Math 10B  Section 9.2


Consider the infinite series

\[ 5 - 10 + 20 - 40 + 80 - . \]

Observe that there is a common ratio of a term divided by the previous term:

\[ \frac{-2}{5} = \frac{-10}{20} = \frac{-40}{80} = \ldots. \]

So we call such an infinite series a geometric series. Because of this, we can rewrite the infinite series as:

\[ 5 + 5 (-2) + 5 (-2)^2 + 5 (-2)^3 + 5 (-2)^4 + \cdots \]

and by the distributive law, this is

\[ 5 \left( 1 + (-2) + (-2)^2 + (-2)^3 + (-2)^4 + \cdots \right). \]

The general form of an infinite geometric series is

\[ a + ax + ax^2 + ax^3 + \cdots = a \left( 1 + x + x^2 + x^3 + \cdots \right). \]

In the example, \( a = 5 \) and \( x = -2. \)

Let \( n \) be a positive integer. We have the general formula for a finite geometric series (p. 500):

\[ a + ax + ax^2 + \cdots + ax^{n-1} = \frac{a(1 - x^n)}{1 - x}. \]

If \( |x| < 1 \), then \( \lim_{n \to \infty} x^n = 0 \) and the sum limits to a number. We say that the series converges. Namely, the sum of the infinite geometric series is

\[ a + ax + ax^2 + \cdots + ax^{n-1} + ax^n + \cdots = \frac{a}{1 - x}. \]

On the other hand, if \( |x| \geq 1 \), then the series does not converge. For example,

1. If \( x = 1 \), then we get

\[ a + a + a + \cdots, \]

which keeps on getting larger in magnitude (the sum of the first \( n \) terms is \( na \)).

2. If \( x = -1 \), then we get

\[ a - a + a - a + a - \cdots. \]

If \( n = 1, 3, 5, \ldots \) (odd), then the sum of the first \( n \) terms is \( a \). On the other hand, if \( n = 2, 4, 6, \ldots \) (even), then the sum of the first \( n \) terms is 0. So the finite sums oscillate between \( a \) and 0, and hence the infinite series does not converge.

3. When \( |x| > 1 \), the terms actually grow: \( ax^n \) gets larger and larger in magnitude. Hence the infinite series does not converge.
Example 1. Consider the infinite geometric series

\[ 5 + \frac{5}{-2} + \frac{5}{(-2)^2} + \frac{5}{(-2)^3} + \frac{5}{(-2)^4} + \cdots. \]

Here, \( a = 5 \) and \( x = -\frac{1}{2} \). So this series sums (converges) to

\[ \frac{5}{1 - (-\frac{1}{2})} = \frac{5}{\frac{3}{2}} = \frac{10}{3}. \]

Example 2. On the other hand, the infinite series

\[ 5 + 5(-2) + 5(-2)^2 + 5(-2)^3 + 5(-2)^4 + \cdots \]

does not converge.

Example 3. For what values of \( z \) does the series

\[ 2 - 4z + 8z^2 - 16z^3 + \cdots \]

converge?

We rewrite the series as:

\[ 2 + 2(-2z) + 2(-2z)^2 + 2(-2z)^3 + \cdots. \]

So \( a = 2 \) and \( x = -2z \). The series converges when \( |x| < 1 \), that is, when \( 1 > |-2z| = 2|z| \), that is, \( |z| < \frac{1}{2} \).

The series diverges when \( |x| \geq 1 \), that is, when \( 1 \leq |-2z| = 2|z| \), that is, \( |z| \geq \frac{1}{2} \).