}dxdy\\ &= \int\int \rho\left(x,y\right)1_{\rho_X(x)>0}dxdy\\ &= \int \rho_X\left(x\right)1_{\rho_X(x)>0}dx = \int \rho_X\left(x\right)dx\\ &= 1 = \int\int \rho\left(x,y\right)dxdy,\end{aligned}$$

or equivalently,

$$\int\int\left[\rho\left(x,y\right) - \rho_{(Y|X)}\left(y,x\right)\rho_X\left(x\right)\right]dxdy = 0$$

which implies the result. ■

Theorem 5.27. *Keeping the notation above,for all or all bounded or non-negative, U, we have $\mathbb{E}\left[U(Y,X)|X\right] = h(X)$ where*

$$h(x) = \int U(y,x)\,\rho_{(Y|X)}(y,x)\,dy \tag{5.10}$$

$$= \begin{cases} \frac{\int U(y,x)\rho_{(Y,X)}(y,x)dy}{\int \rho_{(Y,X)}(y,x)dy} & \text{if} \quad \int \rho_{(Y,X)}(y,x)\,dy > 0 \\ 0 & \text{otherwise} \end{cases}. \tag{5.11}$$

In the future we will usually denote $h(x)$ informally by $\mathbb{E}\left[U(Y,x)|X=x\right]$,[2] *so that*

$$\mathbb{E}\left[U(Y,x)|X=x\right] := \int U(y,x)\,\rho_{(Y|X)}(y,x)\,dy. \tag{5.12}$$

Proof. We are looking for $h : S \to \mathbb{R}$ such that

$$\mathbb{E}\left[U(Y,X)f(X)\right] = \mathbb{E}\left[h(X)f(X)\right] \text{ for all bounded } f.$$

Using Lemma 5.26, we find

$$\begin{aligned}
\mathbb{E}\left[U(Y,X)f(X)\right] &= \int\int U(y,x)f(x)\,\rho_{(Y,X)}(y,x)\,dydx \\
&= \int\int U(y,x)f(x)\,\rho_{(Y|X)}(y,x)\,\rho_X(x)\,dydx \\
&= \int\left[\int U(y,x)\,\rho_{(Y|X)}(y,x)\,dy\right]f(x)\,\rho_X(x)\,dx \\
&= \int h(x)f(x)\,\rho_X(x)\,dx \\
&= \mathbb{E}\left[h(X)f(X)\right]
\end{aligned}$$

where h is given as in Eq. (5.10).

■

Example 5.28 (Durrett 8.15, p. 145). Suppose that X and Y have joint density $\rho(x,y) = 8xy \cdot 1_{0<y<x<1}$. We wish to compute $\mathbb{E}\left[u(X,Y)|Y\right]$. To this end we compute

$$\rho_Y(y) = \int_{\mathbb{R}} 8xy \cdot 1_{0<y<x<1}dx = 8y\int_{x=y}^{x=1} x\cdot dx = 8y\cdot\frac{x^2}{2}|_y^1 = 4y\cdot\left(1-y^2\right).$$

Therefore,

$$\rho_{X|Y}(x,y) = \frac{\rho(x,y)}{\rho_Y(y)} = \frac{8xy\cdot 1_{0<y<x<1}}{4y\cdot(1-y^2)} = \frac{2x\cdot 1_{0<y<x<1}}{(1-y^2)}$$

and so

$$\mathbb{E}\left[u(X,Y)|Y=y\right] = \int_{\mathbb{R}}\frac{2x\cdot 1_{0<y<x<1}}{(1-y^2)}u(x,y)\,dx = 2\frac{1_{0<y<1}}{1-y^2}\int_y^1 u(x,y)\ \ xdx$$

and so

$$\mathbb{E}\left[u(X,Y)|Y\right] = 2\frac{1}{1-Y^2}\int_Y^1 u(x,Y)\,xdx.$$

is the best approximation to $u(X,Y)$ be a function of Y alone.

Proposition 5.29. *Suppose that X, Y are independent random functions, then*

$$\mathbb{E}\left[U(Y,X)|X\right] = h(X)$$

where

$$h(x) := \mathbb{E}\left[U(Y,x)\right].$$

Proof. I will prove this in the continuous distribution case and leave the discrete case to the reader. (The theorem is true in general but requires measure theory in order to prove it in full generality.) The independence assumption is equivalent to $\rho_{(Y,X)}(y,x) = \rho_Y(y)\,\rho_X(x)$. Therefore,

$$\rho_{(Y|X)}(y,x) = \begin{cases} \rho_Y(y) & \text{if } \rho_X(x) > 0 \\ 0 & \text{if } \rho_X(x) = 0 \end{cases}$$

and therefore $\mathbb{E}\left[U(Y,X)|X\right] = h_0(X)$ where

$$\begin{aligned}
h_0(x) &= \int U(y,x)\,\rho_{(Y|X)}(y,x)\,dy \\
&= 1_{\rho_X(x)>0}\int U(y,x)\,\rho_Y(y)\,dy = 1_{\rho_X(x)>0}\mathbb{E}\left[U(Y,x)\right] \\
&= 1_{\rho_X(x)>0}h(x).
\end{aligned}$$

If f is a bounded function of x, then

$$\begin{aligned}
\mathbb{E}\left[h_0(X)f(X)\right] &= \int h_0(x)f(x)\,\rho_X(x)\,dx = \int_{\{x:\rho_X(x)>0\}} h_0(x)f(x)\,\rho_X(x)\,dx \\
&= \int_{\{x:\rho_X(x)>0\}} h(x)f(x)\,\rho_X(x)\,dx = \int h(x)f(x)\,\rho_X(x)\,dx \\
&= \mathbb{E}\left[h(X)f(X)\right].
\end{aligned}$$

So for all practical purposes, $h(X) = h_0(X)$, i.e. $h(X) = h_0(X)$ – a.s. (Indeed, take $f(x) = \operatorname{sgn}(h(x) - h_0(x))$ in the above equation to learn that $\mathbb{E}\left|h(X) - h_0(X)\right| = 0$. ■

[2] **Warning:** this is **not** consistent with Eq. (5.1) as $P(X=x) = 0$ for continuous distributions.

5.4 Conditional Variances

Definition 5.30 (Conditional Variance). *Suppose that $Y \in L^2(P)$ and $X : \Omega \to S$ are given. We define*

$$\operatorname{Var}(Y|X) = \mathbb{E}\left[Y^2|X\right] - \left(\mathbb{E}\left[Y|X\right]\right)^2 \tag{5.13}$$

$$= \mathbb{E}\left[\left(Y - \mathbb{E}\left[Y|X\right]\right)^2 |X\right] \tag{5.14}$$

to be the ***conditional variance of*** Y ***given*** X.

Theorem 5.31. *Suppose that $Y \in L^2(P)$ and $X : \Omega \to S$ are given, then*

$$\operatorname{Var}(Y) = \mathbb{E}\left[\operatorname{Var}(Y|X)\right] + \operatorname{Var}\left(\mathbb{E}\left[Y|X\right]\right).$$

Proof. Taking expectations of Eq. (5.13) implies,

$$\begin{aligned}\mathbb{E}\left[\operatorname{Var}(Y|X)\right] &= \mathbb{E}\mathbb{E}\left[Y^2|X\right] - \mathbb{E}\left(\mathbb{E}\left[Y|X\right]\right)^2 \\ &= \mathbb{E}Y^2 - \mathbb{E}\left(\mathbb{E}\left[Y|X\right]\right)^2 = \operatorname{Var}(Y) + \left(\mathbb{E}Y\right)^2 - \mathbb{E}\left(\mathbb{E}\left[Y|X\right]\right)^2.\end{aligned}$$

The result follows from this identity and the fact that

$$\operatorname{Var}\left(\mathbb{E}\left[Y|X\right]\right) = \mathbb{E}\left(\mathbb{E}\left[Y|X\right]\right)^2 - \left(\mathbb{E}\mathbb{E}\left[Y|X\right]\right)^2 = \mathbb{E}\left(\mathbb{E}\left[Y|X\right]\right)^2 - \left(\mathbb{E}Y\right)^2.$$

■

5.5 Summary on Conditional Expectation Properties

Let Y and X be random variables such that $\mathbb{E}Y^2 < \infty$ and h be function from the range of X to $\mathbb{R}$. Then the following are equivalent:

1. $h(X) = \mathbb{E}(Y|X)$, i.e. $h(X)$ is the conditional expectation of Y given X.
2. $\mathbb{E}(Y - h(X))^2 \le \mathbb{E}(Y - g(X))^2$ for all functions g, i.e. $h(X)$ is the best approximation to Y among functions of X.
3. $\mathbb{E}(Y \cdot g(X)) = \mathbb{E}(h(X) \cdot g(X))$ for all functions g, i.e. $Y - h(X)$ is orthogonal to all functions of X. Moreover, this condition uniquely determines $h(X)$.

The methods for computing $\mathbb{E}(Y|X)$ are given in the next two propositions.

Proposition 5.32 (Discrete Case). *Suppose that Y and X are discrete random variables and $p(y,x) := P(Y = y, X = x)$. Then $\mathbb{E}(Y|X) = h(X)$, where*

$$h(x) = \mathbb{E}(Y|X = x) = \frac{\mathbb{E}(Y : X = x)}{P(X = x)} = \frac{1}{p_X(x)} \sum_y y p(y,x) \tag{5.15}$$

and $p_X(x) = P(X = x)$ is the marginal distribution of X which may be computed as $p_X(x) = \sum_y p(y,x)$.

Proposition 5.33 (Continuous Case). *Suppose that Y and X are random variables which have a joint probability density $\rho(y,x)$ (i.e. $P(Y \in dy, X \in dx) = \rho(y,x)dydx$). Then $\mathbb{E}(Y|X) = h(X)$, where*

$$h(x) = \mathbb{E}(Y|X = x) := \frac{1}{\rho_X(x)} \int_{-\infty}^{\infty} y\rho(y,x)dy \tag{5.16}$$

and $\rho_X(x)$ is the marginal density of X which may be computed as

$$\rho_X(x) = \int_{-\infty}^{\infty} \rho(y,x)dy.$$

Intuitively, in all cases, $\mathbb{E}(Y|X)$ on the set $\{X = x\}$ is $\mathbb{E}(Y|X = x)$. This intuitions should help motivate some of the basic properties of $\mathbb{E}(Y|X)$ summarized in the next theorem.

Theorem 5.34. *Let Y, Y_1, Y_2 and X be random variables. Then:*

1. $\mathbb{E}(Y_1 + Y_2|X) = \mathbb{E}(Y_1|X) + \mathbb{E}(Y_2|X)$.
2. $\mathbb{E}(aY|X) = a\mathbb{E}(Y|X)$ *for all constants a.*
3. $\mathbb{E}(f(X)Y|X) = f(X)\mathbb{E}(Y|X)$ *for all functions f.*
4. $\mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}Y$.
5. *If Y and X are independent then $\mathbb{E}(Y|X) = \mathbb{E}Y$.*
6. *If $Y \ge 0$ then $\mathbb{E}(Y|X) \ge 0$.*

Remark 5.35. Property 4 in Theorem 5.34 turns out to be a very powerful method for computing expectations. I will finish this summary by writing out Property 4 in the discrete and continuous cases:

$$\mathbb{E}Y = \sum_x \mathbb{E}(Y|X = x)p_X(x) \qquad (\textbf{Discrete Case})$$

where

$$\mathbb{E}(Y|X = x) = \begin{cases} \frac{\mathbb{E}(Y 1_{X=x})}{P(X=x)} & \text{if} \quad P(X = x) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}\left[U(Y,X)\right] = \int \mathbb{E}(U(Y,X)|X = x)\rho_X(x)dx, \qquad (\textbf{Continuous Case})$$

where

$$\mathbb{E}\left[U(Y,x)|X = x\right] := \int U(y,x)\rho_{(Y|X)}(y,x)\,dy$$

and

$$\rho_{(Y|X)}(y,x) = \begin{cases} \frac{\rho_{(Y,X)}(y,x)}{\rho_X(x)} & \text{if } \rho_X(x) > 0 \\ 0 & \text{if } \rho_X(x) = 0 \end{cases}.$$

6

Random Sums

Suppose that $\{X_i\}_{i=1}^\infty$ is a collection of random variables and let

$$S_n := \begin{cases} X_1 + \cdots + X_n & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}.$$

Given a $\mathbb{Z}_+$ – valued random variable, N, we wish to consider the random sum;

$$S_N = X_1 + \cdots + X_N.$$

We are now going to suppose for the rest of this subsection that N is independent of $\{X_i\}_{i=1}^\infty$ and for $f \geq 0$ we let

$$Tf(n) := \mathbb{E}[f(S_n)] \text{ for all } n \in \mathbb{N}_0.$$

Theorem 6.1. *Suppose that N is independent of $\{X_i\}_{i=1}^\infty$ as above. Then for any positive function f, we have,*

$$\mathbb{E}[f(S_N)] = \mathbb{E}[Tf(N)].$$

Moreover this formula holds for any f such that

$$\mathbb{E}[|f(S_N)|] = \mathbb{E}[T|f|(N)] < \infty.$$

Proof. If $f \geq 0$ we have,

$$\begin{aligned} \mathbb{E}[f(S_N)] &= \sum_{n=0}^\infty \mathbb{E}[f(S_N) : S_N = n] = \sum_{n=0}^\infty \mathbb{E}[f(S_n) : S_N = n] \\ &= \sum_{n=0}^\infty \mathbb{E}[f(S_n)] P(S_N = n) = \sum_{n=0}^\infty (Tf)(n) P(S_N = n) \\ &= \mathbb{E}[Tf(N)]. \end{aligned}$$

The moreover part follows from general non-sense not really covered in this course. ■

Theorem 6.2. *Suppose that $\{X_i\}_{i=1}^\infty$ are uncorrelated $L^2(P)$ – random variables with $\mu = \mathbb{E}X_i$ and $\sigma^2 = \operatorname{Var}(X_i)$ independent of i. Assuming that $N \in L^2(P)$ is independent of the $\{X_i\}$, then*

$$\mathbb{E}[S_N] = \mu \cdot \mathbb{E}N \tag{6.1}$$

and

$$\operatorname{Var}(S_N) = \sigma^2 \mathbb{E}[N] + \mu^2 \operatorname{Var}(N). \tag{6.2}$$

Proof. Taking $f(x) = x$ in Theorem 6.1 using $Tf(n) = \mathbb{E}[S_n] = n \cdot \mu$ we find,

$$\mathbb{E}[S_N] = \mathbb{E}[\mu \cdot N] = \mu \cdot \mathbb{E}N$$

as claimed. Next take $f(x) = x^2$ in Theorem 6.1 using

$$Tf(n) = \mathbb{E}\left[S_n^2\right] = \operatorname{Var}(S_n) + (\mathbb{E}S_n)^2 = \sigma^2 n + (n \cdot \mu)^2,$$

we find that

$$\begin{aligned} \mathbb{E}\left[S_N^2\right] &= \mathbb{E}\left[\sigma^2 N + \mu^2 N^2\right] \\ &= \sigma^2 \mathbb{E}[N] + \mu^2 \mathbb{E}\left[N^2\right]. \end{aligned}$$

Combining these results shows,

$$\begin{aligned} \operatorname{Var}(S_N) &= \sigma^2 \mathbb{E}[N] + \mu^2 \mathbb{E}\left[N^2\right] - \mu^2 (\mathbb{E}N)^2 \\ &= \sigma^2 \mathbb{E}[N] + \mu^2 \operatorname{Var}(N). \end{aligned}$$

■

Example 6.3 (Karlin and Taylor E.3.1. p77). A six-sided die is rolled, and the number N on the uppermost face is recorded. Then a fair coin is tossed N times, and the total number Z of heads to appear is observed. Determine the mean and variance of Z by viewing Z as a random sum of N Bernoulli random variables. Determine the probability mass function of Z, and use it to find the mean and variance of Z.

We have $Z = S_N = X_1 + \cdots + X_N$ where $X_i = 1$ if heads on the ith toss and zero otherwise. In this case

$$\begin{aligned} \mathbb{E}X_1 &= \frac{1}{2}, \\ \operatorname{Var}(X_1) &= \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \end{aligned}$$

$$\begin{aligned} \mathbb{E}N &= \frac{1}{6}(1 + \cdots + 6) = \frac{1}{6}\frac{7 \cdot 6}{2} = \frac{7}{2}, \\ \mathbb{E}N^2 &= \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) = \frac{91}{6} \\ \operatorname{Var}(N) &= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}. \end{aligned}$$

Therefore,

$$\mathbb{E}Z = \mathbb{E}X_1 \cdot \mathbb{E}N = \frac{1}{2}\cdot\frac{7}{2} = \frac{7}{4}$$
$$\operatorname{Var}(Z) = \frac{1}{4}\cdot\frac{7}{2} + \left(\frac{1}{2}\right)^2 \cdot \frac{35}{12} = \frac{77}{48} = 1.604\,2.$$

Alternatively, we have

$$\begin{aligned} P(Z=k) &= \sum_{n=1}^{6} P(Z=k|N=n)\,P(N=n) \\ &= \frac{1}{6}\sum_{n=k\vee 1}^{6} P(Z=k|N=n) \\ &= \frac{1}{6}\sum_{n=k\vee 1}^{6} \binom{n}{k}\left(\frac{1}{2}\right)^n. \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}Z &= \sum_{k=0}^{6} kP(Z=k) = \sum_{k=1}^{6} kP(Z=k) \\ &= \sum_{k=1}^{6} k\frac{1}{6}\sum_{n=k}^{6}\binom{n}{k}\left(\frac{1}{2}\right)^n = \frac{7}{4} \end{aligned}$$

and

$$\mathbb{E}Z^2 = \sum_{k=0}^{6} k^2 P(Z=k) = \sum_{k=1}^{6} k^2\frac{1}{6}\sum_{n=k}^{6}\binom{n}{k}\left(\frac{1}{2}\right)^n = \frac{14}{3}$$

so that

$$\operatorname{Var}(Z) = \frac{14}{3} - \left(\frac{7}{4}\right)^2 = \frac{77}{48}.$$

We have,

$$P(Z=0) = \frac{1}{6}\sum_{n=1}^{6}\binom{n}{0}\left(\frac{1}{2}\right)^n = \frac{21}{128}$$
$$P(Z=1) = \frac{1}{6}\sum_{n=1}^{6}\binom{n}{1}\left(\frac{1}{2}\right)^n = \frac{5}{16}$$
$$P(Z=2) = \frac{1}{6}\sum_{n=2}^{6}\binom{n}{2}\left(\frac{1}{2}\right)^n = \frac{33}{128}$$
$$P(Z=3) = \frac{1}{6}\sum_{n=3}^{6}\binom{n}{3}\left(\frac{1}{2}\right)^n = \frac{1}{6}$$
$$P(Z=4) = \frac{1}{6}\sum_{n=4}^{6}\binom{n}{4}\left(\frac{1}{2}\right)^n = \frac{29}{384}$$
$$P(Z=5) = \frac{1}{6}\sum_{n=5}^{6}\binom{n}{5}\left(\frac{1}{2}\right)^n = \frac{1}{48}$$
$$P(Z=6) = \frac{1}{6}\sum_{n=6}^{6}\binom{n}{6}\left(\frac{1}{2}\right)^n = \frac{1}{384}.$$

Remark 6.4. If the $\{X_i\}$ are i.i.d., we may work out the moment generating function, $mgf_{S_N}(t) := \mathbb{E}\left[e^{tS_N}\right]$ as follows. Conditioning on $N=n$ shows,

$$\begin{aligned} \mathbb{E}\left[e^{tS_N}|N=n\right] &= \mathbb{E}\left[e^{tS_n}|N=n\right] = \mathbb{E}\left[e^{tS_n}\right] \\ &= \left[\mathbb{E}e^{tX_1}\right]^n = \left[mgf_{X_1}(t)\right]^n \end{aligned}$$

so that

$$\mathbb{E}\left[e^{tS_N}|N\right] = \left[mgf_{X_1}(t)\right]^N = e^{N\ln\left(mgf_{X_1}(t)\right)}.$$

Taking expectations of this equation using the law of total expectation gives,

$$mgf_{S_N}(t) = mgf_N\left(\ln\left(mgf_{X_1}(t)\right)\right).$$

Exercise 6.1 (Karlin and Taylor II.3.P2). For each given p, let Z have a binomial distribution with parameters p and N. Suppose that N is itself binomially distributed with parameters q and M. Formulate Z as a random sum and show that Z has a binomial distribution with parameters pq and M.

Solution to Exercise (Karlin and Taylor II.3.P2). Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. Bernoulli random variables with $P(X_i=1)=p$ and $P(X_i=0)=1-p$. Then $Z \overset{d}{=} X_1+\cdots+X_N$. We now compute

$$
\begin{aligned}
P(Z=k) &= \sum_{n=k}^{M} P(Z=k|N=n)\,P(N=n) \\
&= \sum_{l=0}^{M-k} P(Z=k|N=k+l)\,P(N=k+l) \\
&= \sum_{l=0}^{M-k} P(Z=k|N=k+l)\,P(N=k+l) \\
&= \sum_{l=0}^{M-k} p^k(1-p)^{k+l-k}\binom{k+l}{k}\cdot\binom{M}{k+l}q^{k+l}(1-q)^{M-(k+l)} \\
&= (pq)^k\sum_{l=0}^{M-k}(1-p)^l\,\frac{M!}{k!l!\,(M-k-l)!}q^l(1-q)^{M-k-l} \\
&= \binom{M}{k}(pq)^k\sum_{l=0}^{M-k}\frac{(M-k)!}{l!\,(M-k-l)!}\left[(1-p)\,q\right]^l(1-q)^{M-k-l} \\
&= \binom{M}{k}(pq)^k\sum_{l=0}^{M-k}\binom{M-k}{l}\left[(1-p)\,q\right]^l(1-q)^{M-k-l} \\
&= \binom{M}{k}(pq)^k\left[(1-p)\,q+(1-q)\right]^{M-k} \\
&= \binom{M}{k}(pq)^k\left[1-pq\right]^{M-k}
\end{aligned}
$$

as claimed. See page 58-59 of the notes where this is carried out.

Alternatively. Let $\{\xi_i\}$ be i.i.d. Bernoulli random variables with parameter q and $\{\eta_i\}$ be i.i.d. Bernoulli random variables with parameter p independent of the $\{\xi_i\}$. Then let $N=\eta_1+\dots+\eta_M$ and $Z=\xi_1\eta_1+\dots+\xi_M\eta_M$. Notice that $\{\xi_i\eta_i\}_{i=1}^M$ are Bernoulli random variables with parameter pq so that Z is Binomial with parameters pq and M. Further N is binomial with parameters p and M. Let $B(i_1,\dots,i_n)$ be the event where $\eta_{i_1}=\eta_{i_2}=\dots=\eta_{i_n}=1$ with all others being zero, then

$$
\{N=n\}=\cup_{i_1<\dots<i_n}B(i_1,\dots,i_n)
$$

so that

$$
\begin{aligned}
P(Z=k|N=n) &= \frac{\sum_{i_1<\dots<i_n}P(\{Z=k\}\cap B(i_1,\dots,i_n))}{\sum_{i_1<\dots<i_n}P(B(i_1,\dots,i_n))} \\
&= \frac{\sum_{i_1<\dots<i_n}P(Z=k|B(i_1,\dots,i_n))\,P(B(i_1,\dots,i_n))}{\sum_{i_1<\dots<i_n}P(B(i_1,\dots,i_n))} \\
&= \frac{\sum_{i_1<\dots<i_n}\binom{n}{k}q^k(1-q)^{n-k}P(B(i_1,\dots,i_n))}{\sum_{i_1<\dots<i_n}P(B(i_1,\dots,i_n))} \\
&= \binom{n}{k}q^k(1-q)^{n-k}
\end{aligned}
$$

and this gives another more intuitive proof of the result.

Part I

Markov Chains

7

Markov Chains Basics

For this chapter, let S be a finite or at most countable **state space** and $p : S \times S \to [0,1]$ be a **Markov kernel,** i.e.

$$\sum_{y \in S} p(x,y) = 1 \text{ for all } i \in S. \tag{7.1}$$

A **probability** on S is a function, $\pi : S \to [0,1]$ such that $\sum_{x \in S} \pi(x) = 1$. Further, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\Omega := S^{\mathbb{N}_0} = \{\omega = (s_0, s_1, \dots) : s_j \in S\},$$

and for each $n \in \mathbb{N}_0$, let $X_n : \Omega \to S$ be given by

$$X_n(s_0, s_1, \dots) = s_n.$$

Notation 7.1 *We will denote $(X_0, X_1, X_2, \dots)$ by X.*

Definition 7.2 (Markov probabilities). *A (time homogeneous)* ***Markov probability***[1]*, P, on Ω with transition kernel, p, is probability on Ω such that*

$$\begin{aligned} &P(X_{n+1} = x_{n+1} | X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) \\ &\quad = P(X_{n+1} = x_{n+1} | X_n = x_n) = p(x_n, x_{n+1}) \end{aligned} \tag{7.2}$$

where $\{x_j\}_{j=1}^{n+1}$ are allowed to range over S and n over $\mathbb{N}_0$. The identity in Eq. (7.2) is only to be checked on for those $x_j \in S$ such that $P(X_0 = x_0, X_1 = x_1, \dots, X_n = x_n) > 0$. (Poetically, a Markov chain does not remember its past, its future moves are determined only by its present location and not how it got there.)

If a Markov probability P is given we will often refer to $\{X_n\}_{n=0}^{\infty}$ as a Markov chain. The condition in Eq. (7.2) may also be written as,

$$\mathbb{E}[f(X_{n+1}) \mid X_0, X_1, \dots, X_n] = \mathbb{E}[f(X_{n+1}) \mid X_n] = \sum_{y \in S} p(X_n, y) f(y) \tag{7.3}$$

for all $n \in \mathbb{N}_0$ and any bounded function, $f : S \to \mathbb{R}$.

Proposition 7.3 (Markov joint distributions). *If P is a Markov probability as in Definition 7.2 and $\pi(x) := P(X_0 = x)$, then for all $n \in \mathbb{N}_0$ and $\{x_j\} \subset S$,*

$$P(X_0 = x_0, \dots, X_n = x_n) = \pi(x_0) p(x_0, x_1) \dots p(x_{n-1}, x_n). \tag{7.4}$$

Conversely if $\pi : S \to [0,1]$ is a probability and $\{X_n\}_{n=0}^{\infty}$ is a sequence of random variables satisfying Eq. (7.4) for all n and $\{x_j\} \subset S$, then $(\{X_n\}, P, p)$ satisfies Definition 7.2.

Proof. ($\Longrightarrow$) This formal proof is by induction on n. I will do the case $n = 1$ and $n = 2$ here. For $n = 1$, if $\pi(x_0) = P(X_0 = x_0) = 0$ then both sides of Eq. (7.4) are zero and there is nothing to prove. If $\pi(x_0) = P(X_0 = x_0) > 0$, then

$$\begin{aligned} P(X_0 = x_0, X_1 = x_1) &= P(X_1 = x_1 | X_0 = x_0) P(X_0 = x_0) \\ &= \pi(x_0) \cdot p(x_0, x_1). \end{aligned}$$

Now for the case $n = 2$. Let $p := P(X_0 = x_0, X_1 = x_1) = \pi(x_0) \cdot p(x_0, x_1)$. If $p = 0$ then again both sides of Eq. (7.4) while if $p > 0$ we have by assumption and the case $n = 1$ that

$$\begin{aligned} &P(X_0 = x_0, X_1 = x_1, X_2 = x_2) \\ &\quad = P(X_2 = x_2 | X_0 = x_0, X_1 = x_1,) \cdot P(X_0 = x_0, X_1 = x_1) \\ &\quad = P(X_2 = x_2 | X_1 = x_1) \cdot P(X_0 = x_0, X_1 = x_1) \\ &\quad = p(x_1, x_2) \cdot \pi(x_0) p(x_0, x_1) = \pi(x_0) p(x_0, x_1) p(x_1, x_2). \end{aligned}$$

The formal induction argument is now left to the reader.

($\Longleftarrow$) If

$$\pi(x_0) p(x_0, x_1) \dots p(x_{n-1}, x_n) = P(X_0 = x_0, \dots, X_n = x_n) > 0,$$

[1] The set Ω is sufficiently big that it is no longer so easy to give a rigorous definition of a probability on Ω. For the purposes of this class, a **probability on** Ω should be taken to mean an assignment, $P(A) \in [0,1]$ for all subsets, $A \subset \Omega$, such that $P(\emptyset) = 0$, $P(\Omega) = 1$, and

$$P(A) = \sum_{n=1}^{\infty} P(A_n)$$

whenever $A = \cup_{n=1}^{\infty} A_n$ with $A_n \cap A_m = \emptyset$ for all $m \neq n$. (There are technical problems with this definition which are addressed in a course on "measure theory." We may safely ignore these problems here.)

then by Eq. (7.4) and the definition of conditional probabilities we find,

$$\begin{aligned}&P\left(X_{n+1}=x_{n+1}|X_0=x_0,X_1=x_1,\dots,X_n=x_n\right)\\&=\frac{P\left(X_0=x_0,X_1=x_1,\dots,X_n=x_n,X_{n+1}=x_{n+1}\right)}{P\left(X_0=x_0,\dots,X_n=x_n\right)}\\&=\frac{\pi\left(x_0\right)p\left(x_0,x_1\right)\dots p\left(x_{n-1},x_n\right)p\left(x_n,x_{n+1}\right)}{\pi\left(x_0\right)p\left(x_0,x_1\right)\dots p\left(x_{n-1},x_n\right)}=p\left(x_n,x_{n+1}\right)\end{aligned}$$

as desired. ■

Fact 7.4 *To each probability π on S there is a unique Markov probability, P_π, on Ω such that $P_\pi\left(X_0=x\right)=\pi\left(x\right)$ for all $x\in X$. Moreover, P_π is uniquely determined by Eq. (7.4).*

Notation 7.5 *We will abbreviate the expectation $\left(\mathbb{E}_{P_\pi}\right)$ with respect to P_π by $\mathbb{E}_\pi$. Moreover if*

$$\pi\left(y\right)=\delta_x\left(y\right):=\begin{cases}1 \text{ if } x=y\\0 \text{ if } x\neq y\end{cases}, \tag{7.5}$$

we will write P_x for $P_\pi=P_{\delta_x}$ and $\mathbb{E}_x$ for $\mathbb{E}_{\delta_x}$

For a general probability, π, on S, it follows from Proposition 7.3 and Corollary 7.6 that

$$P_\pi=\sum_{x\in S}\pi\left(x\right)P_x \text{ and } \mathbb{E}_\pi=\sum_{x\in S}\pi\left(x\right)\mathbb{E}_x. \tag{7.6}$$

Corollary 7.6. *If π is a probability on S and $u:S^{n+1}\rightarrow\mathbb{R}$ is a bounded or non-negative function, then*

$$\mathbb{E}_\pi\left[u\left(X_0,\dots,X_n\right)\right]=\sum_{x_0,\dots,x_n\in S}u\left(x_0,\dots,x_n\right)\pi\left(x_0\right)p\left(x_0,x_1\right)\dots p\left(x_{n-1},x_n\right).$$

Definition 7.7 (Matrix multiplication). *If $q:S\times S\rightarrow[0,1]$ is another Markov kernel we let $p\cdot q:S\times S\rightarrow[0,1]$ be defined by*

$$\left(p\cdot q\right)\left(x,y\right):=\sum_{z\in S}p\left(x,z\right)q\left(z,y\right). \quad (!) \tag{7.7}$$

We also let

$$p^n:=\overbrace{p\cdot p\cdot\dots\cdot p}^{n\text{ - times}}.$$

If $\pi:S\rightarrow[0,1]$ is a probability we let $\left(\pi\cdot q\right):S\rightarrow[0,1]$ be defined by

$$\left(\pi\cdot q\right)\left(y\right):=\sum_{x\in S}\pi\left(x\right)q\left(x,y\right).$$

As the definition suggests, $p\cdot q$ is the multiplication of matrices and $\pi\cdot q$ is the multiplication of a row vector π with a matrix q. It is easy to check that $\pi\cdot q$ is still a probability and $p\cdot q$ and p^n are Markov kernels. A key point to keep in mind is that a Markov process is completely specified by its transition kernel, $p:S\times S\rightarrow[0,1]$. For example we have the following method for computing $P_x\left(X_n=y\right)$.

Lemma 7.8. *Keeping the above notation, $P_x\left(X_n=y\right)=p^n\left(x,y\right)$ and more generally,*

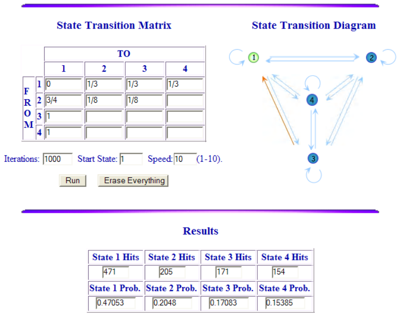
$$P_\pi\left(X_n=y\right)=\sum_{x\in S}\pi\left(x\right)p^n\left(x,y\right)=\left(\pi\cdot p^n\right)\left(y\right).$$

Proof. We have from Eq. (7.4) that

$$\begin{aligned}P_x\left(X_n=y\right)&=\sum_{x_0,\dots,x_{n-1}\in S}P_x\left(X_0=x_0,X_1=x_1,\dots,X_{n-1}=x_{n-1},X_n=y\right)\\&=\sum_{x_0,\dots,x_{n-1}\in S}\delta_x\left(x_0\right)p\left(x_0,x_1\right)\dots p\left(x_{n-2},x_{n-1}\right)p\left(x_{n-1},y\right)\\&=\sum_{x_1,\dots,x_{n-1}\in S}p\left(x,x_1\right)\dots p\left(x_{n-2},x_{n-1}\right)p\left(x_{n-1},y\right)=p^n\left(x,y\right).\end{aligned}$$

The formula for $P_\pi\left(X_n=y\right)$ easily follows from this formula. ■

To get a feeling for Markov chains, I suggest the reader play around with the simulation provided by Stefan Waner and Steven R. Costenoble at `www.zweigmedia.com/RealWorld/markov/markov.html` – see Figure 7.1 below.



State Transition Matrix

		TO			
		1	2	3	4
FROM	1	0	1/3	1/3	1/3
	2	3/4	1/8	1/8	
	3	1			
	4	1			

Iterations: 1000 Start State: 1 Speed: 10 (1-10).

State Transition Diagram

Results

State 1 Hits	State 2 Hits	State 3 Hits	State 4 Hits
471	205	171	154
State 1 Prob.	**State 2 Prob.**	**State 3 Prob.**	**State 4 Prob.**
0.47053	0.2048	0.17083	0.15385

Fig. 7.1. See `www.zweigmedia.com/RealWorld/markov/markov.html` for a Markov chain simulator for chains with a state space of 4 elements or less. The user describes the chain by filling in the transition matrix **P**.

7.1 Examples

Notation 7.9 *Associated to a transition kernel, p, is a **jump graph (or jump diagram)** gotten by taking S as the set of vertices and then for $x, y \in S$, draw an arrow from x to y if $p(x, y) > 0$ and label this arrow by the value $p(x, y)$.*

Example 7.10. The transition matrix,

$$\mathbf{P} = \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

is represented by the jump diagram in Figure 7.2.

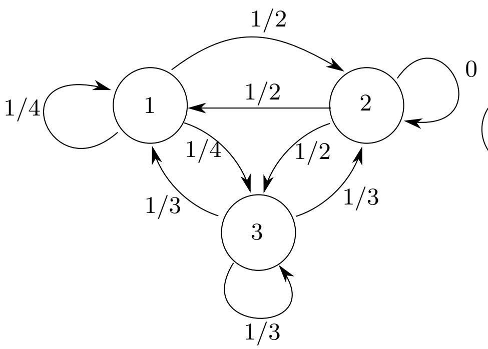


Fig. 7.2. A simple 3 state jump diagram. We typically abbreviate the jump diagram on the left by the one on the right. That is we infer by conservation of probability there has to be probability 1/4 of staying at 1, 1/3 of staying at 3 and 0 probability of staying at 2.

Example 7.11. The jump diagram for

$$\mathbf{P} = \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

is shown in Figure 7.3.

Example 7.12. Suppose that $S = \{1, 2, 3\}$, then

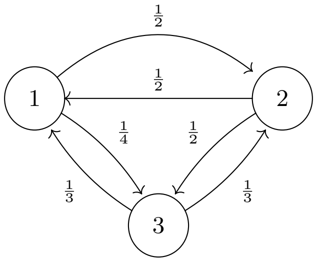


Fig. 7.3. In the above diagram there are jumps from 1 to 1 with probability 1/4 and jumps from 3 to 3 with probability 1/3 which are not explicitly shown but must be inferred by conservation of probability.

$$\mathbf{P} = \begin{array}{c c} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

has the jump graph given by 7.2.

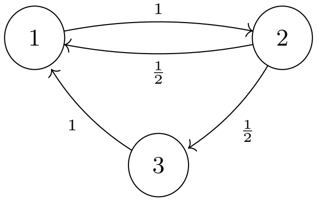


Fig. 7.4. A simple 3 state jump diagram.

Example 7.13 (Ehrenfest Urn Model). Let a beaker filled with a particle fluid mixture be divided into two parts A and B by a semipermeable membrane. Let $X_n = (\#\text{ of particles in } A)$ which we assume evolves by choosing a particle at random from $A \cup B$ and then replacing this particle in the opposite bin from which it was found. Modeling $\{X_n\}$ as a Markov process we find,

$$P(X_{n+1} = j \mid X_n = i) = \begin{cases} 0 & \text{if } j \notin \{i-1, i+1\} \\ \frac{i}{N} & \text{if } \quad j = i - 1 \\ \frac{N-i}{N} & \text{if } \quad j = i + 1 \end{cases} =: q(i, j)$$

As these probabilities do not depend on n, $\{X_n\}$ is a time homogeneous Markov chain.

Exercise 7.1. Consider a rat in a maze consisting of 7 rooms which is laid out as in the following figure.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & & \end{bmatrix}$$

In this figure rooms are connected by either vertical or horizontal adjacent passages only, so that 1 is connected to 2 and 4 but not to 5 and 7 is only connected to 4. At each time $t \in \mathbb{N}_0$ the rat moves from her current room to one of the adjacent rooms with equal probability (the rat always changes rooms at each time step). Find the one step 7×7 transition matrix, q, with entries given by $\mathbf{P}_{ij} := P(X_{n+1} = j | X_n = i)$, where X_n denotes the room the rat is in at time n.

Solution to Exercise (7.1). The rat moves to an adjacent room from nearest neighbor locations probability being $1/D$ where D is the number of doors in the room where the rat is currently located. The transition matrix is therefore,

$$\mathbf{P} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \left[\begin{array}{ccccccc} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}. \tag{7.8}$$

and the corresponding jump diagram is given in Figure 7.5.

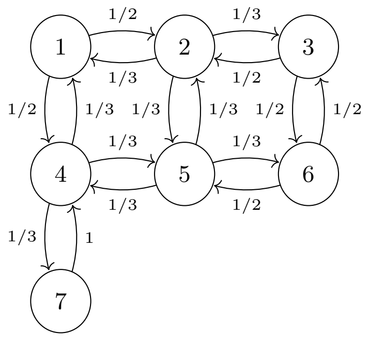


Fig. 7.5. The jump diagram for our rat in the maze.

Exercise 7.2 (2 - step MC). Consider the following simple (i.e. no-brainer) two state "game" consisting of moving between two sites labeled 1 and 2. At each site you find a coin with sides labeled 1 and 2. The probability of flipping a 2 at site 1 is $a \in (0,1)$ and a 1 at site 2 is $b \in (0,1)$. If you are at site i at time n, then you flip the coin at this site and move or stay at the current site as indicated by coin toss. We summarize this scheme by the "jump diagram" of Figure **??**. It is reasonable to suppose that your location, X_n, at time n is modeled by a Markov process with state space, $S = \{1, 2\}$. Explain (briefly) why this is a time homogeneous chain and find the one step transition probabilities,

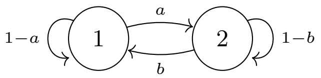


Fig. 7.6. The generic jump diagram for a two state Markov chain.

$$p(i,j) = P(X_{n+1} = j | X_n = i) \text{ for } i, j \in S.$$

Use your result and basic linear (matrix) algebra to compute, $\lim_{n\to\infty} P(X_n = 1)$. Your answer should be independent of the possible starting distributions, $\pi = (\pi_1, \pi_2)$ for X_0 where $\pi_i := P(X_0 = i)$.

Example 7.14. As we will see in concrete examples (see the homework and the text), many Markov chains arise in the following general fashion. Let S and T be discrete sets, $\alpha : S \times T \to S$ be a function, $\{\xi_n\}_{n=1}^\infty$ be i.i.d. random functions with values in T. Then given a random function, X_0 independent of the $\{\xi_n\}_{n=1}^\infty$ with values in S define X_n inductively by $X_{n+1} = \alpha(X_n, \xi_{n+1})$ for $n = 0, 1, 2, \ldots$. We will see that $\{X_n\}_{n=0}^\infty$ satisfies the Markov property with

$$p(x,y) = P(\{\alpha(x,\xi) = y\})$$

where $\xi \overset{d}{=} \xi_n$. To verify this is a Markov process first observe that notice that ξ_{n+1} is independent of $\{X_k\}_{k=0}^n$ as X_k depends on $(X_0, \xi_1, \ldots, \xi_k)$ for all k. Therefore

$$\begin{aligned} P[X_{n+1} = x_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n] \\ = P[\alpha(X_n, \xi_{n+1}) = x_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n] \\ = P[\alpha(x_n, \xi_{n+1}) = x_{n+1} \mid X_0 = x_0, \ldots, X_n = x_n] \\ = P(\alpha(x_n, \xi_{n+1}) = x_{n+1}) = p(x_n, x_{n+1}). \end{aligned}$$

Example 7.15 (Random Walks on the line). Suppose we have a walk on the line with probability of jumping to the right (left) is p $(q = 1 - p)$. In this case we have

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccccc} \cdots & -1 & 0 & 1 & 2 & \cdots \end{array} \\ \left[\begin{array}{cccccc} \ddots & \ddots & & & & \\ \ddots & 0 & p & & & \\ & q & 0 & p & & \\ & & q & 0 & p & \\ & & & q & 0 & \ddots \\ & & & & \ddots & \ddots \end{array}\right] \end{array} \begin{array}{c} \\ \vdots \\ -1 \\ 0 \\ 1 \\ 2 \\ \vdots \end{array},$$

i.e.

$$\mathbf{P}_{ij} = \begin{cases} p & \text{if} \quad j = i+1 \\ q & \text{if} \quad j = i-1 \\ 0 \text{ otherwise} & \end{cases}$$

The jump diagram for such a walk is given in Figure 7.7.This fits into Example 7.14 by taking $S = \mathbb{Z}$, $T = \{\pm 1\}$, $F(s,t) = s + t$, and $\xi_n \overset{d}{=} \xi$ where $P(\xi = +1) = p$ and $P(\xi = -1) = q = 1 - p$.

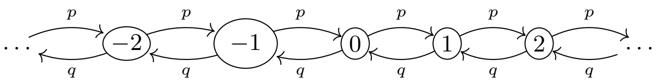


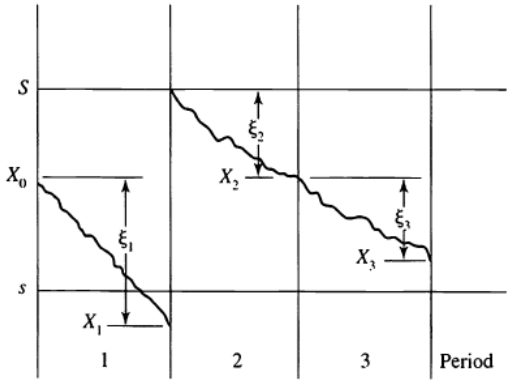
Fig. 7.7. The jump diagram for a possibly biassed simple random walk on the line.

Example 7.16 (See III.3.1 of Karlin and Taylor). Let ξ_n denote the demand of a commodity during the n^{th} – period. We will assume that $\{\xi_n\}_{n=1}^\infty$ are i.i.d. with $P(\xi_n = k) = a_k$ for $k \in \mathbb{N}_0$. Let X_n denote the quantity of stock on hand at the end of the n^{th} – period which is subject to the following replacement policy. We choose $s, S \in \mathbb{N}_0$ with $s < S$, if $X_n \le s$ we immediately replace the stock to have S on hand at the beginning of the next period while if $X_n > s$ we do not add any stock. Thus,

$$X_{n+1} = \begin{cases} X_n - \xi_{n+1} \text{ if } s < X_n \le S \\ S - \xi_{n+1} \text{ if } \quad X_n \le s, \end{cases}$$

see Figure 3.1 on p. 106 of the book (also repeated below). Notice that we allow the stock to go negative indicating the demand is not met. It now follows that

$$P(X_{n+1} = y | X_n = x) = \begin{cases} P(\xi_{n+1} = x - y) \text{ if } s < x \le S \\ P(\xi_{n+1} = S - y) \text{ if } \quad x \le s \end{cases}$$
$$= \begin{cases} a_{x-y} \text{ if } s < x \le S \\ a_{S-y} \text{ if } \quad x \le s \end{cases}$$



Example 7.17 (Discrete queueing model). Let $X_n = \#$ of people in line at time n, $\{\xi_n\}$ be i.i.d. be the number of customers arriving for service in a period and assume one person is served if there are people in the queue (think of a taxi stand). Therefore, $X_{n+1} = (X_n - 1)_+ + \xi_n$ and assuming that $P(\xi_n = k) = a_k$ for all $k \in \mathbb{N}_0$ we have,

$$P(X_{n+1} = j \mid X_n = i) = \begin{cases} 0 & \text{if } j < i - 1 \\ P(\xi_n = 0) = a_0 & \text{if } j = i - 1 \\ P(\xi_n = j - (i-1)) = a_{j-i+1} & \text{if } j \ge i \end{cases}$$

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \left[\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & \cdots & \cdots \\ a_0 & a_1 & a_2 & \cdots & \cdots & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots & \cdots \\ 0 & 0 & a_0 & a_1 & a_2 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}.$$

Remark 7.18 (Memoryless property of the geometric distribution). Suppose that $\{X_i\}$ are i.i.d. Bernoulli random variables with $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$ and $N = \inf\{i \ge 1 : X_i = 1\}$. Then $P(N = k) = P(X_1 = 0, \dots, X_{k-1} = 0, X_k = 1) = (1-p)^{k-1} p$, so that N is geometric with parameter p. Using this representation we easily and intuitively see that

$$\begin{aligned} P(N = n + k | N > n) &= \frac{P(X_1 = 0, \dots, X_{n+k-1} = 0, X_{n+k} = 1)}{P(X_1 = 0, \dots, X_n = 0)} \\ &= P(X_{n+1} = 0, \dots, X_{n+k-1} = 0, X_{n+k} = 1) \\ &= P(X_1 = 0, \dots, X_{k-1} = 0, X_k = 1) = P(N = k). \end{aligned}$$

This can be verified by first principles as well;

$$\begin{aligned}P\left(N=n+k|N>n\right)&=\frac{P\left(N=n+k\right)}{P\left(N>n\right)}=\frac{p\left(1-p\right)^{n+k-1}}{\sum_{k>n}p\left(1-p\right)^{k-1}}\\&=\frac{p\left(1-p\right)^{n+k-1}}{\sum_{j=0}^{\infty}p\left(1-p\right)^{n+j}}=\frac{\left(1-p\right)^{n+k-1}}{\left(1-p\right)^{n}\sum_{j=0}^{\infty}\left(1-p\right)^{j}}\\&=\frac{\left(1-p\right)^{k-1}}{\frac{1}{1-(1-p)}}=p\left(1-p\right)^{k-1}=P\left(N=k\right).\end{aligned}$$

Exercise 7.3 (III.3.P4. (Queueing model)). Consider the queueing model of Section 3.4. of Karlin and Taylor. Now suppose that at most a single customer arrives during a single period, but that the service time of a customer is a random variable Z with the geometric probability distribution

$$P\left(Z=k\right)=\alpha\left(1-\alpha\right)^{k-1}\text{ for }k\in\mathbb{N}.$$

Specify the transition probabilities for the Markov chain whose state is the number of customers waiting for service or being served at the start of each period. Assume that the probability that a customer arrives in a period is β and that no customer arrives with probability $1-\beta$.

Solution to Exercise (III.3.P4). Notice that the probability that the service of customer currently being served is finished at the end of the current period is $\alpha=P\left(Z=m+1|Z>m\right)$; this is the memoryless property of the geometric distribution. A $k\to k$ transition can happen in two ways: (i) a new customer arrives and the customer being served finishes, or (ii) no new customer arrives and the customer in service does not finish. The total probability of a $k\to k$ transition is therefore

$$\beta\cdot\alpha+(1-\beta)(1-\alpha)=1-\alpha-\beta+2\alpha\beta.$$

(If $k=0$ this formula must be emended; the probability of a $0\to 0$ transition is simply $1-\beta$.) A $k\to k+1$ transition occurs if a new customer arrives but the customer in service does not finish; this has probability $(1-\alpha)\beta$ (β if $k=0$). Finally, for $k\geq 1$, the probability of a $k\to k-1$ transition is $\alpha(1-\beta)$, see Figure 7.8 for the jump diagram.

Proposition 7.19 (Historical MC). *Suppose that* $\{X_n\}_{n=0}^{\infty}$ *is a Markov chain with transition probabilities,* $p\left(x,y\right)$ *for* $x,y\in S$. *Then for any* $m\in\mathbb{N}$,

$$Y_n:=\left(X_n,X_{n+1},\dots,X_{n+m}\right)$$

is a Markov chain with values in S^{m+1} *whose transition kernel,* q, *is given by*

$$q\left(\left(a_0,\dots,a_m\right),\left(b_0,\dots,b_m\right)\right)=\delta\left(b_0,a_1\right)\dots\delta\left(b_{m-1},a_m\right)p\left(a_m,b_m\right).$$

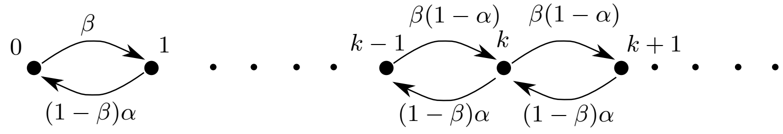


Fig. 7.8. A jump diagram for a simple queueing model.

Proof. Let me give the proof for $m=2$ only as this should suffice to explain the ideas. We have,

$$\begin{aligned}&P\left(Y_{n+1}=\left(b_0,b_1,b_2\right)|Y_n=\left(a_0,a_1,a_2\right),Y_{n-1}=*,\dots,Y_0=*\right)=\\&=P\left(\left(X_{n+1},X_{n+2},X_{n+3}\right)=\left(b_0,b_1,b_2\right)\middle|\begin{array}{c}\left(X_n,X_{n+1},X_{n+2}\right)=\left(a_0,a_1,a_2\right)\\Y_{n-1}=*,\dots,Y_0=*\end{array}\right)\\&=P\left(\left(X_{n+1},X_{n+2},X_{n+3}\right)=\left(b_0,b_1,b_2\right)\middle|\begin{array}{c}\left(X_n,X_{n+1},X_{n+2}\right)=\left(a_0,a_1,a_2\right)\\X_{n-1}=*,\dots,X_0=*\end{array}\right)\\&=P\left(\left(a_1,a_2,X_{n+3}\right)=\left(b_0,b_1,b_2\right)\middle|\begin{array}{c}\left(X_n,X_{n+1},X_{n+2}\right)=\left(a_0,a_1,a_2\right)\\X_{n-1}=*,\dots,X_0=*\end{array}\right)\\&=\delta\left(b_0,a_1\right)\delta\left(b_1,a_2\right)P\left(X_{n+3}=b_2|X_{n+2}=a_2,X_{n+1}=*,\dots,X_0=*\right)\\&=\delta\left(a_0,b_1\right)\delta\left(a_2,b_1\right)p\left(a_2,b_2\right).\end{aligned}$$

■

Example 7.20. Suppose we flip a fair coin repeatedly and would like to find the first time the pattern HHT appears. To do this we will later examine the Markov chain, $Y_n=\left(X_n,X_{n+1},X_{n+2}\right)$ where $\{X_n\}_{n=0}^{\infty}$ is the sequence of unbiased independent coin flips with values in $\{H,T\}$. The state space for Y_n is

$$S=\left\{TTT\ \ THT\ \ TTH\ THH\ HHH\ HTT\ HTH\ HHT\right\}.$$

The transition matrix for recording three flips in a row of a fair coin is

$$\mathbf{P}=\frac{1}{2}\left[\begin{array}{c|cccccccc}&TTT&THT&TTH&THH&HHH&HTT&HTH&HHT\\\hline TTT&1&0&1&0&0&0&0&0\\THT&0&0&0&0&0&1&1&0\\TTH&0&1&0&1&0&0&0&0\\THH&0&0&0&0&1&0&0&1\\HHH&0&0&0&0&1&0&0&1\\HTT&1&0&1&0&0&0&0&0\\HTH&0&1&0&1&0&0&0&0\\HHT&0&0&0&0&0&1&1&0\end{array}\right].$$

7.2 Hitting Times

We assume the $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with values in S and transition kernel $\mathbf{P}$. I will often write $p(x,y)$ for $\mathbf{P}_{xy}$. We are going to further assume that $B \subset S$ is non-empty proper subset of S and $A = S \setminus B$.

Definition 7.21 (Hitting times). *Given a subset $B \subset S$ we let T_B be the first time $\{X_n\}$ hits B, i.e.*

$$T_B = \min\{n : X_n \in B\}$$

*with the convention that $T_B = \infty$ if $\{n : X_n \in B\} = \emptyset$. We call T_B the **first hitting time** of B by $X = \{X_n\}_n$.*

Observe that

$$\begin{aligned}\{T_B = n\} &= \{X_0 \notin B, \dots, X_{n-1} \notin B, X_n \in B\} \\ &= \{X_0 \in A, \dots, X_{n-1} \in A, X_n \in B\}\end{aligned}$$

and

$$\{T_B > n\} = \{X_0 \in A, \dots, X_{n-1} \in A, X_n \in A\}$$

so that $\{T_B = n\}$ and $\{T_B > n\}$ only depends on $(X_0, \dots, X_n)$. A random time, $T : \Omega \to \mathbb{N} \cup \{0, \infty\}$, with either of these properties is called a **stopping time.**

Lemma 7.22. *For any random time $T : \Omega \to \mathbb{N} \cup \{0, \infty\}$ we have*

$$P(T = \infty) = \lim_{n\to\infty} P(T > n) \text{ and } \mathbb{E}T = \sum_{k=0}^{\infty} P(T > k).$$

Proof. The first equality is a consequence of the continuity of P and the fact that

$$\{T > n\} \downarrow \{T = \infty\}.$$

The second equality is proved as follows;

$$\begin{aligned}\mathbb{E}T &= \sum_{m>0} mP(T = m) = \sum_{0<k\le m<\infty} P(T = m) \\ &= \sum_{k=1}^{\infty} P(T \ge k) = \sum_{k=0}^{\infty} P(T > k).\end{aligned}$$

■

Notation 7.23 *Let $\mathbf{Q}$ be $\mathbf{P}$ restricted to A, i.e. $\mathbf{Q}_{x,y} = \mathbf{P}_{x,y}$ for all $x, y \in A$. In particular we have*

$$\mathbf{Q}^N_{x,y} := \sum_{x_1,\dots,x_{N-1}\in A} Q_{x,x_1} Q_{x_1,x_2} \dots Q_{x_{N-1},y} \text{ for all } x, y \in A.$$

Corollary 7.24. *Continuing the notation introduced above, for any $x \in A$ we have*

$$P_x(T_B = \infty) = \lim_{N\to\infty} \sum_{y\in A} \mathbf{Q}^N_{x,y}$$

and

$$\mathbb{E}_x[T_B] = \sum_{N=0}^{\infty} \sum_{y\in A} \mathbf{Q}^N_{x,y}$$

with the convention that

$$\mathbf{Q}^0_{x,y} = \delta_{x,y} = \begin{cases} 1 \text{ if } x = y \\ 0 \text{ if } x \ne y \end{cases}.$$

Proof. The results follow from Lemma 7.22 after observing that

$$\begin{aligned}P_x(T_B > N) &= P_x(X_0 \in A, \dots, X_N \in A) \\ &= \sum_{x_1,\dots,x_N\in A} p(x,x_1)\, p(x_1,x_2) \dots p(x_{N-1},x_N) = \sum_{y\in A} \mathbf{Q}^N_{x,y}. \quad (7.9)\end{aligned}$$

■

Proposition 7.25. *Suppose that $B \subset S$ is non-empty proper subset of S and $A = S \setminus B$. Further suppose there is some $\alpha < 1$ such that $P_x(T_B = \infty) \le \alpha$ for all $x \in A$, then $P_x(T_B = \infty) = 0$ for all $x \in A$. [In words; if there is a "uniform" chance that X hits B starting from any site, then X will surely hit B.]*

Proof. Taking $N = m + n$ in Eq. (7.9) shows

$$P_x(T_B > m + n) = \sum_{y,z\in A} \mathbf{Q}^m_{x,y} \mathbf{Q}^n_{y,z} = \sum_{y\in A} \mathbf{Q}^m_{x,y} P_y(T_B > n). \quad (7.10)$$

Letting $n \to \infty$ (using D.C.T.) in this equation shows,

$$\begin{aligned}P_x(T_B = \infty) &= \sum_{y\in A} \mathbf{Q}^m_{x,y} P_y(T_B = \infty) \\ &\le \alpha \sum_{y\in A} \mathbf{Q}^m_{x,y} = \alpha P_x(T_B > n).\end{aligned}$$

Finally letting $n \to \infty$ shows $P_x(T_B = \infty) \le \alpha P_x(T_B = \infty)$, i.e. $P_x(T_B = \infty) = 0$ for all $x \in A$. ■

We will see in examples later that it is possible for $P_x(T_B = \infty) = 0$ while $\mathbb{E}_x T_B = \infty$. The next theorem gives a criteria which avoids this scenario.

Theorem 7.26. *Suppose that $B \subset S$ is non-empty proper subset of S and $A = S \setminus B$. Further suppose there is some $\alpha < 1$ and $n < \infty$ such that $P_x(T_B > n) \le \alpha$ for all $x \in A$, then*

$$\mathbb{E}_x(T_B) \le \frac{n}{1-\alpha} < \infty$$

for all $x \in A$. [In words; if there is a "uniform" chance that X hits B starting from any site within a fixed number of steps, then the expected hitting time of B is finite and bounded independent of the starting point.]

Proof. From Eq. (7.10) for any $m \in \mathbb{N}$ we have

$$P_x(T_B > m+n) = \sum_{y \in A} \mathbf{Q}^m_{x,y} P_y(T_B > n) \le \alpha \sum_{y \in A} \mathbf{Q}^m_{x,y} = \alpha P_x(T_B > m).$$

One easily uses this relationship to show inductively that

$$P_x(T_B > kn) \le \alpha^k \text{ for all } k = 0, 1, 2 \dots.$$

We then have,

$$\begin{aligned}\mathbb{E}_x T_B &= \sum_{k=0}^{\infty} P(T_B > k) \le \sum_{k=0}^{\infty} nP(T_B > kn) \\ &\le \sum_{k=0}^{\infty} n\alpha^k = \frac{n}{1-\alpha} < \infty,\end{aligned}$$

wherein we have used,

$$P(T_B > kn + m) \le P(T_B > kn) \text{ for } m = 0, \dots, n-1.$$

■

Corollary 7.27. *If $A = S \setminus B$ is a finite set and $P_x(T_B = \infty) < 1$ for all $x \in A$, then $\mathbb{E}_x T_B < \infty$ for all $x \in A$.*

Proof. Let $\alpha_0 = \max_{x \in A} P_x(T = \infty) < 1$. Now fix $\alpha \in (\alpha_0, 1)$. Using

$$\alpha_0 \ge P_x(T = \infty) = \downarrow \lim_{n \to \infty} P_x(T > n)$$

we will have $P_x(T > m) \le \alpha$ for $m \ge N_x$ for some $N_x < \infty$. Taking $n := \max\{N_x : x \in A\} < \infty$ (A is a finite set), we will have $P_x(T > n) \le \alpha$ for all $x \in A$ and we may now apply Theorem 7.26. ■

8

Markov Conditioning

We assume the $\{X_n\}_{n=0}^{\infty}$ is a Markov chain with values in S and transition kernel $\mathbf{P}$ and $\pi : S \to [0,1]$ is a probability on S. As usual we write P_π for the unique probability satisfying Eq. (7.4) and we will often write $p(x,y)$ for $\mathbf{P}_{xy}$.

Theorem 8.1 (Markov conditioning). *Let π be a probability on S, $F(X) = F(X_0, X_1, \dots)$ be a random variable*[1] *depending on X. Then for each $m \in \mathbb{N}$ we have*

$$\mathbb{E}_\pi \left[F(X_0, X_1, \dots)\right] = \mathbb{E}_\pi \left[\mathbb{E}^{(Y)}_{X_m} F(X_0, X_1, \dots X_{m-1}, Y_0, Y_1, \dots)\right] \tag{8.1}$$

where $\mathbb{E}^{(Y)}_x$ denotes the expectation with respect to an independent copy, Y, of the chain X which starts at $x \in S$. To be more explicit,

$$\mathbb{E}_\pi \left[F(X_0, X_1, \dots)\right] = \mathbb{E}_\pi \left[h(X_0, \dots, X_m)\right]$$

where for all $x_0, \dots, x_m \in S$,

$$h(x_0, \dots, x_m) := \mathbb{E}_{x_m}\left[F(x_0, \dots, x_{m-1}, X_0, X_1, \dots)\right].$$

[In words, given $X_0, \dots, X_m$, $(X_m, X_{m+1}, \dots)$ has the same distribution as independent copy $(Y_0, Y_1, \dots)$ of the chain X where Y required to start at X_m.]

Alternatively stated: *if $x_0, x_1, \dots, x_m \in S$ with $P_\pi(X_0 = x_0, \dots, X_m = x_m) > 0$, then*

$$\begin{aligned} &\mathbb{E}_\pi \left[F(X_0, X_1, \dots) | X_0 = x_0, \dots, X_m = x_m\right] \\ &\quad = \mathbb{E}_{x_m}\left[F(x_0, x_1, \dots, x_{m-1}, X_0, X_1, \dots)\right] \end{aligned} \tag{8.2}$$

or equivalently put,

$$\mathbb{E}_\pi \left(\left[F(X_0, X_1, \dots) | X_0, \dots, X_m\right]\right) = \mathbb{E}^{(Y)}_{X_m}\left[F(X_0, X_1, \dots, X_{m-1}, Y_0, Y_1, \dots)\right]. \tag{8.3}$$

Proof. Fact: by "limiting" arguments beyond the scope of this course it suffices to prove Eq. (8.1) for $F(X)$ of the form, $F(X) = F(X_0, X_1, \dots, X_N)$ with $N < \infty$. Now for such a function we have,

$$\begin{aligned} &\mathbb{E}_\pi \left[F(X_0, X_1, \dots, X_N) : X_0 = x_0, \dots, X_m = x_m\right] \\ &\quad = \sum_{x_{m+1}, \dots, x_N \in S} F(x_0, \dots, x_m, x_{m+1}, \dots, x_N) \begin{bmatrix} \pi(x_0)\, p(x_0, x_1) \dots p(x_{m-1}, x_m) \cdot \\ p(x_m, x_{m+1}) \dots p(x_{N-1}, x_N) \end{bmatrix} \\ &\quad = P_\pi(X_0 = x_0, \dots, X_m = x_m) \cdot \\ &\qquad \cdot \sum_{x_{m+1}, \dots, x_N \in S} F(x_0, \dots, x_m, x_{m+1}, \dots, x_N)\, p(x_m, x_{m+1}) \dots p(x_{N-1}, x_N) \\ &\quad = P_\pi(X_0 = x_0, \dots, X_m = x_m) \\ &\qquad \cdot \sum_{y_1, \dots, y_{N-m} \in S} F(x_0, \dots, x_m, y_1, y_2, \dots, y_{N-m})\, p(x_m, y_1) \dots p(y_{N-m-1}, y_{N-m}) \\ &\quad = P_\pi(X_0 = x_0, \dots, X_m = x_m)\, h(x_0, \dots, x_m). \end{aligned} \tag{8.4}$$

Summing this equation on $x_0, \dots, x_m$ in S gives Eq. (8.1) and dividing this equation by $P_\pi(X_0 = x_0, \dots, X_m = x_m)$ proves Eq. (8.2). ■

To help cement the ideas above, let me pause to write out the above argument in the special case where $m = 2$ and $N = 5$. In this case we have;

$$\begin{aligned} &\mathbb{E}_\pi \left[F(X_0, X_1, \dots, X_5) : X_0 = x_0, X_1 = x_1, X_2 = x_2\right] \\ &\quad = \sum_{x_3, x_4, x_5 \in S} F(x_0, x_1, x_2, x_3, x_4, x_5) \begin{bmatrix} \pi(x_0)\, p(x_0, x_1)\, p(x_1, x_2) \cdot \\ p(x_2, x_3)\, p(x_3, x_4)\, p(x_4, x_5) \end{bmatrix} \\ &\quad = P_\pi(X_0 = x_0, X_1 = x_1, X_2 = x_2) \cdot \\ &\qquad \cdot \sum_{x_3, x_4, x_5 \in S} F(x_0, x_1, x_2, x_3, x_4, x_5) \left[p(x_2, x_3)\, p(x_3, x_4)\, p(x_4, x_5)\right] \\ &\quad = P_\pi(X_0 = x_0, X_1 = x_1, X_2 = x_2) \\ &\qquad \cdot \sum_{y_1, y_2, y_3 \in S} F(x_0, x_1, x_2, y_1, y_2, y_3) \left[p(x_2, y_1)\, p(y_1, y_2)\, p(y_2, y_3)\right] \\ &\quad = P_\pi(X_0 = x_0, X_1 = x_1, X_2 = x_2) \cdot \mathbb{E}^{(Y)}_{x_2}\left[F(x_0, x_1, Y_0, Y_1, Y_2, Y_3)\right]. \end{aligned}$$

Let us now use Theorem 8.1 to give variants of the proofs of our hitting time results above. In what follows π will denote a probability on S.

Corollary 8.2. *Let $B \subset S$ and T_B be as above, then for $n, m \in \mathbb{N}$ we have*

[1] In this theorem we assume that F is either bounded or non-negative.

$$P_\pi\left(T_B > m+n\right) = \mathbb{E}_\pi\left[1_{T_B>m} P_{X_m}\left[T_B > n\right]\right]. \tag{8.5}$$

Proof. Using Theorem 8.1,

$$\begin{aligned} P_\pi\left(T_B > m+n\right) &= \mathbb{E}_\pi\left[1_{T_B(X)>m+n}\right] \\ &= \mathbb{E}_\pi\left[\mathbb{E}^{(Y)}_{X_m}\left[1_{T_B(X_0,\dots,X_{m-1},Y_0,Y_1,\dots)>m+n}\right]\right] \\ &= \mathbb{E}_\pi\left[\mathbb{E}^{(Y)}_{X_m}\left[1_{T_B(X)>m}\cdot 1_{T_B(Y)>n}\right]\right] \\ &= \mathbb{E}_\pi\left[1_{T_B(X)>m}\mathbb{E}^{(Y)}_{X_m}\left[1_{T_B(Y)>n}\right]\right] = \mathbb{E}_\pi\left[1_{T_B>m}P_{X_m}\left[T_B>n\right]\right]. \end{aligned}$$

■

Corollary 8.3. *Suppose that $B \subset S$ is non-empty proper subset of S and $A = S\setminus B$. Further suppose there is some $\alpha < 1$ such that $P_x\left(T_B = \infty\right) \le \alpha$ for all $x \in A$, then $P_\pi\left(T_B = \infty\right) = 0$. [In words; if there is a "uniform" chance that X hits B starting from any site, then X will surely hit B from any point in A.]*

Proof. Since $T_B = 0$ on $\{X_0 \in B\}$ we in fact have $P_x\left(T_B = \infty\right) \le \alpha$ for all $x \in S$. Letting $n \to \infty$ in Eq. (8.5) shows,

$$P_\pi\left(T_B = \infty\right) = \mathbb{E}_\pi\left[1_{T_B>m}P_{X_m}\left[T_B = \infty\right]\right] \le \mathbb{E}_\pi\left[1_{T_B>m}\alpha\right] = \alpha P_\pi\left(T_B > m\right).$$

Now letting $m \to \infty$ in this equation shows $P_\pi\left(T_B = \infty\right) \le \alpha P_\pi\left(T_B = \infty\right)$ from which it follows that $P_\pi\left(T_B = \infty\right) = 0$. ■

Corollary 8.4. *Suppose that $B \subset S$ is non-empty proper subset of S and $A = S\setminus B$. Further suppose there is some $\alpha < 1$ and $n < \infty$ such that $P_x\left(T_B > n\right) \le \alpha$ for all $x \in A$, then*

$$\mathbb{E}_\pi\left(T_B\right) \le \frac{n}{1-\alpha} < \infty$$

for all $x \in A$. [In words; if there is a "uniform" chance that X hits B starting from any site within a fixed number of steps, then the expected hitting time of B is finite and bounded independent of the starting distribution.]

Proof. Again using $T_B = 0$ on $\{X_0 \in B\}$ we may conclude that $P_x\left(T_B > n\right) \le \alpha$ for all $x \in S$. Letting $m = kn$ in Eq. (8.5) shows

$$P_\pi\left(T_B > kn+n\right) = \mathbb{E}_\pi\left[1_{T_B>kn}P_{X_m}\left[T_B>n\right]\right] \le \mathbb{E}_\pi\left[1_{T_B>kn}\cdot\alpha\right] = \alpha P_\pi\left(T_B > kn\right).$$

Iterating this equation using the fact that $P_\pi\left(T_B > 0\right) \le 1$ shows $P_\pi\left(T_B > kn\right) \le \alpha^k$ for all $k \in \mathbb{N}_0$. Therefore with the aid of Lemma 7.22 and the observation,

$$P\left(T_B > kn+m\right) \le P\left(T_B > kn\right) \text{ for } m = 0,\dots,n-1,$$

we find,

$$\begin{aligned} \mathbb{E}_x T_B &= \sum_{k=0}^\infty P\left(T_B > k\right) \le \sum_{k=0}^\infty nP\left(T_B > kn\right) \\ &\le \sum_{k=0}^\infty n\alpha^k = \frac{n}{1-\alpha} < \infty. \end{aligned}$$

■

Corollary 8.5. *If $A = S\setminus B$ is a finite set and $P_x\left(T_B = \infty\right) < 1$ for all $x \in A$, then $\mathbb{E}_\pi T_B < \infty$.*

Proof. Let $\alpha_0 = \max_{x\in A} P_x\left(T = \infty\right) < 1$. Now fix $\alpha \in \left(\alpha_0, 1\right)$. Using

$$\alpha_0 \ge P_x\left(T = \infty\right) = \downarrow \lim_{n\to\infty} P_x\left(T > n\right)$$

we will have $P_x\left(T > m\right) \le \alpha$ for $m \ge N_x$ for some $N_x < \infty$. Taking $n := \max\{N_x : x \in A\} < \infty$ (A is a finite set), we will have $P_x\left(T > n\right) \le \alpha$ for all $x \in A$ and we may now apply Corollary 8.4. ■

8.1 First Step Analysis

The next theorem (which is a special case of Theorem 8.1) is the basis of the first step analysis developed in this section.

Theorem 8.6 (First step analysis). *Let $F(X) = F\left(X_0, X_1, \dots\right)$ be some function of the paths $\left(X_0, X_1, \dots\right)$ of our Markov chain, then for all $x, y \in S$ with $p(x,y) > 0$ we have*

$$\mathbb{E}_x\left[F\left(X_0, X_1, \dots\right)|X_1 = y\right] = \mathbb{E}_y\left[F\left(x, X_0, X_1, \dots\right)\right] \tag{8.6}$$

and

$$\begin{aligned} \mathbb{E}_x\left[F\left(X_0, X_1, \dots\right)\right] &= \mathbb{E}_{p(x,\cdot)}\left[F\left(x, X_0, X_1, \dots\right)\right] \\ &= \sum_{y\in S} p(x,y)\,\mathbb{E}_y\left[F\left(x, X_0, X_1, \dots\right)\right]. \end{aligned} \tag{8.7}$$

Proof. Equation (8.6) follows directly from Theorem 8.1,

$$\begin{aligned} \mathbb{E}_x\left[F\left(X_0, X_1, \dots\right)|X_1 = y\right] &= \mathbb{E}_x\left[F\left(X_0, X_1, \dots\right)|X_0 = x, X_1 = y\right] \\ &= \mathbb{E}_y\left[F\left(x, X_0, X_1, \dots\right)\right]. \end{aligned}$$

Equation (8.7) now follows from Eq. (8.6), the law of total expectation, and the fact that $P_x\left(X_1 = y\right) = p(x,y)$. ■

Let us now suppose for until further notice that B is a non-empty proper subset of S, $A = S\ \setminus B$, and $T_B = T_B(X)$ is the first hitting time of B by X.

Notation 8.7 *Given a transition matrix* $\mathbf{P} = (p(x,y))_{x,y\in S}$ *we let* $Q = (p(x,y))_{x,y\in A}$ *and* $\mathbf{R} := (p(x,y))_{x\in A, y\in B}$ *so that, schematically,*

$$\mathbf{P} = \begin{array}{cc|c} A & B & \\ \hline \mathbf{Q} & \mathbf{R} & A \\ * & * & B \end{array}.$$

Remark 8.8. To construct the matrix $\mathbf{Q}$ and $\mathbf{R}$ from $\mathbf{P}$, let $\mathbf{P}'$ be $\mathbf{P}$ with the rows corresponding to B omitted. To form $\mathbf{Q}$ from $\mathbf{P}'$, remove the columns of $\mathbf{P}'$ corresponding to B and to form $\mathbf{R}$ from $\mathbf{P}'$, remove the columns of $\mathbf{P}'$ corresponding to A.

Example 8.9. If $S = \{1,2,3,4,5,6,7\}$, $A = \{1,2,4,5,6\}$, $B = \{3,7\}$, and

$$\mathbf{P} = \begin{array}{ccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 1 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 2 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 3 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 4 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 5 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{array},$$

then

$$\mathbf{P}' = \begin{array}{ccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \hline 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 1 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 2 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 4 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 5 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 6 \end{array}.$$

Deleting the 3 and 7 columns of $\mathbf{P}'$ gives

$$\mathbf{Q} = \mathbf{P}_{A,A} = \begin{array}{ccccc|c} 1 & 2 & 4 & 5 & 6 & \\ \hline 0 & 1/2 & 1/2 & 0 & 0 & 1 \\ 1/3 & 0 & 0 & 1/3 & 0 & 2 \\ 1/3 & 0 & 0 & 1/3 & 0 & 4 \\ 0 & 1/3 & 1/3 & 0 & 1/3 & 5 \\ 0 & 0 & 0 & 1/2 & 0 & 6 \end{array}$$

and deleting the $1, 2, 4, 5,$ and 6 columns of $\mathbf{P}'$ gives

$$\mathbf{R} = \mathbf{P}_{A,B} = \begin{array}{cc|c} 3 & 7 & \\ \hline 0 & 0 & 1 \\ 1/3 & 0 & 2 \\ 0 & 1/3 & 4 \\ 0 & 0 & 5 \\ 1/2 & 0 & 6 \end{array}.$$

Theorem 8.10 (Hitting distributions). *Let* $h : B \to \mathbb{R}$ *be a bounded or non-negative function and let* $u : S \to \mathbb{R}$ *be defined by*

$$u(x) := \mathbb{E}_x\left[h(X_{T_B}) : T_B < \infty\right] \text{ for } x \in A.$$

Then $u = h$ *on* B *and*

$$u(x) = \sum_{y\in A} p(x,y)\,u(y) + \sum_{y\in B} p(x,y)\,h(y) \text{ for all } x \in A. \tag{8.8}$$

In matrix notation this becomes

$$\mathbf{u} = \mathbf{Q}u + \mathbf{Rh} \implies \mathbf{u} = (I - \mathbf{Q})^{-1}\mathbf{Rh},$$

i.e.

$$\mathbb{E}_x\left[h(X_{T_B}) : T_B < \infty\right] = \left[(I-\mathbf{Q})^{-1}\mathbf{Rh}\right]_x \text{ for all } x \in A. \tag{8.9}$$

As a special case if $h(s) = \delta_y(s)$ *for some* $y \in B$*, then Eq. (8.9) becomes,*

$$P_x\left(X_{T_B} = y : T_B < \infty\right) = \left[(I-\mathbf{Q})^{-1}\mathbf{R}\right]_{x,y}. \tag{8.10}$$

Proof. To shorten the notation we will use the convention that $h(X_{T_B}) = 0$ if $T_B = \infty$ so that we may simply write $u(x) := \mathbb{E}_x[h(X_{T_B})]$. Let

$$F(X_0, X_1, \dots) = h\left(X_{T_B(X)}\right) = h\left(X_{T_B(X)}\right) 1_{T_B(X)<\infty},$$

then for $x \in A$ we have $F(x, X_0, X_1, \dots) = F(X_0, X_1, \dots)$. Therefore by the first step analysis (Theorem 8.6) we learn

$$\begin{aligned} u(x) = \mathbb{E}_x h\left(X_{T_B(X)}\right) &= \mathbb{E}_x F(x, X_1, \dots) = \sum_{y\in S} p(x,y)\,\mathbb{E}_y F(x, X_0, X_1, \dots) \\ &= \sum_{y\in S} p(x,y)\,\mathbb{E}_y F(X_0, X_1, \dots) = \sum_{y\in S} p(x,y)\,\mathbb{E}_y\left[h\left(X_{T_B(X)}\right)\right] \\ &= \sum_{y\in A} p(x,y)\,\mathbb{E}_y\left[h\left(X_{T_B(X)}\right)\right] + \sum_{y\in B} p(x,y)\,h(y) \\ &= \sum_{y\in A} p(x,y)\,u(y) + \sum_{y\in B} p(x,y)\,h(y). \end{aligned}$$

■

Theorem 8.11 (Travel averages). *Given $g : A \to [0,\infty]$, let $w(x) := \mathbb{E}_x\left[\sum_{n<T_B} g(X_n)\right]$. Then $w(x)$ satisfies*

$$w(x) = \sum_{y\in A} p(x,y)\,w(y) + g(x) \text{ for all } x\in A. \tag{8.11}$$

In matrix notation this becomes,

$$\mathbf{w} = \mathbf{Q}\mathbf{w} + \mathbf{g} \implies \mathbf{w} = (I-\mathbf{Q})^{-1}\mathbf{g}$$

so that

$$\mathbb{E}_x\left[\sum_{n<T_B} g(X_n)\right] = \left[(I-\mathbf{Q})^{-1}\mathbf{g}\right]_x.$$

The following two special cases are of most interest;

1. *Suppose $g(x) = \delta_y(x)$ for some $y\in A$, then $\sum_{n<T_B} g(X_n) = \sum_{n<T_B}\delta_y(X_n)$ is the number of visits of the chain to y and*

$$\begin{aligned}&\mathbb{E}_x(\#\text{ visits to } y \text{ before hitting } B)\\ &= \mathbb{E}_x\left[\sum_{n<T_B}\delta_y(X_n)\right] = (I-\mathbf{Q})^{-1}_{x,y}.\end{aligned}$$

2. *Suppose that $g(x) = 1$, then $\sum_{n<T_B} g(X_n) = T_B$ and we may conclude that*

$$\mathbb{E}_x[T_B] = \left[(I-\mathbf{Q})^{-1}\mathbf{1}\right]_x$$

where $\mathbf{1}$ is the column vector consisting of all ones.

Proof. Let $F(X_0,X_1,\dots) = \sum_{n<T_B(X_0,X_1,\dots)} g(X_n)$ be the sum of the values of g along the chain before its first exit from A, i.e. entrance into B. With this interpretation in mind, if $x\in A$, it is easy to see that

$$\begin{aligned}F(x,X_0,X_1,\dots) &= \begin{cases} g(x) & \text{if } X_0\in B\\ g(x)+F(X_0,X_1,\dots) & \text{if } X_0\in A\end{cases}\\ &= g(x) + 1_{X_0\in A}\cdot F(X_0,X_1,\dots).\end{aligned}$$

Therefore by the first step analysis (Theorem 8.6) it follows that

$$\begin{aligned}w(x) = \mathbb{E}_xF(X_0,X_1,\dots) &= \sum_{y\in S} p(x,y)\,\mathbb{E}_yF(x,X_0,X_1,\dots)\\ &= \sum_{y\in S} p(x,y)\,\mathbb{E}_y\left[g(x) + 1_{X_0\in A}\cdot F(X_0,X_1,\dots)\right]\\ &= g(x) + \sum_{y\in A} p(x,y)\,\mathbb{E}_y\left[F(X_0,X_1,\dots)\right]\\ &= g(x) + \sum_{y\in A} p(x,y)\,w(y).\end{aligned}$$

■

8.2 Finite state space examples

Example 8.12. Consider the Markov chain determined by

$$\mathbf{P} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 0 & 1/3 & 1/3 & 1/3 \\ 3/4 & 1/8 & 1/8 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Notice that 3 and 4 are absorbing states. Let $h_i = P_i(X_n \text{ hits } 3) = P(X_n \text{ hits 3 before 4})$ for $i = 1,2,3,4$. Clearly $h_3 = 1$ while $h_4 = 0$ and by the first step analysis we have

$$\begin{aligned}h_1 &= \frac{1}{3}h_2 + \frac{1}{3}h_3 + \frac{1}{3}h_4 = \frac{1}{3}h_2 + \frac{1}{3}\\ h_2 &= \frac{3}{4}h_1 + \frac{1}{8}h_2 + \frac{1}{8}h_3 = \frac{3}{4}h_1 + \frac{1}{8}h_2 + \frac{1}{8}\end{aligned}$$

i.e.

$$\begin{aligned}h_1 &= \frac{1}{3}h_2 + \frac{1}{3}\\ h_2 &= \frac{3}{4}h_1 + \frac{1}{8}h_2 + \frac{1}{8}\end{aligned}$$

which have solutions,

$$\begin{aligned}P_1(X_n \text{ hits } 3) = h_1 &= \frac{8}{15} \cong 0.533\,33\\ P_2(X_n \text{ hits } 3) = h_2 &= \frac{3}{5}.\end{aligned}$$

Similarly if we let $h_i = P_i(X_n \text{ hits } 4)$ instead, from the above equations with $h_3 = 0$ and $h_4 = 1$, we find

$$\begin{aligned}h_1 &= \frac{1}{3}h_2 + \frac{1}{3}\\ h_2 &= \frac{3}{4}h_1 + \frac{1}{8}h_2\end{aligned}$$

which has solutions,

$$P_1\left(X_n \text{ hits } 4\right) = h_1 = \frac{7}{15} \text{ and}$$
$$P_2\left(X_n \text{ hits } 4\right) = h_2 = \frac{2}{5}.$$

Of course we did not really need to compute these, since

$$P_1\left(X_n \text{ hits } 3\right) + P_1\left(X_n \text{ hits } 4\right) = 1 \text{ and}$$
$$P_2\left(X_n \text{ hits } 3\right) + P_2\left(X_n \text{ hits } 4\right) = 1.$$

We can do these computations using the matrix formalism as well. For this we have

$$\mathbf{P}' = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & 1/3 & 1/3 & 1/3 \\ 3/4 & 1/8 & 1/8 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array},$$

$$\mathbf{Q} = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{cc} 0 & 1/3 \\ 3/4 & 1/8 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array}, \text{ and } \mathbf{R} = \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \left[\begin{array}{cc} 1/3 & 1/3 \\ 1/8 & 0 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array}.$$

Matrix manipulations now shows,

$$\mathbb{E}_i\left(\# \text{ visits to } j \text{ before hitting } \{3,4\}\right) = (I - \mathbf{Q})^{-1} = \begin{array}{c|cc} i\backslash j & 1 & 2 \\ 1 & \frac{7}{5} & \frac{8}{15} \\ 2 & \frac{6}{5} & \frac{8}{5} \end{array},$$

$$\mathbb{E}_i T_{\{3,4\}} = (I - \mathbf{Q})^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{array}{c|c} i & \\ 1 & \frac{29}{15} \\ 2 & \frac{14}{5} \end{array} \text{ and}$$

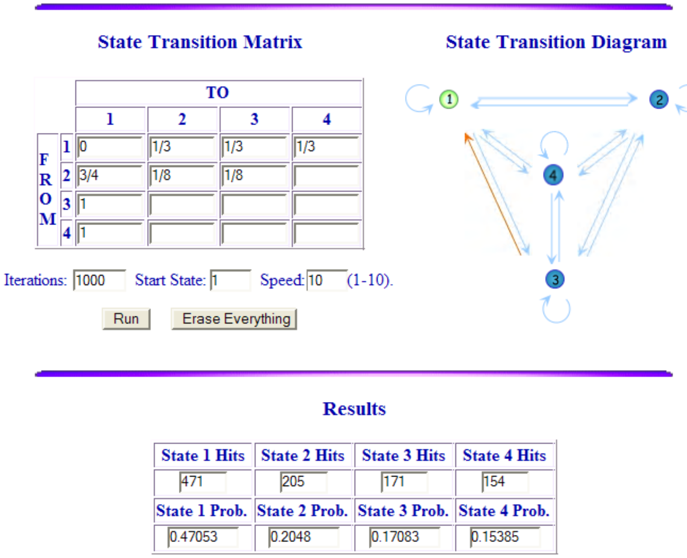
$$P_i\left(X_{T_{\{3,4\}}} = j\right) = (I - \mathbf{Q})^{-1}\mathbf{R} = \begin{array}{c|cc} i\backslash j & 3 & 4 \\ 1 & \frac{8}{15} & \frac{7}{15} \\ 2 & \frac{3}{5} & \frac{2}{5} \end{array}.$$

The output of one simulation from `www.zweigmedia.com/RealWorld/markov/markov.html` is in Figure 8.1 below.

Example 8.13. Let us continue the rat in the maze Exercise 7.1 and now suppose that room 3 contains food while room 7 contains a mouse trap.

$$\left[\begin{array}{ccc} 1 & 2 & 3 \text{ (food)} \\ 4 & 5 & 6 \\ 7 \text{ (trap)} & & \end{array}\right].$$

Recall that the transition matrix for this chain with sites 3 and 7 absorbing is given by,



State Transition Matrix

		TO			
		1	2	3	4
FROM	1	0	1/3	1/3	1/3
	2	3/4	1/8	1/8	
	3	1			
	4	1			

Iterations: 1000 Start State: 1 Speed: 10 (1-10).

Run | Erase Everything

State Transition Diagram

Results

State 1 Hits	State 2 Hits	State 3 Hits	State 4 Hits
471	205	171	154
State 1 Prob.	**State 2 Prob.**	**State 3 Prob.**	**State 4 Prob.**
0.47053	0.2048	0.17083	0.15385

Fig. 8.1. In this run, rather than making sites 3 and 4 absorbing, we have made them transition back to 1. I claim now to get an approximate value for $P_1\left(X_n \text{ hits } 3\right)$ we should compute: (State 3 Hits)/(State 3 Hits + State 4 Hits). In this example we will get $171/(171 + 154) = 0.526\,15$ which is a little lower than the predicted value of 0.533. You can try your own runs of this simulator.

$$\mathbf{P} = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array},$$

see Figure 8.2 for the corresponding jump diagram for this chain.

We would like to compute the probability that the rat reaches the food before he is trapped. To answer this question we let $A = \{1,2,4,5,6\}$, $B = \{3,7\}$, and $T := T_B$ be the first hitting time of B. Then deleting the 3 and 7 rows of $\mathbf{P}$ leaves the matrix,

$$\mathbf{P}' = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{array}.$$

Deleting the 3 and 7 columns of $\mathbf{P}'$ gives

$$\mathbf{Q} = \mathbf{P}_{A,A} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{array}$$

and deleting the $1, 2, 4, 5,$ and 6 columns of $\mathbf{P}'$ gives

$$\mathbf{R} = \mathbf{P}_{A,B} = \begin{array}{c} \begin{array}{cc} 3 & 7 \end{array} \\ \left[\begin{array}{cc} 0 & 0 \\ 1/3 & 0 \\ 0 & 1/3 \\ 0 & 0 \\ 1/2 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{array}.$$

Therefore,

$$I - \mathbf{Q} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{3} & 1 & 0 & -\frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix},$$

and using a computer algebra package we find

$$\left(\mathbb{E}_i\left[X_{T_B} = j\right]\right) = (I - \mathbf{Q})^{-1} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{ccccc} \frac{11}{6} & \frac{5}{4} & \frac{5}{4} & 1 & \frac{1}{3} \\ \frac{5}{6} & \frac{7}{4} & \frac{3}{4} & 1 & \frac{1}{3} \\ \frac{5}{6} & \frac{3}{4} & \frac{7}{4} & 1 & \frac{1}{3} \\ \frac{2}{3} & 1 & 1 & 2 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 1 & \frac{4}{3} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{array}.$$

In particular we may conclude,

$$\begin{bmatrix} \mathbb{E}_1 T \\ \mathbb{E}_2 T \\ \mathbb{E}_4 T \\ \mathbb{E}_5 T \\ \mathbb{E}_6 T \end{bmatrix} = (I - \mathbf{Q})^{-1} \mathbf{1} = \begin{bmatrix} \frac{17}{3} \\ \frac{14}{3} \\ \frac{14}{3} \\ \frac{16}{3} \\ \frac{11}{3} \end{bmatrix},$$

and

$$\begin{bmatrix} P_1(X_T = 3) & P_1(X_T = 7) \\ P_2(X_T = 3) & P_2(X_T = 3) \\ P_4(X_T = 3) & P_4(X_T = 3) \\ P_5(X_T = 3) & P_5(X_T = 3) \\ P_6(X_T = 3) & P_6(X_T = 7) \end{bmatrix} = (I - \mathbf{Q})^{-1} \mathbf{R} = \begin{array}{c} \begin{array}{cc} 3 & 7 \end{array} \\ \left[\begin{array}{cc} \frac{7}{12} & \frac{5}{12} \\ \frac{3}{4} & \frac{1}{4} \\ \frac{5}{12} & \frac{7}{12} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{6} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{array}.$$

.

Since the event of hitting 3 before 7 is the same as the event $\{X_T = 3\}$, the desired hitting probabilities are

$$\begin{bmatrix} P_1(X_T = 3) \\ P_2(X_T = 3) \\ P_4(X_T = 3) \\ P_5(X_T = 3) \\ P_6(X_T = 3) \end{bmatrix} = \begin{bmatrix} \frac{7}{12} \\ \frac{3}{4} \\ \frac{5}{12} \\ \frac{2}{3} \\ \frac{5}{6} \end{bmatrix}.$$

We can also derive these hitting probabilities from scratch using the first step analysis. In order to do this let

$$h_i = P_i(X_T = 3) = P_i(X_n \text{ hits 3 (food) before 7(trapped)}).$$

By the first step analysis we will have,

$$\begin{aligned} h_i &= \sum_j P_i(X_T = 3 | X_1 = j) P_i(X_1 = j) \\ &= \sum_j p(i,j) P_i(X_T = 3 | X_1 = j) \\ &= \sum_j p(i,j) P_j(X_T = 3) \\ &= \sum_j p(i,j) h_j \end{aligned}$$

where $h_3 = 1$ and $h_7 = 0$. Looking at the jump diagram (Figure 8.2) we easily find

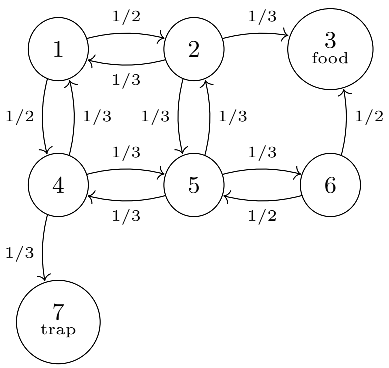


Fig. 8.2. The jump diagram for our proverbial rat in the maze. Here we assume the rat is "absorbed" at sites 3 and 7

$$\begin{aligned}
h_1 &= \frac{1}{2}\left(h_2 + h_4\right) \\
h_2 &= \frac{1}{3}\left(h_1 + h_3 + h_5\right) = \frac{1}{3}\left(h_1 + 1 + h_5\right) \\
h_4 &= \frac{1}{3}\left(h_1 + h_5 + h_7\right) = \frac{1}{3}\left(h_1 + h_5\right) \\
h_5 &= \frac{1}{3}\left(h_2 + h_4 + h_6\right) \\
h_6 &= \frac{1}{2}\left(h_3 + h_5\right) = \frac{1}{2}\left(1 + h_5\right)
\end{aligned}$$

and the solutions to these equations are (as seen before) given by

$$\left[h_1 = \frac{7}{12}, h_2 = \frac{3}{4}, h_4 = \frac{5}{12}, h_5 = \frac{2}{3}, h_6 = \frac{5}{6}\right]. \tag{8.12}$$

Similarly, if

$$k_i := P_i\left(X_T = 7\right) = P_i\left(X_n \text{ is trapped before dinner}\right),$$

we need only use the above equations with h replaced by k and now taking $k_3 = 0$ and $k_7 = 1$ to find,

$$\begin{aligned}
k_1 &= \frac{1}{2}\left(k_2 + k_4\right) \\
k_2 &= \frac{1}{3}\left(k_1 + k_5\right) \\
k_4 &= \frac{1}{3}\left(k_1 + k_5 + 1\right) \\
k_5 &= \frac{1}{3}\left(k_2 + k_4 + k_6\right) \\
k_6 &= \frac{1}{2}k_5
\end{aligned}$$

and then solve to find,

$$\left[k_1 = \frac{5}{12}, k_2 = \frac{1}{4}, k_4 = \frac{7}{12}, k_5 = \frac{1}{3}, k_6 = \frac{1}{6}\right]. \tag{8.13}$$

Notice that the sum of the hitting probabilities in Eqs. (8.12) and (8.13) add up to 1 as they should.

Exercise 8.1 (III.4.P11 on p.132). An urn contains two red and two green balls. The balls are chosen at random, one by one, and removed from the urn. The selection process continues until all of the green balls have been removed from the urn. What is the probability that a single red ball is in the urn at the time that the last green ball is chosen?

Solution to Exercise (III.4.P11 on p.132). Let's choose the states to be $(G, R) = (i, j)$ with $i, j = 0, 1, 2$ so that $(1, 2)$ implies that there is one green ball and two red balls in the urn. Let $B = \{(0, 0), (0, 1), (0, 2\}$,

$$T = T_B = \min\{n \geq 0 : X_n = (0, 0) \text{ or } (0, 1) \text{ or } (0, 2)\}.$$

We wish to compute $P(X_T = (0, 1)|X_0 = (2, 2))$. The transition matrix for this chain is given by;

$$\mathbf{P} = \left[\begin{array}{ccccccccc}
(0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 2/3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2/3 & 0 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0
\end{array}\right]
\begin{array}{c}
\\ (0,0) \\ (0,1) \\ (0,2) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2)
\end{array}.$$

Using the matrix method;

$$\mathbf{Q} = \left[\begin{array}{cccccc} (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{array}\right] \begin{array}{c} \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \end{array}$$

$$\mathbf{R} = \left[\begin{array}{cccc} & (0,0) & (0,1) & (0,2) \\ (1,0) & 1 & 0 & 0 \\ (1,1) & 0 & 1/2 & 0 \\ (1,2) & 0 & 0 & 1/3 \\ (2,0) & 0 & 0 & 0 \\ (2,1) & 0 & 0 & 0 \\ (2,2) & 0 & 0 & 0 \end{array}\right].$$

So

$$P_{(a,b)}\left[X_{T_B} = (c,d)\right] = (I - \mathbf{Q})^{-1}\mathbf{R} = \begin{array}{c} (a,b) \setminus (c,d) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \end{array} \begin{array}{c} (0,0)\ (0,1)\ (0,2) \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{\mathbf{1}}{\mathbf{3}} & \frac{1}{6} \end{bmatrix} \end{array}$$

and therefore,

$$P_{(2,2)}(X_T = (0,1)) = P(X_T = (0,1)|X_0 = (2,2)) = 1/3.$$

Example 8.14 (A modified rat maze). Here is the modified maze,

$$\begin{bmatrix} 1 & 2 & 3(\text{food}) \\ 4 & 5 & \\ 6(\text{trap}) & & \end{bmatrix}.$$

The transition matrix with 3 and 6 made into absorbing states[2] is:

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array},$$

$$\mathbf{Q} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 1/3 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \end{array}, \qquad \mathbf{R} = \begin{array}{c} \begin{array}{cc} 3 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1/3 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \end{array}$$

$$(I_4 - \mathbf{Q})^{-1} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 2 & \frac{3}{2} & \frac{3}{2} & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \end{array},$$

$$(I_4 - \mathbf{Q})^{-1}\mathbf{R} = \begin{array}{c} \begin{array}{cc} 3 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \end{array},$$

$$(I_4 - \mathbf{Q})^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 5 \\ 6 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 4 \\ 5 \end{array}.$$

So for example, $P_4(X_T = 3(\text{food})) = 1/3$, $E_4(\text{Number of visits to 1}) = 1$, $E_5(\text{Number of visits to 2}) = 3/2$ and $E_1T = E_5T = 6$ and $E_2T = E_4T = 5$. Therefore,

$$\begin{aligned} h_6 &= \frac{1}{2}(1 + h_5) \\ h_5 &= \frac{1}{3}(h_2 + h_4 + h_6) \\ h_4 &= \frac{1}{2}h_1 \\ h_2 &= \frac{1}{3}(1 + h_1 + h_5) \\ h_1 &= \frac{1}{2}(h_2 + h_4). \end{aligned}$$

[2] It is not necessary to make states 3 and 6 absorbing. In fact it does matter at all what the transition probabilities are for the chain for leaving either of the states 3 or 6 since we are going to stop when we hit these states. This is reflected in the fact that the first thing we will do in the first step analysis is to delete rows 3 and 6 from P. Making 3 and 6 absorbing simply saves a little ink.

The solutions to these equations are,

$$h_1 = \frac{4}{9},\ h_2 = \frac{2}{3},\ h_4 = \frac{2}{9},\ h_5 = \frac{5}{9},\ h_6 = \frac{7}{9}. \tag{8.14}$$

Similarly if $h_i = P_i(X_n \text{ hits 7 before 3})$ we have $h_7 = 1$, $h_3 = 0$ and

$$\begin{aligned} h_6 &= \frac{1}{2}h_5 \\ h_5 &= \frac{1}{3}(h_2 + h_4 + h_6) \\ h_4 &= \frac{1}{2}(h_1 + 1) \\ h_2 &= \frac{1}{3}(h_1 + h_5) \\ h_1 &= \frac{1}{2}(h_2 + h_4) \end{aligned}$$

whose solutions are

$$h_1 = \frac{5}{9},\ h_2 = \frac{1}{3},\ h_4 = \frac{7}{9},\ h_5 = \frac{4}{9},\ h_6 = \frac{2}{9}. \tag{8.15}$$

Notice that the sum of the hitting probabilities in Eqs. (8.14) and (8.15) add up to 1 as they should.

8.3 Random Walk Exercises

Exercise 8.2 (Uniqueness of solutions to 2nd order recurrence relations). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and suppose $\{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\}$ solves the second order homogeneous recurrence relation:

$$au(x+1) + bu(x) + cu(x-1) = 0 \tag{8.16}$$

for $\alpha < x < \beta$. Show; if u and w both satisfy Eq. (8.16) and $u = w$ on two consecutive points in $(\alpha, \beta) \cap \mathbb{Z}$, then $u(x) = w(x)$ for all $x \in [\alpha, \beta] \cap \mathbb{Z}$.

Exercise 8.3 (General solutions to 2nd order recurrence relations). Let a, b, c be real numbers with $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{Z} \cup \{\pm\infty\}$ with $\alpha < \beta$, and suppose $\{u(x) : x \in [\alpha, \beta] \cap \mathbb{Z}\}$ solves the second order homogeneous recurrence relation:for $\alpha < x < \beta$. Show:

1. for any $\lambda \in \mathbb{C}$,
$$a\lambda^{x+1} + b\lambda^x + c\lambda^{x-1} = \lambda^{x-1}p(\lambda) \tag{8.17}$$
where $p(\lambda) = a\lambda^2 + b\lambda + c$ is the **characteristic polynomial** associated to Eq. (8.16).
Let $\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ be the roots of $p(\lambda)$ and suppose for the moment that $b^2 - 4ac \neq 0$. From Eq. (8.16) it follows that for any choice of $A_\pm \in \mathbb{R}$, the function,
$$w(x) := A_+\lambda_+^x + A_-\lambda_-^x,$$
solves Eq. (8.16) for all $x \in \mathbb{Z}$.
2. Show there is a unique choice of constants, $A_\pm \in \mathbb{R}$, such that the function $u(x)$ is given by
$$u(x) := A_+\lambda_+^x + A_-\lambda_-^x \text{ for all } \alpha \leq x \leq \beta.$$
3. Now suppose that $b^2 = 4ac$ and $\lambda_0 := -b/(2a)$ is the double root of $p(\lambda)$. Show for any choice of A_0 and A_1 in $\mathbb{R}$ that
$$w(x) := (A_0 + A_1 x)\lambda_0^x$$
solves Eq. (8.16) for all $x \in \mathbb{Z}$. **Hint:** Differentiate Eq. (8.17) with respect to λ and then set $\lambda = \lambda_0$.
4. Again show that any function u solving Eq. (8.16) is of the form $u(x) = (A_0 + A_1 x)\lambda_0^x$ for $\alpha \leq x \leq \beta$ for some unique choice of constants $A_0, A_1 \in \mathbb{R}$.

In the next couple of exercises you are going to use first step analysis to show that a simple unbiased random walk on $\mathbb{Z}$ is null recurrent. We let $\{X_n\}_{n=0}^\infty$ be the Markov chain with values in $\mathbb{Z}$ with transition probabilities given by

$$P(X_{n+1} = x \pm 1 | X_n = x) = 1/2 \text{ for all } n \in \mathbb{N}_0 \text{ and } x \in \mathbb{Z}.$$

Further let $a, b \in \mathbb{Z}$ with $a < 0 < b$ and

$$T_{a,b} := \min\{n : X_n \in \{a, b\}\} \text{ and } T_b := \inf\{n : X_n = b\}.$$

We know by Corollary 8.5 that $\mathbb{E}_0[T_{a,b}] < \infty$ from which it follows that $P(T_{a,b} < \infty) = 1$ for all $a < 0 < b$.

Exercise 8.4. Let $w_x := P_x\left(X_{T_{a,b}} = b\right) := P\left(X_{T_{a,b}} = b | X_0 = x\right)$.

1. Use first step analysis to show for $a < x < b$ that
$$w_x = \frac{1}{2}(w_{x+1} + w_{x-1}) \tag{8.18}$$
provided we define $w_a = 0$ and $w_b = 1$.

2. Use the results of Exercises 8.2 and 8.3 to show
$$P_x\left(X_{T_{a,b}}=b\right)=w_x=\frac{1}{b-a}(x-a). \tag{8.19}$$
3. Let
$$T_b:=\begin{cases}\min\{n:X_n=b\} & \text{if} \quad \{X_n\} \text{ hits } b\\ \infty & \text{otherwise}\end{cases}$$
be the first time $\{X_n\}$ hits b. Explain why, $\left\{X_{T_{a,b}}=b\right\}\subset\{T_b<\infty\}$ and use this along with Eq. (8.19) to conclude that $P_x(T_b<\infty)=1$ for all $x<b$. (By symmetry this result holds true for all $x\in\mathbb{Z}$.)

Exercise 8.5. The goal of this exercise is to give a second proof of the fact that $P_x(T_b<\infty)=1$. Here is the outline:

1. Let $w_x:=P_x(T_b<\infty)$. Again use first step analysis to show that w_x satisfies Eq. (8.18) for all x with $w_b=1$.
2. Use Exercises 8.2 and 8.3 to show that there is a constant, c, such that
$$w_x=c(x-b)+1 \text{ for all } x\in\mathbb{Z}.$$
3. Explain why c must be zero to again show that $P_x(T_b<\infty)=1$ for all $x\in\mathbb{Z}$.

Exercise 8.6. Let $T=T_{a,b}$ and $u_x:=\mathbb{E}_xT:=\mathbb{E}[T|X_0=x]$.

1. Use first step analysis to show for $a<x<b$ that
$$u_x=\frac{1}{2}(u_{x+1}+u_{x-1})+1 \tag{8.20}$$
with the convention that $u_a=0=u_b$.
2. Show that
$$u_x=A_0+A_1x-x^2 \tag{8.21}$$
solves Eq. (8.20) for any choice of constants A_0 and A_1.
3. Choose A_0 and A_1 so that u_x satisfies the boundary conditions, $u_a=0=u_b$. Use this to conclude that
$$\mathbb{E}_xT_{a,b}=-ab+(b+a)x-x^2=-a(b-x)+bx-x^2. \tag{8.22}$$

Remark 8.15. Notice that $T_{a,b}\uparrow T_b=\inf\{n:X_n=b\}$ as $a\downarrow-\infty$, and so passing to the limit as $a\downarrow-\infty$ in Eq. (8.22) shows
$$\mathbb{E}_xT_b=\infty \text{ for all } x<b.$$

Combining the last couple of exercises together shows that $\{X_n\}$ is "null - recurrent."

Exercise 8.7. Let $T=T_b$. The goal of this exercise is to give a second proof of the fact and $u_x:=\mathbb{E}_xT=\infty$ for all $x\neq b$. Here is the outline. Let $u_x:=\mathbb{E}_xT\in[0,\infty]=[0,\infty)\cup\{\infty\}$.

1. Note that $u_b=0$ and, by a first step analysis, that u_x satisfies Eq. (8.20) for all $x\neq b$ – allowing for the possibility that some of the u_x may be infinite.
2. Argue, using Eq. (8.20), that if $u_x<\infty$ for some $x<b$ then $u_y<\infty$ for all $y<b$. Similarly, if $u_x<\infty$ for some $x>b$ then $u_y<\infty$ for all $y>b$.
3. If $u_x<\infty$ for all $x>b$ then u_x must be of the form in Eq. (8.21) for some A_0 and A_1 in $\mathbb{R}$ such that $u_b=0$. However, this would imply, $u_x=\mathbb{E}_xT\to-\infty$ as $x\to\infty$ which is impossible since $\mathbb{E}_xT\geq0$ for all x. Thus we must conclude that $\mathbb{E}_xT=u_x=\infty$ for all $x>b$. (A similar argument works if we assume that $u_x<\infty$ for all $x<b$.)

Exercise 8.8 (Biased random walks I). Let $p\in(1/2,1)$ and consider the biased random walk $\{X_n\}_{n\geq0}$ on the $S=\mathbb{Z}$ where $X_n=\xi_0+\xi_1+\dots+\xi_n$, $\{\xi_i\}_{i=1}^\infty$ are i.i.d. with $P(\xi_i=1)=p\in(0,1)$ and $P(\xi_i=-1)=q:=1-p$, and $\xi_0=x$ for some $x\in\mathbb{Z}$. Let $T=T_{\{0\}}$ be the first hitting time of $\{0\}$ and $u(x):=P_x(T<\infty)$.

Example 8.16. a) Use the first step analysis to show
$$u(x)=pu(x+1)+qu(x-1) \text{ for } x\neq0 \text{ and } u(0)=1. \tag{8.23}$$

b) Use Eq. (8.23) along with Exercises 8.2 and 8.3 to show for some $a_\pm\in\mathbb{R}$ that
$$u(x)=(1-a_+)+a_+(q/p)^x \text{ for } x\geq0 \text{ and} \tag{8.24}$$
$$u(x)=(1-a_-)+a_-(q/p)^x \text{ for } x\leq0. \tag{8.25}$$

c) By considering the limit as $x\to-\infty$ conclude that $a_-=0$ and $u(x)=1$ for all $x<0$, i.e. $P_x(T_0<\infty)=1$ for all $x\leq0$.

Exercise 8.9 (Biased random walks II). The goal of this exercise is to evaluate $P_x(T_0<\infty)$ for $x\geq0$. To do this let $B_n:=\{0,n\}$ and $T_n:=T_{\{0,n\}}$. Let $h(x):=P_x(X_{T_n}=0)$ where $\{X_{T_n}=0\}$ is the event of hitting 0 before n.

a) Use the first step analysis to show
$$h(x)=ph(x+1)+qh(x-1) \text{ with } h(0)=1 \text{ and } h(n)=0.$$

b) Show the unique solution to this equation is given by
$$P_x(X_{T_n}=0)=h(x)=\frac{(q/p)^x-(q/p)^n}{1-(q/p)^n}.$$

c) Argue that

$$P_x(T<\infty)=\lim_{n\to\infty}P_x(\{X_{T_n}=0\})=(q/p)^x<1 \text{ for all } x>0.$$

Example 8.17 (Biased random walks II). Continue the notation in Exercise 8.8. Let us start to compute $\mathbb{E}_xT$. Since $P_x(T=\infty)>0$ for $x>0$ we already know that $\mathbb{E}_xT=\infty$ for all $x>0$. Nevertheless we will deduce this fact again here. Letting $u(x)=\mathbb{E}_xT$ it follows by the first step analysis that, for $x\neq 0$,

$$\begin{aligned} u(x)&=p\,[1+u(x+1)]+q\,[1+u(x-1)]\\ &=pu(x+1)+qu(x-1)+1 \end{aligned} \tag{8.26}$$

with $u(0)=0$. Notice $u(x)=\infty$ is a solution to this equation while if $u(n)<\infty$ for some $n\neq 0$ then Eq. (8.26) implies that $u(x)<\infty$ for all $x\neq 0$ with the same sign as n. A particular solution to this equation may be found by trying $u(x)=\alpha x$ to learn,

$$\alpha x=p\alpha(x+1)+q\alpha(x-1)+1=\alpha x+\alpha(p-q)+1$$

which is valid for all x provided $\alpha=(q-p)^{-1}$. The general **finite** solution to Eq. (8.26) is therefore,

$$u(x)=(q-p)^{-1}x+a+b\,(q/p)^x. \tag{8.27}$$

Using the boundary condition, $u(0)=0$ allows us to conclude that $a+b=0$ and therefore,

$$u(x)=u_a(x)=(q-p)^{-1}x+a\,[1-(q/p)^x]. \tag{8.28}$$

Notice that $u_a(x)\to-\infty$ as $x\to+\infty$ no matter how a is chosen and therefore we must conclude that the desired solution to Eq. (8.26) is $u(x)=\infty$ for $x>0$ as we already mentioned. In the next exercise you will compute $\mathbb{E}_xT$ for $x<0$.

Exercise 8.10 (Biased random walks II). Continue the notation in Example 8.17. Using the outline below, show

$$\mathbb{E}_xT=\frac{|x|}{p-q} \text{ for } x\leq 0. \tag{8.29}$$

In the following outline n is a negative integer, T_n is the first hitting time of n so that $T_{\{n,0\}}=T_n\wedge T=\min\{T,T_n\}$ is the first hitting time of $\{n,0\}$. By Corollary 8.5 we know that $u(x):=\mathbb{E}_x\left[T_{\{n,0\}}\right]<\infty$ for all $n\leq x\leq 0$ and by a first step analysis one sees that $u(x)$ still satisfies Eq. (8.26) for $n<x<0$ and has boundary conditions $u(n)=0=u(0)$.

a) From Eq. (8.28) we know that

$$\mathbb{E}_x\left[T_{\{n,0\}}\right]=u_a(x)=(q-p)^{-1}x+a\,[1-(q/p)^x].$$

Use $u(n)=0$ in order to show

$$a=a_n=\frac{n}{(1-(q/p)^n)(p-q)}$$

and therefore,

$$\mathbb{E}_x\left[T_{\{n,0\}}\right]=\frac{1}{p-q}\left[|x|+n\frac{1-(q/p)^x}{1-(q/p)^n}\right] \text{ for } n\leq x\leq 0.$$

b) Argue that $\mathbb{E}_xT=\lim_{n\to-\infty}\mathbb{E}_x[T_n\wedge T]$ and use this and part a) to prove Eq. (8.29).

8.4 Computations avoiding the first step analysis

You may skip the rest of this chapter!!

Theorem 8.18. *Let n denote a non-negative integer. If $h:B\to\mathbb{R}$ is measurable and either bounded or non-negative, then*

$$\mathbb{E}_x[h(X_n):T_B=n]=\left(Q_A^{n-1}Q\,[1_Bh]\right)(x)$$

and

$$\mathbb{E}_x[h(X_{T_B}):T_B<\infty]=\left(\sum_{n=0}^{\infty}Q_A^nQ\,[1_Bh]\right)(x). \tag{8.30}$$

If $g:A\to\mathbb{R}_+$ is a measurable function, then for all $x\in A$ and $n\in\mathbb{N}_0$,

$$\mathbb{E}_x[g(X_n)\,1_{n<T_B}]=(Q_A^ng)(x).$$

In particular we have

$$\mathbb{E}_x\left[\sum_{n<T_B}g(X_n)\right]=\sum_{n=0}^{\infty}(Q_A^ng)(x)=:u(x), \tag{8.31}$$

where by convention, $\sum_{n<T_B}g(X_n)=0$ when $T_B=0$.

Proof. Let $x\in A$. In computing each of these quantities we will use;

$$\begin{aligned} \{T_B>n\}&=\{X_i\in A \text{ for } 0\leq i\leq n\} \text{ and}\\ \{T_B=n\}&=\{X_i\in A \text{ for } 0\leq i\leq n-1\}\cap\{X_n\in B\}. \end{aligned}$$

From the second identity above it follows that for

$$\begin{aligned}\mathbb{E}_x\left[h\left(X_n\right):T_B=n\right]&=\mathbb{E}_x\left[h\left(X_n\right):\left(X_1,\dots,X_{n-1}\right)\in A^{n-1},\,X_n\in B\right]\\&=\sum_{n=1}^{\infty}\int_{A^{n-1}\times B}\prod_{j=1}^{n}Q\left(x_{j-1},dx_j\right)h\left(x_n\right)\\&=\left(Q_A^{n-1}Q\left[1_Bh\right]\right)(x)\end{aligned}$$

and therefore

$$\begin{aligned}\mathbb{E}_x\left[h\left(X_{T_B}\right):T_B<\infty\right]&=\sum_{n=1}^{\infty}\mathbb{E}_x\left[h\left(X_n\right):T_B=n\right]\\&=\sum_{n=1}^{\infty}Q_A^{n-1}Q\left[1_Bh\right]=\sum_{n=0}^{\infty}Q_A^nQ\left[1_Bh\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{E}_x\left[g\left(X_n\right)1_{n<T_B}\right]&=\int_{A^n}Q\left(x,dx_1\right)Q\left(x_1,dx_2\right)\dots Q\left(x_{n-1},dx_n\right)g\left(x_n\right)\\&=\left(Q_A^ng\right)(x)\end{aligned}$$

and therefore,

$$\begin{aligned}\mathbb{E}_x\left[\sum_{n=0}^{\infty}g\left(X_n\right)1_{n<T_B}\right]&=\sum_{n=0}^{\infty}\mathbb{E}_x\left[g\left(X_n\right)1_{n<T_B}\right]\\&=\sum_{n=0}^{\infty}\left(Q_A^ng\right)(x).\end{aligned}$$

■

In practice it is not so easy to sum the series in Eqs. (8.30) and (8.31). Thus we would like to have another way to compute these quantities. Since $\sum_{n=0}^{\infty}Q_A^n$ is a geometric series, we expect that

$$\sum_{n=0}^{\infty}Q_A^n=\left(I-Q_A\right)^{-1}$$

which is basically correct at least when $\left(I-Q_A\right)$ is invertible. This suggests that if $u(x)=\mathbb{E}_x\left[h\left(X_{T_B}\right):T_B<\infty\right]$, then (see Eq. (8.30))

$$u=Q_Au+Q\left[1_Bh\right]\text{ on }A,\tag{8.32}$$

and if $u(x)=\mathbb{E}_x\left[\sum_{n<T_B}g\left(X_n\right)\right]$, then (see Eq. (8.31))

$$u=Q_Au+g\text{ on }A.\tag{8.33}$$

That these equations are valid was the content of Corollaries **??** and 8.11 above. below which we will prove using the “first step” analysis in the next theorem. We will give another direct proof in Theorem 8.23 below as well.

Lemma 8.19. *Keeping the notation above we have*

$$\mathbb{E}_xT=\sum_{n=0}^{\infty}\sum_{y\in A}Q^n(x,y)\text{ for all }x\in A,\tag{8.34}$$

where $\mathbb{E}_xT=\infty$ is possible.

Proof. By definition of T we have for $x\in A$ and $n\in\mathbb{N}_0$ that,

$$\begin{aligned}P_x(T>n)&=P_x\left(X_1,\dots,X_n\in A\right)\\&=\sum_{x_1,\dots,x_n\in A}p\left(x,x_1\right)p\left(x_1,x_2\right)\dots p\left(x_{n-1},x_n\right)\\&=\sum_{y\in A}Q^n(x,y).\end{aligned}\tag{8.35}$$

Therefore Eq. (8.34) now follows from Lemma 7.22 and Eq. (8.35). ■

Proposition 8.20. *Let us continue the notation above and let us further assume that A is a finite set and*

$$P_x(T<\infty)=P\left(X_n\in B\text{ for some }n\right)>0\;\forall\;x\in A.\tag{8.36}$$

Under these assumptions, $\mathbb{E}_xT<\infty$ for all $x\in A$ and in particular $P_x(T<\infty)=1$ for all $x\in A$. In this case we may may write Eq. (8.34) as

$$\left(\mathbb{E}_xT\right)_{x\in A}=(I-Q)^{-1}\mathbf{1}\tag{8.37}$$

where $\mathbf{1}(x)=1$ for all $x\in A$.

Proof. Since $\{T>n\}\downarrow\{T=\infty\}$ and $P_x(T=\infty)<1$ for all $x\in A$ it follows that there exists an $m\in\mathbb{N}$ and $0\le\alpha<1$ such that $P_x(T>m)\le\alpha$ for all $x\in A$. Since $P_x(T>m)=\sum_{y\in A}Q^m(x,y)$ it follows that the row sums of Q^m are all less than $\alpha<1$. Further observe that

$$\begin{aligned}\sum_{y\in A}Q^{2m}(x,y)&=\sum_{y,z\in A}Q^m(x,z)Q^m(z,y)=\sum_{z\in A}Q^m(x,z)\sum_{y\in A}Q^m(z,y)\\&\le\sum_{z\in A}Q^m(x,z)\alpha\le\alpha^2.\end{aligned}$$

Similarly one may show that $\sum_{y\in A} Q^{km}(x,y) \le \alpha^k$ for all $k \in \mathbb{N}$. Therefore from Eq. (8.35) with m replaced by km, we learn that $P_x(T > km) \le \alpha^k$ for all $k \in \mathbb{N}$ which then implies that

$$\sum_{y\in A} Q^n(x,y) = P_x(T>n) \le \alpha^{\left\lfloor \frac{n}{k} \right\rfloor} \text{ for all } n \in \mathbb{N},$$

where $\lfloor t \rfloor = m \in \mathbb{N}_0$ if $m \le t < m+1$, i.e. $\lfloor t \rfloor$ is the nearest integer to t which is smaller than t. Therefore, we have

$$\mathbb{E}_x T = \sum_{n=0}^{\infty} \sum_{y\in A} Q^n(x,y) \le \sum_{n=0}^{\infty} \alpha^{\left\lfloor \frac{n}{m} \right\rfloor} \le m \cdot \sum_{l=0}^{\infty} \alpha^l = m\frac{1}{1-\alpha} < \infty.$$

So it only remains to prove Eq. (8.37). From the above computations we see that $\sum_{n=0}^{\infty} Q^n$ is convergent. Moreover,

$$(I-Q)\sum_{n=0}^{\infty} Q^n = \sum_{n=0}^{\infty} Q^n - \sum_{n=0}^{\infty} Q^{n+1} = I$$

and therefore $(I-Q)$ is invertible and $\sum_{n=0}^{\infty} Q^n = (I-Q)^{-1}$. Finally,

$$(I-Q)^{-1}\mathbf{1} = \sum_{n=0}^{\infty} Q^n \mathbf{1} = \left(\sum_{n=0}^{\infty}\sum_{y\in A} Q^n(x,y)\right)_{x\in A} = (\mathbb{E}_x T)_{x\in A}$$

as claimed. ■

Remark 8.21. Let $\{X_n\}_{n=0}^{\infty}$ denote the fair random walk on $\{0,1,2,\dots\}$ with 0 being an absorbing state. Using the first homework problems, see Remark **??**, we learn that $\mathbb{E}_i T = \infty$ for all $i > 0$. This shows that we can not in general drop the assumption that A ($A = \{1,2,\dots\}$ in this example) is a finite set the statement of Proposition 8.20.

8.4.1 General facts about sub-probability kernels

Definition 8.22. *Suppose $(A,\mathcal{A})$ is a measurable space. A* ***sub-probability kernel*** *on $(A,\mathcal{A})$ is a function $\rho : A \times \mathcal{A} \to [0,1]$ such that $\rho(\cdot, C)$ is $\mathcal{A}/\mathcal{B}_{\mathbb{R}}$ – measurable for all $C \in \mathcal{A}$ and $\rho(x,\cdot) : \mathcal{A} \to [0,1]$ is a measure for all $x \in A$.*

As with probability kernels we will identify ρ with the linear map, $\rho : \mathcal{A}_b \to \mathcal{A}_b$ given by

$$(\rho f)(x) = \rho(x,f) = \int_A f(y)\,\rho(x,dy).$$

Of course we have in mind that $\mathcal{A} = \mathcal{S}_A$ and $\rho = Q_A$. In the following lemma let $\|g\|_\infty := \sup_{x\in A} |g(x)|$ for all $g \in \mathcal{A}_b$.

Theorem 8.23. *Let ρ be a sub-probability kernel on a measurable space $(A,\mathcal{A})$ and define $u_n(x) := (\rho^n 1)(x)$ for all $x \in A$ and $n \in \mathbb{N}_0$. Then;*

1. *u_n is a decreasing sequence so that $u := \lim_{n\to\infty} u_n$ exists and is in $\mathcal{A}_b$. (When $\rho = Q_A$, $u_n(x) = P_x(T_B > n) \downarrow u(x) = P(T_B = \infty)$ as $n \to \infty$.)*
2. *The function u satisfies $\rho u = u$.*
3. *If $w \in \mathcal{A}_b$ and $\rho w = w$ then $|w| \le \|w\|_\infty u$. In particular the equation, $\rho w = w$, has a non-zero solution $w \in \mathcal{A}_b$ iff $u \neq 0$.*
4. *If $u = 0$ and $g \in \mathcal{A}_b$, then there is at most one $w \in \mathcal{A}_b$ such that $w = \rho w + g$.*
5. *Let*
$$U := \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \rho^n 1 : A \to [0,\infty] \tag{8.38}$$
and suppose that $U(x) < \infty$ for all $x \in A$. Then for each $g \in \mathcal{S}_b$,
$$w = \sum_{n=0}^{\infty} \rho^n g \tag{8.39}$$
is absolutely convergent,
$$|w| \le \|g\|_\infty U, \tag{8.40}$$
$\rho(x,|w|) < \infty$ for all $x \in A$, and w solves $w = \rho w + g$. Moreover if v also solves $v = \rho v + g$ and $|v| \le CU$ for some $C < \infty$ then $v = w$.
Observe that when $\rho = Q_A$,
$$U(x) = \sum_{n=0}^{\infty} P_x(T_B > n) = \sum_{n=0}^{\infty} \mathbb{E}_x(1_{T_B > n}) = \mathbb{E}_x\left(\sum_{n=0}^{\infty} 1_{T_B > n}\right) = \mathbb{E}_x[T_B].$$
6. *If $g : A \to [0,\infty]$ is any measurable function then*
$$w := \sum_{n=0}^{\infty} \rho^n g : A \to [0,\infty]$$
is a solution to $w = \rho w + g$. (It may be that $w \equiv \infty$ though!) Moreover if $v : A \to [0,\infty]$ satisfies $v = \rho v + g$ then $w \le v$. Thus w is the minimal non-negative solution to $v = \rho v + g$.
7. *If there exists $\alpha < 1$ such that $u \le \alpha$ on A then $u = 0$. (When $\rho = Q_A$, this states that $P_x(T_B = \infty) \le \alpha$ for all $x \in A$ implies $P_x(T_A = \infty) = 0$ for all $x \in A$.)*
8. *If there exists an $\alpha < 1$ and an $n \in \mathbb{N}$ such that $u_n = \rho^n 1 \le \alpha$ on A, then there exists $C < \infty$ such that*
$$u_k(x) = \left(\rho^k 1\right)(x) \le C\beta^k \text{ for all } x \in A \text{ and } k \in \mathbb{N}_0$$

where $\beta := \alpha^{1/n} < 1$. In particular, $U \le C\left(1-\beta\right)^{-1}$ and $u = 0$ under this assumption.
(When $\rho = Q_A$ this assertion states; if $P_x\left(T_B > n\right) \le \alpha$ for all $\alpha \in A$, then $P_x\left(T_B > k\right) \le C\beta^k$ and $\mathbb{E}_x T_B \le C\left(1-\beta\right)^{-1}$ for all $k \in \mathbb{N}_0$.)

Proof. We will prove each item in turn.

1. First observe that $u_1\left(x\right) = \rho\left(x, A\right) \le 1 = u_0\left(x\right)$ and therefore,
$$u_{n+1} = \rho^{n+1} 1 = \rho^n u_1 \le \rho^n 1 = u_n.$$
We now let $u := \lim_{n\to\infty} u_n$ so that $u : A \to [0,1]$.
2. Using DCT we may let $n \to \infty$ in the identity, $\rho u_n = u_{n+1}$ in order to show $\rho u = u$.
3. If $w \in \mathcal{A}_b$ with $\rho w = w$, then
$$|w| = |\rho^n w| \le \rho^n |w| \le \|w\|_\infty \rho^n 1 = \|w\|_\infty \cdot u_n.$$
Letting $n \to \infty$ shows that $|w| \le \|w\|_\infty u$.
4. If $w_i \in \mathcal{A}_b$ solves $w_i = \rho w_i + g$ for $i = 1, 2$ then $w := w_2 - w_1$ satisfies $w = \rho w$ and therefore $|w| \le Cu = 0$.
5. Let $U := \sum_{n=0}^\infty u_n = \sum_{n=0}^\infty \rho^n 1 : A \to [0, \infty]$ and suppose $U\left(x\right) < \infty$ for all $x \in A$. Then $u_n\left(x\right) \to 0$ as $n \to \infty$ and so bounded solutions to $\rho u = u$ are necessarily zero. Moreover we have, for all $k \in \mathbb{N}_0$, that
$$\rho^k U = \sum_{n=0}^\infty \rho^k u_n = \sum_{n=0}^\infty u_{n+k} = \sum_{n=k}^\infty u_n \le U. \tag{8.41}$$
Since the tails of convergent series tend to zero it follows that $\lim_{k\to\infty} \rho^k U = 0$.
Now if $g \in \mathcal{S}_b$, we have
$$\sum_{n=0}^\infty |\rho^n g| \le \sum_{n=0}^\infty \rho^n |g| \le \sum_{n=0}^\infty \rho^n \|g\|_\infty = \|g\|_\infty \cdot U < \infty \tag{8.42}$$
and therefore $\sum_{n=0}^\infty \rho^n g$ is absolutely convergent. Making use of Eqs. (8.41) and (8.42) we see that
$$\sum_{n=1}^\infty \rho\left|\rho^n g\right| \le \|g\|_\infty \cdot \rho U \le \|g\|_\infty U < \infty$$
and therefore (using DCT),
$$w = \sum_{n=0}^\infty \rho^n g = g + \sum_{n=1}^\infty \rho^n g$$
$$= g + \rho \sum_{n=1}^\infty \rho^{n-1} g = g + \rho w,$$
i.e. w solves $w = g + \rho w$.
If $v : A \to \mathbb{R}$ is measurable such that $|v| \le CU$ and $v = g + \rho v$, then $y := w - v$ solves $y = \rho y$ with $|y| \le \left(C + \|g\|_\infty\right) U$. It follows that
$$|y| = |\rho^n y| \le \left(C + \|g\|_\infty\right) \rho^n U \to 0 \text{ as } n \to \infty,$$
i.e. $0 = y = w - v$.
6. If $g \ge 0$ we may always define w by Eq. (8.39) allowing for $w\left(x\right) = \infty$ for some or even all $x \in A$. As in the proof of the previous item (with DCT being replaced by MCT), it follows that $w = \rho w + g$. If $v \ge 0$ also solves $v = g + \rho v$, then
$$v = g + \rho\left(g + \rho v\right) = g + \rho g + \rho^2 v$$
and more generally by induction we have
$$v = \sum_{k=0}^n \rho^k g + \rho^{n+1} v \ge \sum_{k=0}^n \rho^k g.$$
Letting $n \to \infty$ in this last equation shows that $v \ge w$.
7. If $u \le \alpha < 1$ on A, then by item 3. with $w = u$ we find that
$$u \le \|u\|_\infty \cdot u \le \alpha u$$
which clearly implies $u = 0$.
8. If $u_n \le \alpha < 1$, then for any $m \in \mathbb{N}$ we have,
$$u_{n+m} = \rho^m u_n \le \alpha \rho^m 1 = \alpha u_m.$$
Taking $m = kn$ in this inequality shows, $u_{(k+1)n} \le \alpha u_{kn}$. Thus a simple induction argument shows $u_{kn} \le \alpha^k$ for all $k \in \mathbb{N}_0$. For general $l \in \mathbb{N}_0$ we write $l = kn + r$ with $0 \le r < n$. We then have,
$$u_l = u_{kn+r} \le u_{kn} \le \alpha^k = \alpha^{\frac{l-r}{n}} = C\alpha^{l/n}$$
where $C = \alpha^{-\frac{n-1}{n}}$.

■

Corollary 8.24. *If $h : B \to [0,\infty]$ is measurable, then $u(x) := \mathbb{E}_x\left[h\left(X_{T_B}\right) : T_B < \infty\right]$ is the unique minimal non-negative solution to Eq. (8.32) while if $g : A \to [0,\infty]$ is measurable, then $u(x) = \mathbb{E}_x\left[\sum_{n<T_B} g(X_n)\right]$ is the unique minimal non-negative solution to Eq. (8.33).*

Exercise 8.11. Keeping the notation of Exercise 8.8 and 8.10. Use Corollary 8.24 to show again that $P_x(T_B < \infty) = (q/p)^x$ for all $x > 0$ and $\mathbb{E}_x T_0 = x/(q-p)$ for $x < 0$. You should do so without making use of the extraneous hitting times, T_n for $n \neq 0$.

Solution to Exercise (8.11). From Eq. (8.23) of Exercise 8.8 we have seen for $x > 1$ that

$$P_x(T_0 < \infty) = a + (1-a)(q/p)^x$$

for some $a \in [0,1]$. Since

$$\frac{d}{da}\left[a + (1-a)(q/p)^x\right] = 1 - (q/p)^x > 0,$$

the right side will be smallest when $a = 0$ and therefore we may (Corollary 8.24) conclude that

$$P_x(T_0 < \infty) = (q/p)^x \text{ for all } x > 0.$$

Similarly from Eq. (8.28) of Exercise 8.10 we have seen that if $\mathbb{E}_x T_0 < \infty$ for some and hence all $x < 0$ then

$$\mathbb{E}_x T_0 = (q-p)^{-1} x + a\left[1 - (q/p)^x\right]$$

for some $a \leq 0$. Since the right side of this equation is minimized by taking $a = 0$ we again have by Corollary 8.24 that

$$\mathbb{E}_x T_0 = (q-p)^{-1} x \text{ for all } x < 0.$$

Corollary 8.25. *If $P_x(T_B = \infty) = 0$ for all $x \in A$ and $h : B \to \mathbb{R}$ is a bounded measurable function, then $u(x) := \mathbb{E}_x\left[h\left(X_{T_B}\right)\right]$ is the **unique** solution to Eq. (8.32).*

Corollary 8.26. *Suppose now that $A = B^c$ is a finite subset of S such that $P_x(T_B = \infty) < 1$ for all $x \in A$. Then there exists $C < \infty$ and $\beta \in (0,1)$ such that $P_x(T_B > n) \leq C\beta^n$ and in particular $\mathbb{E}_x T_B < \infty$ for all $x \in A$.*

Proof. Let $\alpha_0 = \max_{x \in A} P_x(T_B = \infty) < 1$. We know that

$$\lim_{n\to\infty} P_x(T_B > n) = P_x(T_B = \infty) \leq \alpha_0 \text{ for all } x \in A.$$

Therefore if $\alpha \in (\alpha_0, 1)$, using the fact that A is a finite set, there exists an n sufficiently large such that $P_x(T_B > n) \leq \alpha$ for all $x \in A$. The result now follows from item 8. of Theorem 8.23. ■

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