Injections

Definition 9.1.1. We say that a function \( f : X \to Y \) is an injection if

\[
\text{If } x_1, x_2 \in X \text{ are such that } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).
\]

I.e., different inputs have different outputs. The (equivalent) contrapositive way to say this is:

\[
\text{If } x_1, x_2 \in X \text{ are such that } f(x_1) = f(x_2), \text{ then } x_1 = x_2.
\]

Exercise 9.4. Suppose that \( f : X \to Y \) and \( g : Y \to Z \) are injections. Prove that \( g \circ f : X \to Z \) is an injection.

Solution. Let \( x_1, x_2 \in X \) be such that \((g \circ f)(x_1) = (g \circ f)(x_2)\). Then \( g(f(x_1)) = g(f(x_2))\). Since \( g \) is an injection, this implies that \( f(x_1) = f(x_2)\). Now, since \( f \) is an injection, this implies \( x_1 = x_2 \). We have proved that \( g \circ f : X \to Z \) is an injection. \( \square \)

The following discussion about a ‘universal property’ of injections is more advanced. It is not necessary to understand this for Problem 19, but it is related.

‘Dual’ to Problem 19 on p. 118. Let \( f : X \to Y \) be a function. Prove that there exists a function \( g : Y \to X \) such that \( g \circ f = I_X \) if and only if \( f \) is an injection. (\( g \) is called a left inverse of \( f \).)

Solution. This statement is an ‘if and only if’ statement, i.e., a ‘\( \iff \)’ statement.

(Proof of \( \Rightarrow \)). Suppose there exists a function \( f : Y \to X \) such that \( g \circ f = I_X \). Suppose \( x_1, x_2 \in X \) are such that \( f(x_1) = f(x_2) \). Then \( g(f(x_1)) = g(f(x_2)) \) (just because \( g \) is a function). Since \( g \circ f = I_X \), this says that \( x_1 = x_2 \).

(Proof of \( \Leftarrow \)). Suppose \( f \) is an injection. We need to define \( g \). Let \( y \in Y \).

Case 1. \( y \in \text{Im } f \). Since \( f \) is an injection, there exists a unique \( x \in X \) such that \( f(x) = y \). Define \( g(y) = x \).

Details: (i) Existence. Such an \( x \) exists by the definition of \( y \in \text{Im } f \).
(ii) Uniqueness. Suppose \( x' \in X \) is also such that \( f(x') = y \). Then \( f(x') = f(x) \) (since \( f \) is an injection, \( x' = x \).

Case 2. \( y \notin \text{Im } f \). Choose any \( x \in X \) and define \( g(y) = x \).

Now that we have defined \( g \), we prove \( g \circ f = I_X \).

Let \( x \in X \). Then since \( f(x) \in \text{Im } f \), we have \( g(f(x)) = x \). This proves \( g \circ f = I_X \). \( \square \)