A fundamental sequence in nature, computer science, mathematics, etc. are the **Fibonacci numbers**. The first fifteen Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

You may have noticed the pattern: $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, $3 + 5 = 8$, $5 + 8 = 13$, etc. I.e., the sum of two successive Fibonacci numbers is equal to the next one.

Denote by $u_n$ the $n^{th}$ Fibonacci number. Then $u_n$ is defined using strong induction by

$$u_1 = 1,$$
$$u_2 = 1,$$
$$u_{k+1} = u_{k-1} + u_k \quad \text{for} \quad k \geq 2.$$ 

So $u_3 = u_1 + u_2 = 2$ ($k = 2$), $u_4 = u_2 + u_3 = 3$ ($k = 3$), etc.

**Proposition 1** For any $n \in \mathbb{Z}^+$, the $5n^{th}$ Fibonacci number $u_{5n}$ is divisible by 5.

**Proof.** We prove this by induction.

**Base case.** $u_5 = u_3 + u_4 = 5$, which is divisible by 5.

**Inductive step.** Suppose that $k \in \mathbb{Z}^+$ is such that $u_{5k}$ is divisible by 5 (inductive hypothesis). Then

$$u_{5(k+1)} = u_{5k+5}$$
$$= u_{5k+3} + u_{5k+4}$$
$$= u_{5k+2} + 2u_{5k+3}$$
$$= 2u_{5k+1} + 3u_{5k+2}$$
$$= 3u_{5k} + 5u_{5k+1}. $$

Since $3u_{5k}$ is divisible by 5 ($u_{5k}$ is divisible by 5) and since $5u_{5k+1}$ is divisible by 5, we conclude that their sum $u_{5(k+1)}$ is divisible by 5 (inductive conclusion).

**Conclusion.** We are done by mathematical induction. □

**Commentary:** Let $P(n)$ be the statement ‘$u_{5n}$ is divisible by 5’. We first observed that $P(1)$ is true (the base case). Then, assuming that $P(k)$ is true for some $k$, we proved that then $P(k + 1)$ is true (inductive step). From mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}$. 