Math 109. Instructor: Chow

Homework #2. Due in class on Wednesday, Jan 23, 2013.

Remarks after problems are for your information and not meant for you to prove in this homework assignment.

Problem 1: Prove (directly) the following:

a. If $n$ is an integer, then there exists an integer $m$ such that

$$(3n + 1)^2 = 3m + 1.$$ 

b. If $a$ is an integer, then there exists an integer $b$ such that

$$(3a + 2)^2 = 3b + 1.$$ 

The above are elementary facts in the spirit of modular arithmetic (Part V of the book).

Problem 2: Prove that for any real numbers $a, b, c, d$,

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2).$$

Hint: a backwards proof (as in section 3.2) may be the easiest.

Remark: Given vectors $[x, y]$ and $[u, v]$, define $[x, y] \cdot [u, v] = xu + yv$. Define $||x, y||^2 = [x, y] \cdot [x, y]$. The above says that $(||a, b|| \cdot ||c, d||)^2 \leq ||[a, b] \cdot [c, d]||^2$, or equivalently, $||a, b|| \cdot ||c, d|| \leq ||a, b|| \cdot ||c, d||$.

Problem 3: Prove by induction on $n$ that, for all positive integers $n$,

$$1^2 + 3^2 + \cdots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}.$$ 

Problem 4: Prove by induction on $n$ that, for all positive integers $n$,

$$1^3 + 3^3 + \cdots + (2n - 1)^3 = n^2(2n^2 - 1).$$

Problem 5: Let $u_1, u_2, u_3, \ldots$ denote the Fibonacci numbers, defined by $u_1 = 1, u_2 = 1$ and $u_{k+1} = u_{k-1} + u_k$ for $k \geq 2$. Prove by induction on $n$ that, for all positive integers $n$,

$u_{4n}$ is divisible by 3.

Problem 6: Let $x$ be a real number greater than $-1$. Prove by induction that for each nonnegative integer $n$,

$$(1 + x)^n \geq 1 + nx.$$ 

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**Problem 7:** A rational number is any real number which can be expressed as the quotient $p/q$ of two integers, where $q$ is not equal to zero. Prove by contradiction that there does not exist a smallest positive rational number.

**Problem 8:** Assume that we know that for any positive real numbers $x_1, x_2, \ldots, x_{100}$ that

$$\frac{x_1 + x_2 + \cdots + x_{100}}{100} \geq (x_1 \cdot x_2 \cdots \cdot x_{100})^{\frac{1}{100}}.$$ 

Prove that for any positive real numbers $x_1, x_2, \ldots, x_{99}$ that

$$\frac{x_1 + x_2 + \cdots + x_{99}}{99} \geq (x_1 \cdot x_2 \cdots \cdot x_{99})^{\frac{1}{99}}.$$ 

Hint: One way to prove this is to use the following idea: given $x_1, x_2, \ldots, x_{99}$, define $x_{100} = (x_1 \cdot x_2 \cdots \cdot x_{99})^{\frac{1}{99}}$.

**Remark:** the above represents an idea in part of the proof of the arithmetic-geometric mean inequality.