Math 109. Instructor: Chow
Homework #4. Due in discussion section on Monday, Feb 4, 2013.

**Problem 1:** Consider the function \( f : \mathbb{R} - \{2\} \rightarrow \mathbb{R} \) defined by
\[
f(x) = \frac{x^2 + 3x - 10}{x - 2}.
\]

(a) Find the unique choice of defining \( f(2) \) so that, with your choice of definition of \( f(2) \), the extended function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous. Continuity is defined in the sense of calculus.

(b) If you haven’t already done this in part (a), explain why this choice works. You don’t have to use \( \varepsilon \)’s and \( \delta \)’s in your explanation, but you should use limits (and perhaps some elementary properties of limits).

**Problem 2:** Is the function \( g : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( g(x) = \sqrt{|x - 3|} \) continuous at \( x = 3 \)? Explain why or why not.

**Problem 3:** We say that a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is periodic with period \( P \) if for each \( x \in \mathbb{R} \) we have \( f(x + P) = x \).

(a) Is the function \( g(x) = \sin(x^2) \) periodic for some period \( P \)? Explain.

(b) Is the function \( h(x) = \sin |x| \) periodic for some period \( P \)? Explain.

(c) Is the function \( k(x) = \sin^2 x \) periodic for some period \( P \)? Explain.

**Problem 4:** For each of the following sequences, tell us whether it converges or not. If it does not converge, then you may simply say so. If it converges, tell us what it converges to. For each answer give only a short explanation. We are not looking for proofs or complete justifications, we just want to make sure you understand how you got your answer.

(a) \( 4 + 2^n \),

(b) \( -7 - \left(-\frac{1}{3}\right)^n \),

(c) \( e^{5-n^2} \),

(d) \( \cos \frac{1}{n} \),

(e) \( \sin \left(\pi n^2\right) \),

(f) \( \ln(e^2 + n^{-3}) \).

Basically, for each of the above sequences we want to know whether, as \( n \) approaches infinity (i.e., gets larger and larger without bound), the expression converges to anything.
**Problem 5:** Define the function \( f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by \( f(x, y) = x^2 + y^2 \).

(a) Prove that for each \( z \in \mathbb{R}_{\geq} \) there exists \((x, y)\) such that \( f(x, y) = z \).

(b) Prove that for each \( z \in \mathbb{R} - \mathbb{R}_{\geq} \) there does not exist \((x, y)\) such that \( f(x, y) = z \).

(c) Prove the following. If \( a \leq b \), then

\[
\{(x, y) \mid f(x, y) \leq a\} \subseteq \{(x, y) \mid f(x, y) \leq b\}.
\]

**Problem 6:** In this problem, all sets will be assumed to be finite (i.e., only contain a finite number of elements). Also, let \( \mathbb{N}_n = \{1, 2, \ldots, n\} = \{a \in \mathbb{Z} \mid 1 \leq a \leq n\}\), which is a set with \( n \) elements.

Given a logical explanation (not a formal proof) for why the following statements are true. This problem is to test your conceptual understanding of various facts about when functions are injective or surjective.

(a) If \( X \subseteq Y \), then \( |X| \leq |Y| \).

(b) If \( f : A \to B \) is an injection, then \( |A| \leq |B| \). For example, if \( f : \mathbb{N}_m \to \mathbb{N}_n \) is an injection, then \( m \leq n \).

(c) If \( g : C \to D \) is a surjection, then \( |C| \geq |D| \). For example, if \( f : \mathbb{N}_m \to \mathbb{N}_n \) is a surjection, then \( m \geq n \).

(d) If \( h : E \to F \) is a bijection, then what is true about \( |E| \) and \( |F| \)? In particular, what is true about \( m \) and \( n \) in the case where \( E = \mathbb{N}_m \) and \( F = \mathbb{N}_n \)?