
(a) Prove by induction on \( n \) that, for all integers \( n \geq 2 \),

\[
1^2 + 3^2 + \cdots + (2n - 3)^2 = \frac{(2n - 1)(2n - 2)(2n - 3)}{6}.
\]

**Ans:** Base case: \( n = 2 \):

\[
1^2 = 1 = \frac{3 \cdot 2 \cdot 1}{6} = \frac{(4 - 1)(4 - 2)(4 - 3)}{6}.
\]

**Inductive step:** Suppose

\[
1^2 + 3^2 + \cdots + (2n - 3)^2 = \frac{(2n - 1)(2n - 2)(2n - 3)}{6}
\]

for some \( n \geq 2 \). Then

\[
1^2 + 3^2 + \cdots + (2n - 3)^2 + (2(n + 1) - 3)^2
= \frac{(2n - 1)(2n - 2)(2n - 3)}{6} + (2n - 1)^2
= \frac{(2n - 1)}{6}((2n - 2)(2n - 3) + 6(2n - 1)).
\]

We compute

\[
(2n - 2)(2n - 3) + 6(2n - 1) = 4n^2 - 10n + 6 + 12n - 6
= 4n^2 + 2n
= .
\]

Hence

\[
1^2 + 3^2 + \cdots + (2n - 3)^2 + (2(n + 1) - 3)^2
= \frac{(2n - 1)2n(2n + 1)}{6}
= \frac{(2(n + 1) - 1)(2(n + 1) - 2)(2(n + 1) - 3)}{6}.
\]

By Mathematical Induction, we are done.

13. Quantifiers.

(a) Prove that for each negative real number \( x \) there exists a negative real number \( y \) such that \( xy - y > 100 \).

**Ans:** Let \( x < 0 \). The inequality is equivalent to

\[
y(x - 1) > 100
\]
Since \( x - 1 < 0 \), this is equivalent to
\[
y < \frac{100}{x - 1}.
\]
So define
\[
y = \frac{100}{x - 1} - 1
\]
to get
\[
y < \frac{100}{x - 1} \quad \text{and} \quad y < 0.
\]
(b) Prove that for each negative real number \( x \), there does not exist a real number \( y \) such that \( y^2 x > 101 \).

**Ans:** Let \( x < 0 \) and \( y \in \mathbb{R} \). Since \( x < 0 \) and \( y^2 \geq 0 \), we have \( y^2 x \leq 0 < 101 \). Hence \( y^2 x \not> 101 \).

14. Let \( X \) and \( Y \) be disjoint sets.

(a) Prove that: If \( C \subseteq Y \), then \( C \cap X = \emptyset \).

**Ans:** Since \( C \subseteq Y \) and \( Y \cap X = \emptyset \), we have \( C \cap X \subseteq Y \cap X = \emptyset \). Since the only subset of the empty set is the empty set, we conclude that \( C \cap X = \emptyset \).

(b) Prove that: If \( B \subseteq X \) and \( C \subseteq Y \), then \((B \cup C) \cap X = B \).

**Ans:** Using (a), we have
\[
(B \cup C) \cap X = (B \cap X) \cup (C \cap X) = (B \cap X) \cup \emptyset = B \cap X = B,
\]
with the last equality since \( B \subseteq X \).

(c) Define the map \( f : \mathcal{P}(X \cup Y) \to \mathcal{P}(X) \times \mathcal{P}(Y) \) by \( f(A) = (A \cap X, A \cap Y) \). Prove that \( f \) is surjective (i.e., onto).

**Ans:** Let \((B, C) \in \mathcal{P}(X) \times \mathcal{P}(Y) \). Then \( B \subseteq X \) and \( C \subseteq Y \). Hence \( B \cup C \subseteq X \cup Y \), i.e., \( B \cup C \in \mathcal{P}(X \cup Y) \). We have
\[
f(B \cup C) = ((B \cup C) \cap X, (B \cup C) \cap Y) = (B, C)
\]
by (b) and the analogous fact: If \( B \subseteq X \) and \( C \subseteq Y \), then \((B \cup C) \cap Y = C \). Hence \( f \) is surjective.

16. **Complete the following direct proof** (this is essentially a special case of the Division Theorem, so you are not allowed to use the Division Theorem).

**Proposition.** For any \( a \in \mathbb{Z}^+ \) there exist integers \( q \) and \( r \) such that
\[
a = 23q + r \quad \text{and} \quad 23 \leq r < 46.
\]

**Proof.** Define the set
\[
S = \{ a - 23\bar{q} \mid \bar{q} \in \mathbb{Z} \text{ and } a - 23\bar{q} \geq 23 \}.
\]
By the well-ordering principle, \( S \) contains a minimum (i.e., least) element.

**Finish the proof in the space below.**
Ans: The above facts at the beginning of the proof are assumed (you do not need to prove them). Let $a - 23q$ be the least element of $S$ and define $r \div a - 23q$. Then

$$a = 23q + r.$$ 

Since $r \in S$, by the definition of $S$ we have

$$r \geq 23.$$ 

Since $r$ is the least element of $S$, $r - 23 = a - 23(q + 1) \notin S$, Hence, by the definition of $S$,

$$a - 23(q + 1) < 23.$$ 

This implies $a - 23q - 23 < 23$, so that $r = a - 23q < 46$. We are done.

18. Suppose $a, b, c \in \mathbb{Z}^+$ and $m, n \in \mathbb{Z}$ are such that $am + bn = 2$ and $a$ is even.

(a) Prove that: If $a$ divides $bc$, then $\frac{a}{2}$ divides $c$.

**Ans:** Multiply $am + bn = 2$ by $c$ to get

$$acm + bcn = 2c.$$ 

Since $a$ divides $bc$, we have $a$ divides $bcn$. Of course, $a$ divides $acm$. So $a$ divides the sum $acm + bcn = 2c$. Since $a$ divides $2c$, we conclude that $\frac{a}{2}$ divides $c$.

(b) Is the statement in part (a) still true if we assume that $am + bn = 4$ instead of $am + bn = 2$ (with the remaining hypotheses the same)?

**If so,** explain why. **If not,** give a counterexample.

**Ans:** Suppose $am + bn = 4$, $a$ is even, and $a$ divides $bc$. The argument no longer works, so we look for a counterexample. Let $a = 4$, $b = 4$, and $c = 1$. To get $am + bn = 4$ we choose $m = 2$ and $n = -1$. Clearly $a = 4$ divides $bc = 4$. But $\frac{a}{2} = 2$ does not divide $c = 1$. 

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