Theorem 5.1 (p. 290). The Fundamental Theorem of Calculus (FTC). If \( f \) is continuous on the interval \([a, b]\) and \( f(t) = F'(t) \), then

\[
\int_{a}^{b} f(t) dt = F(b) - F(a).
\]

If \( F(t) \) is the position function of an object, then \( f(t) = F'(t) \) is its velocity. The Fundamental Theorem of Calculus says that over a time interval,

The change in position equals the integral of the velocity function.

In other words, the Total Change equals the Integral of the (instantaneous) Rate of Changes.

Recall that in §5.2 we defined \( \Delta t = \frac{b-a}{\Delta y} \),

\[
t_0 = a,
\]

\[
t_1 = a + \Delta t,
\]

\[
t_2 = a + 2\Delta t,
\]

\[
\vdots
\]

\[
t_n = a + n\Delta t = b,
\]

and the left-hand Riemann sum is

\[
f(t_0) \Delta t + \cdots + f(t_{n-1}) \Delta t.
\]

We have

\[
f(t_0) \Delta t = \text{velocity at } t_0 \text{ times time lapse } \Delta t = t_1 - t_0
\]

\[
= \text{estimated change in position from time } t_0 \text{ to time } t_1
\]

\[
\approx F(t_1) - F(t_0).
\]

And

\[
f(t_0) \Delta t + f(t_1) \Delta t = \text{estimated change in position from time } t_0 \text{ to time } t_1
\]

\[
+ \text{estimated change in position from time } t_1 \text{ to time } t_2
\]

\[
\approx (F(t_1) - F(t_0)) + (F(t_2) - F(t_1))
\]

\[
= F(t_2) - F(t_0).
\]

Continuing this way, we get

\[
f(t_0) \Delta t + \cdots + f(t_{n-1}) \Delta t \approx F(t_{n-1}) - F(t_0) \approx F(b) - F(a)
\]

since \( t_0 = a \) and \( t_{n-1} = b - \Delta t \approx b \) and \( f \) is continuous. This is a heuristic proof of the Fundamental Theorem of Calculus.

Units \( \int_{a}^{b} f(t) dt \):

1. If \( f(t) \) is distance in feet and \( t \) is time in seconds, then the units of \( \int_{a}^{b} f(t) dt \) are feet \( \cdot \) seconds.
2. If \( f(t) \) is velocity in feet per second and \( t \) is time in seconds, then the units of \( \int_a^b f(t)\,dt \) are feet (like distance).

3. If \( f(t) \) is acceleration in feet per second\(^2\) and \( t \) is time in seconds, then the units of \( \int_a^b f(t)\,dt \) are feet per second (like velocity).

**Applying the FTC: #11:** Here \( F(t) = \ln t \) has derivative \( f(t) = F'(t) = \frac{1}{t} \). By the FTC, we have:

\[
\int_1^5 \frac{1}{t}\,dt = \ln 5 - \ln 1 = [\ln 5] \approx 1.609.
\]

Often, you need to use the Chain Rule to differentiate, such as in #18.

#16. When the velocity is positive, the distance traveled is the definite integral of the velocity.