Test #1 Review Material

Definition 30.1. If \( T(x) = Ax \) is a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) then

\[
\text{Nul}(T) = \{ \mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = 0 \} = \text{Nul}(A)
\]

\[
\text{Ran}(T) = \{ Ax \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n \}
\]

\[
= \{ b \in \mathbb{R}^m : Ax = b \text{ has a solution} \}.
\]

We refer to \( \text{Nul}(T) \) as the null space of \( T \) and \( \text{Ran}(T) \) and the range of \( T \).
We further say:
1. \( T \) is one to one if \( T(x) = T(y) \) implies \( x = y \) or equivalently, for all \( b \in \mathbb{R}^m \) there is at most one solution to \( T(x) = b \).
2. \( T \) is onto if \( \text{Ran}(T) = \mathbb{R}^m \) or equivalently put for every \( b \in \mathbb{R}^m \) there is at least one solution to \( T(x) = b \).

30.1 Things you should know;

1. Linear systems like

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

are equivalent to the matrix equation \( Ax = b \) where

\[
A = [a_1|\ldots|a_n] = \begin{bmatrix}
  a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix} = m \times n \text{ - coefficient matrix},
\]

\[
x = \begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]

and

\[
b = \begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{bmatrix}.
\]

2. \( Ax \) is a linear combination of the columns of \( A \)

\[
Ax = \sum_{i=1}^{n} x_ia_i.
\]

and \( T(x) := Ax \) defines a linear transformation from \( \mathbb{R}^n \to \mathbb{R}^m \), i.e. \( T \) preserves vector addition and scalar multiplication.

3. \( \text{span} \{a_1, \ldots, a_n\} = \{ b \in \mathbb{R}^m : Ax = b \text{ has a solution} \}
\]

\[
= \{ b \in \mathbb{R}^m : Ax = b \text{ is consistent} \}
\]

\[
= \text{Ran}(A) = \{ b = Ax : x \in \mathbb{R}^n \}.
\]

4. \( \text{Nul}(A) := \{ x \in \mathbb{R}^n : Ax = 0 \} = \{ \text{all solutions to the homogeneous equation:} \}
\]

\[
Ax = 0
\]

5. Theorem: If \( Ap = b \) then the general solution to \( Ax = b \) is of the form \( x = p + v_h \) where \( v_h \) is a generic solution to \( Av_h = 0 \), i.e. \( x = p + v_h \) with \( v_h \in \text{Nul}(A) \).

6. You should know the definition of linear independence (dependence), i.e. \( \Gamma := \{a_1, \ldots, a_n\} \subset \mathbb{R}^m \) are linearly independent iff the only solution to \( \sum_{i=1}^{n} x_ia_i = 0 \) is \( x = 0 \). Equivalently put,

\[
\Gamma := \{a_1, \ldots, a_n\} \subset \mathbb{R}^m \text{ are L.I.} \iff \text{Nul}(A) = \{0\}
\]

\( \iff Ax = 0 \) has only the trivial solution.

7. You should know to perform row reduction in order to put a matrix into its reduced row echelon form.

8. You should be able to write down the general solution to the equation \( Ax = b \) and find equation that \( b \) must satisfy so that \( Ax = b \) is consistent.

9. You should be able to find the eigenvalues of \( 2 \times 2 \) matrices, i.e. those \( \lambda \) such that \( \text{Nul}(A - \lambda I) \neq 0 \).

10. Theorem: Let \( A \) be a \( m \times n \) matrix and \( U := \text{ref}(A) \). Then;
a) $Ax = b$ has a solution iff $\text{rref}(\begin{bmatrix} A & b \end{bmatrix})$ does not have a pivot in the last column.

b) $Ax = b$ has a solution for every $b \in \mathbb{R}^m$ iff $\text{span}\{a_1, \ldots, a_n\} = \text{Ran}(A) = \mathbb{R}^m$ iff $U := \text{rref}(A)$ has a pivot in every row, i.e. $U$ does not contain a row of zeros.

c) If $m > n$ (i.e. there are more equations than unknowns), then $Ax = b$ will be inconsistent for some $b \in \mathbb{R}^m$. This is because there can be at most $n$ - pivots and since $m > n$ there must be a row of zeros in $\text{rref}(A)$.

d) $Ax = b$ has at most one solution iff $\text{Nul}(A) = 0$ iff $Ax = 0$ has only the trivial solution iff $\{a_1, \ldots, a_n\}$ are linearly independent, iff $\text{rref}(A)$ has no free variables – i.e. there is a pivot in every column.

e) If $m < n$ (i.e. there are fewer equation than unknowns) then $Ax = 0$ will always have a non-trivial solution or equivalently put the columns of $A$ are necessarily linearly dependent. This is because there can be at most $m$ pivots and so at least one column does not have pivot and there is at least one free variable.
Test #2 Review Material

Things to know;

1. The notion of a vector space and the fact that \( \mathbb{R}^m \) and \( V (\mathbb{D}) = \) functions from \( \mathbb{D} \) to \( \mathbb{R} \) form a subspace.
2. How to determine if \( H \subset V \) is a subspace or not.
3. The notions of linear independence and span of vectors in a vector space.
4. The notions of a basis, dimension, and coordinates relative to a basis.
5. Be able to check if vectors in \( \mathbb{R}^m \) are a basis or not by putting the vectors in the columns of a matrix and row reducing. In particular you should know that \( n \) vectors in \( \mathbb{R}^m \) are always linearly dependent if \( n > m \) and that they do not span \( \mathbb{R}^m \) is \( n < m \).
6. Matrix operations. How to compute \( AB, A + B, ABC = (AB)C \) or \( A (BC) \). You should understand when these operations make sense.
7. The notion of a linear transformation, \( T : V \rightarrow W \).
   a) If \( V = \mathbb{R}^n \) and \( W = \mathbb{R}^m \) then \( T (x) = Ax \) where \( A = [T (e_1) | \ldots | T (e_n)] \).
   b) The derivative and integration operations are linear.
   c) Other examples of linear transformations from the homework, e.g. \( T : \mathcal{P}_2 \rightarrow \mathbb{R}^3 \) with
   \[
   T (p) := \begin{bmatrix}
   p (1) \\
   p (-1) \\
   p (2)
   \end{bmatrix}
   \]
   is linear.
8. \( \text{Nul} (T) = \{ v \in V : T (v) = 0 \} \) – all vectors in the domain which are sent to zero.
   a) \( T \) is one to one iff \( \text{Nul} (T) = \{ 0 \} \).
   b) be able to find a basis for \( \text{Nul} (A) \subset \mathbb{R}^n \) when \( A \) is a \( m \times n \) matrix.
9. \( \text{Ran} (T) = \{ T (v) \in W : v \in V \} \) – the range of \( T \). Equivalently, \( w \in \text{Ran} (T) \) iff there exists a solution, \( v \in V \), to \( T (v) = w \).
   a) Know how to find a basis for \( \text{col} (A) = \text{Ran} (A) \).
   b) Know how to find a basis for \( \text{row} (A) \).
   c) Know that \( \text{dim} \text{row} (A) = \text{dim col} (A) = \text{dim Ran} (A) = \text{rank} (A) \).
10. You should understand the rank-nullity theorem – see Theorem 14 on p. 265.
11. **Theorem.** If \( A \) is not a square matrix then \( A \) is not invertible!
12. **Theorem.** If \( A \) is a \( n \times n \) matrix then the following are equivalent:
   a) \( A \) is invertible.
   b) \( \text{col} (A) = \text{Ran} (A) = \mathbb{R}^n \iff \text{dim Ran} (A) = n = \text{dim row} (A) \)
   c) \( \text{row} (A) = \mathbb{R}^n \)
   d) \( \text{Nul} (A) = \{ 0 \} \iff \text{dim Nul} (A) = 0 \).
   e) \( \text{det} (A) \neq 0 \).
   f) \( \text{rref}(A) = I \).
13. Be able to find \( A^{-1} \) using \([A|I] \sim [A^{-1}|I]\) when \( A \) is invertible.
14. Now that \( Ax = b \) has solution given by \( x = A^{-1}b \) when \( A \) is invertible.
15. Understand how to compute determinants of matrices. You should also know the determinants behavior under row and column operations.
16. For an \( n \times n \) – matrix, \( A \), the following are equivalent:
   a) \( A^{-1} \) does not exist,
   b) \( \text{Nul} (A) \neq \{ 0 \} \),
   c) \( \text{Ran} (A) \neq \mathbb{R}^n \),
   d) \( \text{det} A = 0 \).
Things you should know:

1. You should understand the definition of **Eigenvalues** and **Eigenvectors** for linear transformations and especially for matrices. This includes being able to test if a vector is an eigenvector or not for a given linear transformation.

2. If $T : V \rightarrow V$ is a linear transformation and $\{v_1, \ldots, v_n\}$ are eigenvectors of $T$ such that $T(v_i) = \lambda_i v_i$ with all the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ being distinct, then $\{v_1, \ldots, v_n\}$ is a linearly independent set.

3. The **characteristic polynomial** of a $n \times n$ matrix, $A$, is

   $$ p(\lambda) := \det (A - \lambda I). $$

   You should know:
   a) how to compute $p(\lambda)$ for a given matrix $A$ using the standard methods of evaluating determinants.
   b) The roots $\{\lambda_1, \ldots, \lambda_n\}$ (with possible repetitions) are all of the eigenvalues of $A$. 
   c) The matrix $A$ is diagonalizable if all of the roots of $p(\lambda)$ is distinct, i.e. there will be $n$ -distinct eigenvectors in this case. If there are repeated roots $A$ may or may not be diagonalizable.

4. After finding an eigenvalue, $\lambda_i$, of $A$ you should be able to find a basis of the corresponding eigenvectors to this eigenvalue, i.e. find a basis for Nul $(A - \lambda_i I)$.

5. An $n \times n$ matrix $A$ is diagonalizable (i.e. may be written as $A = PDP^{-1}$ where $D$ is a diagonal matrix and $P$ is an invertible matrix) An $n \times n$ matrix $A$ is diagonalizable iff $A$ has a basis ($B$) of eigenvectors. Moreover if $B = \{v_1, \ldots, v_n\}$, with $Av_i = \lambda_i v_i$, and

   $$ P = [v_1 \ldots | v_n] \text{ and } D = \begin{bmatrix}
   \lambda_1 & 0 & \cdots & 0 \\
   0 & \lambda_2 & \cdots & 0 \\
   \vdots & \ddots & \ddots & \vdots \\
   0 & \cdots & 0 & \lambda_n
\end{bmatrix}, $$

   then $A = PDP^{-1}$.

6. If $A = PDP^{-1}$ as above then $A^k = PD^kP^{-1}$ where

   $$ D^k = \begin{bmatrix}
   \lambda_1^k & 0 & \cdots & 0 \\
   0 & \lambda_2^k & \cdots & 0 \\
   \vdots & \ddots & \ddots & \vdots \\
   0 & \cdots & 0 & \lambda_n^k
\end{bmatrix}. $$

7. The basic definition of the dot product on $\mathbb{R}^n$;

   $$ u \cdot v = \sum_{i=1}^{n} u_i v_i = u^T v = v^T u $$

   and its associated norm or length function, $\|u\| := \sqrt{u \cdot u}$ and know that the distance between $u$ and $v$ is

   $$ \text{dist} (u, v) := \|u - v\| = \sqrt{\sum_{i=1}^{n} |u_i - v_i|^2}.$$

8. The meaning of **orthogonal sets, orthonormal sets, orthogonal bases**, and **orthonormal bases**. Moreover you should know;
   a) If $B = \{u_1, \ldots, u_n\}$ is an orthogonal basis for a subspace, $W \subset \mathbb{R}^m$, then

      $$ \text{proj}_{\text{span}(B)} v = \sum_{i=1}^{n} \frac{v \cdot u_i}{\|u_i\|^2} u_i $$

   is the orthogonal projection of $v$ onto $W$. The vector $\text{proj}_{\text{span}(B)} v$ is the unique vector in $W$ closest to $v$ and also is the unique vector $w \in W$ such that $v - w$ is perpendicular to $W$, i.e. $(v - w) \cdot w' = 0$ for all $w' \in W$.
   b) If $B = \{u_1, \ldots, u_n\}$ is an orthogonal basis for $\mathbb{R}^n$ then

      $$ v = \sum_{i=1}^{n} \frac{v \cdot u_i}{\|u_i\|^2} u_i \text{ for all } v \in \mathbb{R}^n,$$

   i.e.
\[
[v]_B = \begin{bmatrix}
\frac{v \cdot u_1}{\|u_1\|^2} \\
\frac{v \cdot u_2}{\|u_2\|^2} \\
\vdots \\
\frac{v \cdot u_n}{\|u_n\|^2}
\end{bmatrix}
\]

c) If \( U = [u_1 \ldots u_n] \) is a \( n \times n \) matrix, then the following are equivalent:
   i. \( U \) is an orthogonal matrix, i.e. \( U^T U = I = UU^T \) or equivalently put \( U^{-1} = U^T \).
   ii. The columns \( \{u_1, \ldots, u_n\} \) of \( U \) form an orthonormal basis for \( \mathbb{R}^n \).

d) You should be able to carry out the Gram–Schmidt process in order to make an orthonormal basis for a subspace \( W \) out of an arbitrary basis for \( W \).

9. If \( A \) is a \( m \times n \) matrix you should know that \( A^T \) is the unique \( n \times m \) matrix such that
\[
A x \cdot y = x \cdot A^T y \quad \text{for all } x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^m.
\]

10. You should be able to find all least squares solutions to an inconsistent matrix equation, \( Ax = b \) (i.e. \( b \notin \text{Ran}(A) \)) by solving the \textbf{normal equations}, \( A^T Ax = A^T b \). Recall the Least squares theorem states that \( x \in \mathbb{R}^n \) is a minimizer of \( \|Ax - b\| \) iff \( x \) satisfies \( A^T Ax = A^T b \).

11. You should know and be able to show that if \( A \) is a symmetric matrix and \( u \) and \( v \) are eigenvectors of \( A \) with distinct eigenvalues, then \( u \cdot v = 0 \).

12. The \textbf{spectral theorem}; every symmetric \( n \times n \) matrix \( A \) has an orthonormal basis of eigenvectors. Equivalently put \( A \) may be written as \( A = UDU^T \) where \( U \) is an orthogonal matrix and \( D \) is a diagonal matrix. In particular every symmetric matrix may be diagonalized.

13. Given a symmetric matrix \( A \) you should be able to find \( U = [u_1 \ldots u_n] \) as described in the spectral theorem. The method is to find (using the Gram–Schmidt process if necessary) an orthonormal basis of eigenvectors \( \{u_1, \ldots, u_n\} \) of \( A \) and then we take \( U = [u_1 \ldots u_n] \).