Test #1 Review Material

Definition 30.1. *If $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ then*

$$\begin{aligned}\operatorname{Nul}(T) &= \{\mathbf{x} \in \mathbb{R}^n : T(x) = 0\} = \operatorname{Nul}(A)\\ \operatorname{Ran}(T) &= \{A\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n\}\\ &= \{b \in \mathbb{R}^m : Ax = b \text{ has a solution}\}.\end{aligned}$$

*We refer to $\operatorname{Nul}(T)$ as the **null space** of T and $\operatorname{Ran}(T)$ and the **range** of T. We further say;*

1. *T is **one to one** if $T(x) = T(y)$ implies $x = y$ or equivalently, for all $b \in \mathbb{R}^m$ there is at most one solution to $T(x) = b$.*
2. *T is **onto** if $\operatorname{Ran}(T) = \mathbb{R}^m$ or equivalently put for every $b \in \mathbb{R}^m$ there is at least one solution to $T(x) = b$.*

30.1 Things you should know;

1. Linear systems like

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m.\end{aligned}$$

are equivalent to the matrix equation $A\mathbf{x} = b$ where

$$A = [\mathbf{a}_1|\ldots|\mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n}\\ a_{21} & a_{22} & \ldots & a_{2n}\\ \vdots & \ldots & \ldots & \vdots\\ a_{m1} & a_{m2} & \ldots & a_{mn}\end{bmatrix} = m \times n \text{ - } \textbf{coefficient matrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_n\\ \vdots\\ x_n\end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1\\ b_n\\ \vdots\\ b_m\end{bmatrix}.$$

2. $A\mathbf{x}$ is a linear combination of the columns of A –

$$A\mathbf{x} = \sum_{i=1}^{n} x_i a_i.$$

and $T(\mathbf{x}) := A\mathbf{x}$ defines a linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$, i.e. T preserves vector addition and scalar multiplication.

3.

$$\begin{aligned}\operatorname{span}\{a_1, \ldots, a_n\} &= \{b \in \mathbb{R}^m : Ax = b \text{ has a solution}\}\\ &= \{b \in \mathbb{R}^m : Ax = b \text{ is consistent}\}\\ &= \operatorname{Ran}(A) = \{b = Ax : x \in \mathbb{R}^n\}.\end{aligned}$$

4.

$$\begin{aligned}\operatorname{Nul}(A) &:= \{x \in \mathbb{R}^n : Ax = 0\}\\ &= \left\{\begin{matrix}\text{all solutions to the homogeneous equation:}\\ Ax = 0\end{matrix}\right\}\end{aligned}$$

5. **Theorem:** If $Ap = b$ then the general solution to $Ax = b$ is of the form $x = p + v_h$ where v_h is a generic solution to $Av_h = 0$, i.e. $x = p + v_h$ with $v_h \in \operatorname{Nul}(A)$.
6. You should know the definition of **linear independence (dependence),** i.e. $\Gamma := \{a_1, \ldots, a_n\} \subset \mathbb{R}^m$ are linearly independent iff the only solution to $\sum_{i=1}^{n} x_i a_i = 0$ is $\mathbf{x} = 0$. Equivalently put,

$$\begin{aligned}\Gamma := \{a_1, \ldots, a_n\} \subset \mathbb{R}^m \text{ are L.I.} &\iff \operatorname{Nul}(A) = \{0\}\\ &\iff Ax = 0 \text{ has only the trivial solution.}\end{aligned}$$

7. You should know to perform row reduction in order to put a matrix into it reduced row echelon form.
8. You should be able to write down the general solution to the equation $Ax = b$ and find equation that b must satisfy so that $Ax = b$ is consistent.
9. You should be able to find the eigenvalues of 2×2 matrices, i.e. those λ such that $\operatorname{Nul}(A - \lambda I) \neq 0$.
10. **Theorem:** Let A be a $m \times n$ matrix and $U :=\operatorname{rref}(A)$. Then;

a) $Ax = b$ has a solution iff rref$([A|b])$ does not have a pivot in the last column.

b) $Ax = b$ has a solution for every $b \in \mathbb{R}^m$ iff $\operatorname{span}\{a_1, \ldots, a_n\} = \operatorname{Ran}(A) = \mathbb{R}^m$ iff $U :=$rref(A) has a pivot in **every row**, i.e. U does not contain a row of zeros.

c) If $m > n$ (i.e. there are more equations than unknowns), then $Ax = b$ will be inconsistent for some $b \in \mathbb{R}^m$. This is because there can be at most n -pivots and since $m > n$ there must be a row of zeros in rref(A).

d) $Ax = b$ has at most one solution iff $\operatorname{Nul}(A) = 0$ iff $Ax = 0$ has only the trivial solution iff $\{a_1, \ldots, a_n\}$ are linearly independent, iff rref(A) has no free variables – i.e. there is a pivot in every column.

e) If $m < n$ (i.e. there are fewer equation than unknowns) then $Ax = 0$ will always have a non-trivial solution or equivalently put the columns of A are necessarily linearly dependent. This is because there can be at most m pivots and so at least one column does not have pivot and there is at least one free variable.

Test #2 Review Material

Things to know;

1. The notion of a vector space and the fact that $\mathbb{R}^m$ and $V(\mathbb{D})$ = functions from $\mathbb{D}$ to $\mathbb{R}$ form a subspace.
2. How to determine if $H \subset V$ is a subspace or not.
3. The notions of linear independence and span of vectors in a vector space.
4. The notions of a basis, dimension, and coordinates relative to a basis.
5. Be able to check if vectors in $\mathbb{R}^m$ are a basis or not by putting the vectors in the columns of a matrix and row reducing. In particular you should know that n vectors in $\mathbb{R}^m$ are always linearly dependent if $n > m$ and that they do not span $\mathbb{R}^m$ is $n < m$.
6. Matrix operations. How to compute AB, $A+B$, $ABC = (AB)C$ or $A(BC)$. You should understand when these operations make sense.
7. The notion of a linear transformation, $T : V \to W$.
   a) If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ then $T(\mathbf{x}) = A\mathbf{x}$ where $A = [T(e_1)|\ldots|T(e_n)]$.
   b) The derivative and integration operations are linear.
   c) Other examples of linear transformations from the homework, e.g. $T : \mathcal{P}_2 \to \mathbb{R}^3$ with
   $$T(p) := \begin{bmatrix} p(1) \\ p(-1) \\ p(2) \end{bmatrix} \text{ is linear.}$$
8. $\mathrm{Nul}(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = 0\}$ – all vectors in the domain which are sent to zero.
   a) T is one to one iff $\mathrm{Nul}(T) = \{0\}$.
   b) be able to find a basis for $\mathrm{Nul}(A) \subset \mathbb{R}^n$ when A is a $m \times n$ matrix.
9. $\mathrm{Ran}(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\}$ – the range of T. Equivalently, $w \in \mathrm{Ran}(T)$ iff there exists a solution, $\mathbf{v} \in V$, to $T(\mathbf{v}) = w$.
   a) Know how to find a basis for $\mathrm{col}(A) = \mathrm{Ran}(A)$.
   b) Know how to find a basis for $\mathrm{row}(A)$.
   c) Know that $\dim \mathrm{row}(A) = \dim \mathrm{col}(A) = \dim \mathrm{Ran}(A) = \mathrm{rank}(A)$.
10. You should understand the rank-nullity theorem – see Theorem 14 on p. 265.
11. **Theorem.** If A is not a square matrix then A is not invertible!
12. **Theorem.** If A is a $n \times n$ matrix then the following are equivalent.
    a) A is invertible.
    b) $\mathrm{col}(A) = \mathrm{Ran}(A) = \mathbb{R}^n \iff \dim \mathrm{Ran}(A) = n = \dim \mathrm{row}(A)$
    c) $\mathrm{row}(A) = \mathbb{R}^n$
    d) $\mathrm{Nul}(A) = \{0\} \iff \dim \mathrm{Nul}(A) = 0$.
    e) $\det(A) \neq 0$.
    f) $\mathrm{rref}(A) = I$.
13. Be able to find A^{-1} using $[A|I] \sim \left[A^{-1}|I\right]$ when A is invertible.
14. Now that $Ax = b$ has solution given by $x = A^{-1}b$ when A is invertible.
15. Understand how to compute determinants of matrices. You should also know the determinants behavior under row and column operations.
16. For an $n \times n$ – matrix, A, the following are equivalent;
    a) A^{-1} does not exist,
    b) $\mathrm{Nul}(A) \neq \{0\}$,
    c) $\mathrm{Ran}(A) \neq \mathbb{R}^n$,
    d) $\det A = 0$.

After Test #2 Review Material

Things you should know:

1. You should understand the definition of **Eigenvalues** and **Eigenvectors** for linear transformations and especially for matrices. This includes being able to test if a vector is an eigenvector or not for a given linear transformation.
2. If $T : V \to V$ is a linear transformation and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are eigenvectors of T such that $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ with all the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ being distinct, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set.
3. The **characteristic polynomial** of a $n \times n$ matrix, A, is
$$p(\lambda) := \det(A - \lambda I).$$
You should know;
   a) how to compute $p(\lambda)$ for a given matrix A using the standard methods of evaluating determinants.
   b) The roots $\{\lambda_1, \ldots, \lambda_n\}$ (with possible repetitions) are all of the eigenvalues of A.
   c) The matrix A is diagonalizable if all of the roots of $p(\lambda)$ is distinct, i.e. there will be n -distinct eigenvectors in this case. If there are repeated roots A may or may not be diagonalizable.
4. After finding an eigenvalue, λ_i, of A you should be able to find a basis of the corresponding eigenvectors to this eigenvalue, i.e. find a basis for $\operatorname{Nul}(A - \lambda_i I)$.
5. An $n \times n$ matrix A is diagonalizable (i.e. may be written as $A = PDP^{-1}$ where D is a diagonal matrix and P is an invertible matrix) An $n \times n$ matrix A is diagonalizable iff A has a basis $(\mathcal{B})$ of eigenvectors. Moreover if $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, with $Av_i = \lambda_i v_i$,and
$$P = [\mathbf{v}_1 | \ldots | \mathbf{v}_n] \text{ and } D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix},$$
then $A = PDP^{-1}$.
6. If $A = PDP^{-1}$ as above then $A^k = PD^kP^{-1}$ where
$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n^k \end{bmatrix}.$$
7. The basic definition of the dot product on $\mathbb{R}^n$;
$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}$$
and its associated **norm or length function**, $\|\mathbf{u}\| := \sqrt{\mathbf{u} \cdot \mathbf{u}}$ and know that the distance between $\mathbf{u}$ and $\mathbf{v}$ is
$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) := \|\mathbf{u} - \mathbf{v}\| = \sqrt{\sum_{i=1}^n |u_i - v_i|^2}.$$
8. The meaning of **orthogonal sets**, **orthonormal sets**, **orthogonal bases**, and **orthonormal bases**. Moreover you should know;
   a) If $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal basis for a subspace, $W \subset \mathbb{R}^m$, then
$$\operatorname{proj}_{\operatorname{span}(S)} \mathbf{v} = \sum_{i=1}^n \frac{\mathbf{v} \cdot \mathbf{u}_i}{\|\mathbf{u}_i\|^2} \mathbf{u}_i$$
is the orthogonal projection of $\mathbf{v}$ onto W. The vector $\operatorname{proj}_{\operatorname{span}(\mathcal{B})} \mathbf{v}$ is the unique vector in W closest to $\mathbf{v}$ and also is the unique vector $\mathbf{w} \in W$ such that $\mathbf{v} - \mathbf{w}$ is perpendicular to W, i.e. $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{w}' = 0$ for all $\mathbf{w}' \in W$.
   b) If $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthogonal basis for $\mathbb{R}^n$ then
$$\mathbf{v} = \sum_{i=1}^n \frac{\mathbf{v} \cdot \mathbf{u}_i}{\|\mathbf{u}_i\|^2} \mathbf{u}_i \text{ for all } \mathbf{v} \in \mathbb{R}^n,$$
i.e.

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2} \\ \frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}_2\|^2} \\ \vdots \\ \frac{\mathbf{v}\cdot\mathbf{u}_n}{\|\mathbf{u}_n\|^2} \end{bmatrix}.$$

c) If $U = [\mathbf{u}_1|\dots|\mathbf{u}_n]$ is a $n \times n$ matrix, then the following are equivalent;
   i. U is an orthogonal matrix, i.e. $U^TU = I = UU^T$ or equivalently put $U^{-1} = U^T$.
   ii. The columns $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of U form an orthonormal basis for $\mathbb{R}^n$.

d) You should be able to carry out the Gram–Schmidt process in order to make an orthonormal basis for a subspace W out of an arbitrary basis for W.

9. If A is a $m \times n$ matrix you should know that A^T is the unique $n \times m$ matrix such that
$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^T\mathbf{y} \text{ for all } \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{y} \in \mathbb{R}^m.$$
10. You should be able to find all least squares solutions to an inconsistent matrix equation, $A\mathbf{x} = \mathbf{b}$ (i.e. $\mathbf{b} \notin \operatorname{Ran}(A)$ by solving the **normal equations,** $A^TA\mathbf{x} = A^T\mathbf{b}$. Recall the Least squares theorem states that $\mathbf{x} \in \mathbb{R}^n$ is a minimizer of $\|A\mathbf{x} - b\|$ iff $\mathbf{x}$ satisfies $A^TA\mathbf{x} = A^T\mathbf{b}$.
11. You should know and be able to show that if A is a symmetric matrix and u and v are eigenvectors of A with distinct eigenvalues, then $u \cdot v = 0$.
12. The **spectral theorem;** every symmetric $n \times n$ matrix A has an orthonormal basis of eigenvectors. Equivalently put A may be written as $A = UDU^T$ where U is a orthogonal matrix and D is a diagonal matrix. In particular every symmetric matrix may be diagonalized.
13. Given a symmetric matrix A you should be able to find $U = [\mathbf{u}_1|\dots|\mathbf{u}_n]$ as described in the spectral theorem. The method is to find (using the Gram–Schmidt process if necessary) an orthonormal basis of eigenvectors $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ of A and then we take $U = [\mathbf{u}_1|\dots|\mathbf{u}_n]$.