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Theorem 13.1 (General 2×2 inverse). *Let A be a non-zero 2×2 matrix written as*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and set

$$\delta = \det(A) := ad - bc.$$

Then A^{-1} exists iff $\det(A) \neq 0$ and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Proof. If

$$C := \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

then a simple computation shows;

$$CA = AC = \delta I. \tag{13.1}$$

So if A^{-1} exists, then $A \neq 0$ and hence $C \neq 0$. Since

$$\delta A^{-1} = A^{-1}(\delta I) = A^{-1}(AC) = C \tag{13.2}$$

it follows that $\delta \neq 0$.

Conversely if $\delta \neq 0$ then it follows from Eq. (13.2) that $A^{-1} = \frac{1}{\delta} C$ as claimed.

Alternative Proof. Given $y \in \mathbb{R}^2$ we will solve the best we can the equation $A\mathbf{x} = \mathbf{y}$. In components this becomes;

$$ax_1 + bx_2 = y_1 \tag{13.3}$$
$$cx_1 + dx_2 = y_2. \tag{13.4}$$

Performing

$$\begin{aligned} d\,(13.3) - b\,(13.4) &\implies (ad - bc)\,x_1 = dy_1 - by_2 \\ -c\,(13.3) + a\,(13.4) &\implies (ad - bc)\,x_2 = ay_2 - cy_1. \end{aligned}$$

So if $Ax = y$ then,

$$\begin{aligned} \det(A)\,\mathbf{x} &= \begin{bmatrix} dy_1 - by_2 \\ -cy_1 + ay_2 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \mathbf{y}. \end{aligned}$$

Case 1. If $\det(A) = 0$ then the vector y would have to satisfy

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \mathbf{y} = \mathbf{0}$$

and this means A is not onto, i.e. $Ax = y$ does not have a solution for all y and hence A is not invertible.

Case 2. If $\det(A) \neq 0$ we have found

$$x = A^{-1}y = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} y.$$

It is now straight forward to verify that

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det(A)\, I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

so that $A^{-1}A = I = AA^{-1}$. ■

Proposition 13.2 (Properties of the inverse). *Suppose A and B are invertible $n \times n$ matrices. Then;*

1. *A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.*
2. *AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.*

Proof. 1. We need only check that $\left(A^{-1}\right)A = I = A\left(A^{-1}\right)$ which is certainly true.

2. We need only check that $(AB)\left(B^{-1}A^{-1}\right) = I = \left(B^{-1}A^{-1}\right)(AB)$ which is again true.

3. Again it suffices to show $A^T\left(A^{-1}\right)^T = I = \left(A^{-1}\right)^T A^T$. This is seen as follows

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I \text{ and}$$
$$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I^T = I.$$

■

Proposition 13.3. *Suppose that A and B are $n\times n$ matrices such that $AB = I$, then $BA = I$, i.e. $B = A^{-1}$.*

Proof. First proof. If $y \in \mathbb{R}^n$ and $x = By$, then $Ax = ABy = Iy = y$ which shows that A is onto which implies that A has n – pivots which implies $\mathrm{Nul}\,(A) = \{0\}$. If $x \in \mathbb{R}^n$ we have $ABAx = IAx = Ax$, so that $A\,(BAx - x) = 0$. As $\mathrm{Nul}\,(A) = \{0\}$ we may conclude that $BAx - x = 0$ for all $x \in \mathbb{R}^n$, i.e. $BA = I$.

Second proof. If $Bx = 0$ then $0 = A0 = ABx = x$ which implies $\mathrm{Nul}\,(B) = 0$. Therefore B has n - pivots which implies B is onto as well. Thus we may define C so that $Bc_i = e_i$ for all i and for this C we will have $BC = I$. However,

$$C = IC = (AB)\,C = ABC = AI = A$$

and therefore $BA = I$ as desired. ■

Theorem 13.4. *Suppose that A is a $n \times n$ matrix. Then A^{-1} exists iff the equation $Ax = y$ has a unique solution x for all $y \in \mathbb{R}^n$ iff rref$(A) = I$. Moreover*

$$A^{-1} = [b_1|\dots|b_n] =: B$$

where $Ab_i = e_i$ for $i = 1, 2, \dots, n$.

Proof. This has already been proved but here is the proof again. If A^{-1} exists and $Ax = y$, then

$$x = Ix = \left(A^{-1}A\right)x = A^{-1}\,(Ax) = A^{-1}y$$

and if $y \in \mathbb{R}^n$ and $x := A^{-1}y$ we find

$$Ax = A\left(A^{-1}y\right) = \left(AA^{-1}\right)y = Iy = y.$$

Conversely suppose $Ax = y$ has a unique solution for all $y \in \mathbb{R}^n$ and let B be as above. Then

$$AB = [Ab_1|\dots|Ab_n] = [e_1|\dots|e_n] = I.$$

So to finish the proof we must show $BA = I$ as well. Now apply Proposition 13.3. ■

Corollary 13.5. *If A is an $n \times n$ matrix is invertible, then*

$$rref([A|I]) = \left[I|A^{-1}\right].$$

Proof. Recall that $A^{-1}e_i = b_i$ where $Ab_i = e_i$ and hence

$$rref\,([A|e_i]) = [I|b_i] = \left[I|A^{-1}e_i\right].$$

Therefore

$$rref\,([A|e_1|e_2|\dots|e_n]) = \left[I|A^{-1}e_1|A^{-1}e_2|\dots|A^{-1}e_n\right] = \left[I|A^{-1}\right].$$

■

Theorem 13.6 (Invertible Matrix Theorem 9, p. 129). *Let A be a square matrix then the following are equivalent;*

1. *A^{-1} exists.*
2. *a) $A : \mathbb{R}^n \to \mathbb{R}^n$ is one to one, i.e. $Ax = y$ has at most one solution.*
   b) $\mathrm{Nul}\,(A) = \{0\}$
   c) $Ax = 0$ has only the trivial solution $x = 0$.
   d) the columns of A are linearly independent.
   e) rref(A) has a pivot in every column.
   f) rref$(A) = I$
3. *a) $A : \mathbb{R}^n \to \mathbb{R}^n$ is onto.*
   b) $Ax = y$ is consistent for all $y \in \mathbb{R}^m$.
   c) $\mathrm{Ran}\,(A) = \mathrm{col}\,(A) = \mathbb{R}^n$.
   d) rref(A) has a pivot in every row.
   e) rref$(A) = I$
4. *There is a matrix C such that $CA = I$.*
5. *There is a matrix D such that $AD = I$.*
6. *A^T is invertible.*

13.1 Vector Spaces

Let us recall the following basic properties about $\mathbb{R}^n$.

Theorem 13.7 (See top of p. 32). *For all $x, y, z \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$;*

1. ***Associativity of addition:*** $x + (y + z) = (x + y) + z$.
2. ***Commutativity of addition:*** $y + z = z + y$.
3. ***Identity element of addition:*** $0 + y = y$ *for all* y.
4. ***Inverse elements of addition:*** $-y + y = 0$ *for all* $y \in \mathbb{R}^n$.

5. ***Distributivity of scalar multiplication with respect to vector addition:*** $a(y+z) = ay + az$.
6. ***Distributivity of scalar multiplication with respect to field addition*** $(a+b)y = ay + by$.
7. ***Compatibility of scalar multiplication with the multiplication on*** $\mathbb{R}$ $a(by) = (ab)y$.
8. ***Identity element of scalar multiplication*** $1y = y$ *for all* $y \in \mathbb{R}^n$.

We now turn these properties into a definition.

Definition 13.8 (See Definition on p. 217). *A **vector space** is a non-empty set V of objects, called vectors, equipped with an addition operation "+" and scalar (=$\mathbb{R}$ or maybe $\mathbb{C}$) multiplication "$\cdot$" satisfying all of the properties above: i.e. For all $u, v, w \in V$ and $a, b \in \mathbb{R}$;*

1. **Associativity of addition:** $u + (v + w) = (u + v) + w$.
2. **Commutativity of addition:** $v + w = w + v$.
3. **Identity element of addition:** There is a element $0 \in V$ such that $0 + v = v$ for all v.
4. **Inverse elements of addition:** $-v + v = 0$ for all $v \in V$. (In fact $-v = (-1) \cdot v$.)
5. **Distributivity of scalar multiplication with vector addition:** $a \cdot (v + w) = a \cdot v + a \cdot w$.
6. **Distributivity of scalar multiplication with respect to field addition** $(a + b) \cdot v = a \cdot v + b \cdot v$.
7. **Compatibility of scalar multiplication with the multiplication on** $\mathbb{R}$ $a \cdot (b \cdot v) = (ab) \cdot v$.
8. **Identity element of scalar multiplication** $1 \cdot v = v$ for all $v \in V$.

Example 13.9. The infinite blackboard with geometric addition and scalar multiplication.

Example 13.10 (The Main Umbrella Example). Let $\mathbb{D}$ be a non-empty set and let V denote all functions, $f : \mathbb{D} \to \mathbb{R}$. For $f, g \in V$ and $\lambda \in \mathbb{R}$ we define $f + g$ and $\lambda \cdot f$ by

$$\begin{aligned} (f+g)(t) &= f(t) + g(t) \quad \text{(addition in } \mathbb{R}), \\ (\lambda \cdot f)(t) &= \lambda f(t) \quad \text{(multiplication in } \mathbb{R}). \end{aligned}$$

It can now be checked that V is a vector space so that functions have now become vectors! Essentially all other examples of vector spaces we give will be related to an example of this form.

Example 13.11. $\mathbb{R}^3 = \{x : \{1, 2, 3\} \to \mathbb{R}\}$ and more generally $\mathbb{R}^n = \{x : \{1, 2, \ldots, n\} \to \mathbb{R}\}$.

Example 13.12. The vector space of 2×2 matrices;

$$M_{2\times 2} = \{A : A \text{ is a } 2 \times 2 - \text{matrix }\} = \{A : \{(1,1), (1,2), (2,1), (2,2)\} \to \mathbb{R}\}.$$

This can be generalized.

Definition 13.13 (Subspace). *Let V be a vector space. A non-empty subset, $H \subset V$, is a **subspace** of V if H is closed under addition and scalar multiplication.*