Math 10B  Chapter 6. Section 2: Constructing Antiderivatives Analytically

The Mean Value Theorem says that at some time the instantaneous velocity is equal to the average velocity. A simple but important application is the following (see p. 326):

*If $F'(x) = 0$ on an interval, then $F(x) = C$ on this interval, for some constant $C$.***

That is, if the velocity of an object is zero for some period of time, then the object doesn’t move during that time.

Equivalently, the antiderivatives of the zero function are exactly the constant functions.

**Example 1.** Quentin is lazy and has velocity zero from dusk to dawn. Therefore Quentin is stationary from dusk to dawn. **Conclusion:** Quentin will have trouble avoiding vampires.

Nomenclature and notation:
The general antiderivative of a function $f(x)$ is called its **indefinite integral** and denoted by

$$\int f(x) \, dx.$$ 

So if $F(x)$ is a particular antiderivative of $f(x)$, that is $F'(x) = f(x)$, then

$$\int f(x) \, dx = F(x) + C.$$ 

**Example 2.** Some basic antiderivatives are (all given in the book; see pp. 327–328):

- $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$ for $n \neq -1$,
- $\int \frac{1}{x} \, dx = \ln |x| + C$ defined for $x \neq 0$,
- $\int e^x \, dx = e^x + C$,
- $\int \cos x \, dx = \sin x + C$,
- $\int \sin x \, dx = -\cos x + C$.

A key property of the antiderivative, i.e., the indefinite integral, is its linearity as an operator: For functions $f$ and $g$ and a constant $c$,

$$\int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx,$$

$$\int cf(x) \, dx = c \int f(x) \, dx.$$ 

**Example 3.** Find the general antiderivative of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{x}$.
**Answer:** $f(x) = x^{1/2} + x^{-1/2} + \frac{1}{x}$, so its antiderivative is

$$F(x) = \frac{2}{3} x^{3/2} + 2x^{1/2} + \ln |x| + C.$$
Example 4. Find the general antiderivative of \( g(t) = \pi e + \sin \sqrt{2} + \ln 100 \).

Answer: \( g(t) \) is a constant, so the answer is

\[
G(t) = \left( \pi e + \sin \sqrt{2} + \ln 100 \right) t + C.
\]

Example 5 (digression). Define

\[
f(x) = \begin{cases} 
1 & \text{if } x > 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]

Answer: (1) For \( x > 0 \), \( F(x) = x + C_1 \) is the general antiderivative.
(2) For \( x < 0 \), \( F(x) = -x + C_2 \) is the general antiderivative.

Pop Quiz (further digression): The moon is made of green cheese.


(a) True.
(b) False.
(c) None of the above.
(d) None of the above.

Logically and without knowing the question, could (d) be the correct answer?

Example 6. Suppose \( F'(t) = t^2 - 2 \) and \( F(1) = -1 \). Find \( F(t) \).

Answer: That is, we want to find the (unique) antiderivative \( F(t) \) of the function \( t^2 - 2 \) which satisfies \( F(1) = -1 \). The general antiderivative of \( t^2 - 2 \) is

\[
F(t) = \frac{1}{3} t^3 - 2t + C.
\]

We find the constant \( C \) by seeing what the condition \( F(1) = -1 \) says:

\[
-1 = F(1) = \frac{1}{3} \cdot 1^3 - 2 \cdot 1 + C = -\frac{5}{3} + C.
\]

Thus \( C = \frac{2}{3} \), so that

\[
F(t) = \frac{1}{3} t^3 - 2t + \frac{2}{3}.
\]

The graph of \( F'(t) = t^2 - 2 \) is in green and the graph of \( F(t) = \frac{1}{3} t^3 - 2t + \frac{2}{3} \) is in red: