1. The graph of $f$ is given on the left and the graph of $g$ is given on the right. By areas of triangles and rectangles,
\[
\int_0^3 f(x) \, dx = \frac{1}{2} \cdot 1 \cdot 2 + 1 \cdot 2 = 3
\]
and by the information given ($g(x) = 0$ for $1 \leq x \leq 2$ from the graph)
\[
\int_0^3 g(x) \, dx = \int_0^1 g(x) \, dx + \int_1^2 g(x) \, dx + \int_2^3 g(x) \, dx = 0 + 0 + \left( -\frac{2}{3} \right) = -\frac{2}{3}.
\]
So
\[
\int_0^3 (2f(x) + 3g(x)) \, dx = 2 \int_0^3 f(x) \, dx + 3 \int_0^3 g(x) \, dx
\]
\[
= 2 \cdot 3 + 3 \left( -\frac{2}{3} \right) = 4.
\]
(b) Since $f$ is an odd function, we have
\[
\int_{-2}^0 f(x) \, dx = -\int_0^2 f(x) \, dx = -1.
\]
(c) By the Fundamental Theorem of Calculus and $G'(x) = g(x)$,
\[
G(3) = G(0) + \int_0^3 G'(x) \, dx
\]
\[
= G(0) + \int_0^3 g(x) \, dx
\]
\[
= 1 + \left( -\frac{2}{3} \right) = \frac{1}{3}.
\]
2. The hint was the algebra simplification $\frac{ax^2 + bx + c}{x} = ax + b + \frac{c}{x}$. In particular
\[
\int \frac{x^2 + 3x + 1}{x} \, dx = \int \left( x + 3 + \frac{1}{x} \right) \, dx = \frac{1}{2}x^2 + 3x + \ln |x| + C.
\]
3. The hint was that FTC and Chain Rule tell us that
\[
\frac{d}{dx} \int_a^{g(x)} f(t) \, dt = f(g(x)) \cdot g'(x).
\]
In particular,
\[
\frac{d}{dx} \int_3^{\sin x} e^{\sqrt{t}} \, dt = e^{\sqrt{\sin x}} \cos x.
\]
4. The hint was Integration By Parts with \( u = \ln x \) and \( dv = x^3 dx \). Then

\[
du = \frac{1}{x} dx \quad \text{and} \quad v = \frac{1}{4} x^4.
\]

So

\[
\int x^3 \ln x dx = \int u dv = uv - \int v du = \frac{1}{4} x^4 \ln x - \int \frac{1}{4} x^4 \cdot \frac{1}{x} dx.
\]

Now

\[
\int \frac{1}{4} x^4 \cdot \frac{1}{x} dx = \int \frac{1}{4} x^3 dx = \frac{1}{16} x^4 + C
\]

So the answer is:

\[
\int x^3 \ln x dx = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C.
\]

5. The hint is \( u \)-substitution \( u = 4 + x^2 \). This works because of the numerator \( x \). We have

\[
du = 2xdx, \quad \text{so} \quad \frac{1}{2} du = xdx
\]

and hence \((x = 0 \text{ implies } u = 4 \text{ and } x = 1 \text{ implies } u = 5)\)

\[
\int_0^1 \frac{x}{\sqrt{4 + x^2}} dx = \int_4^5 \frac{1}{2\sqrt{u}} du = \sqrt{u}\bigg|_4^5 = \sqrt{5} - 2.
\]

6. The hint is

\[
\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.
\]

We put the right-side over the same denominator as the left, so

\[
\frac{x}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2},
\]

so that

\[
x = A(x+1)^2 + B(x-1)(x+1) + C(x-1).
\]

Taking \( x = 1 \), we get

\[
1 = A \cdot 2^2 + B \cdot 0 \cdot 2 + C \cdot 0 = 4A, \quad \text{so} \quad A = \frac{1}{4}.
\]

Taking \( x = -1 \), we get

\[
-1 = A \cdot 0^2 + B (-2) \cdot 0 + C (-2), \quad \text{so} \quad C = \frac{1}{2}.
\]

We choose another value (any value will work), but \( x = 0 \) is simplest, which yields

\[
0 = A \cdot 1^2 + B (-1) \cdot 1 + C (-1)
\]

\[
= A - B - C
\]

\[
= \frac{1}{4} - B - \frac{1}{2}.
\]
so

\[ B = -\frac{1}{4}. \]

We conclude that

\[ \frac{x}{(x - 1)(x + 1)^2} = \frac{1}{4(x - 1)} - \frac{1}{4(x + 1)} + \frac{1}{2(x + 1)^2}. \]

Therefore

\[
\int \frac{x}{(x - 1)(x + 1)^2} dx = \int \frac{1}{4(x - 1)} dx - \int \frac{1}{4(x + 1)} dx + \int \frac{1}{2(x + 1)^2} dx
\]

\[ = \frac{1}{4} \ln |x - 1| - \frac{1}{4} \ln |x + 1| - \frac{1}{2(x + 1)} + C. \]

7.

(a) We have

\[
\int_{2}^{5} \frac{1}{\sqrt{x - 2}} dx = \lim_{a \to 2^{+}} \int_{a}^{5} \frac{1}{\sqrt{x - 2}} dx
\]

and

\[
\int_{a}^{5} \frac{1}{\sqrt{x - 2}} dx = 2\sqrt{x - 2}\bigg|_{a}^{5} = 2\sqrt{3} - 2\sqrt{a - 2}.
\]

Thus

\[
\int_{2}^{5} \frac{1}{\sqrt{x - 2}} dx = \lim_{a \to 2^{+}} \left( 2\sqrt{3} - 2\sqrt{a - 2} \right)
\]

\[ = 2\sqrt{3} - 2 \lim_{a \to 2^{+}} \sqrt{a - 2}
\]

\[ = 2\sqrt{3} - 2 \cdot 0
\]

\[ = 2\sqrt{3}. \]

(b) Since \( \ln x \leq x \) for \( x \geq 1 \) and since \( x^3 + 1 > 0 \) for \( x \geq 1 \), we have

\[ 0 \leq \frac{\ln x}{x^3 + 1} \leq \frac{x}{x^3 + 1} \leq \frac{1}{x^2} \]

for \( x \geq 1 \). We have that the improper integral

\[ \int_{1}^{\infty} \frac{1}{x^2} dx \]

converges.

Hence, by the Comparison Test for Improper Integrals, we get that

\[ \int_{1}^{\infty} \frac{\ln x}{x^3 + 1} dx \]

converges.
8. The volume of the solid is

\[ V = \int_{0}^{\frac{\pi}{4}} \pi y^2 \, dx = \int_{0}^{\frac{\pi}{4}} \pi \tan x \sec^2 x \, dx = \pi \int_{0}^{\frac{\pi}{4}} \sec x \cdot \sec x \tan x \, dx. \]

Make the \( u \)-substitution with \( u = \sec x \). Then \( du = \sec x \tan x \, dx \). Since \( \sec 0 = 1 \) and \( \sec(\pi/4) = \sqrt{2} \), we have

\[ V = \pi \int_{1}^{\sqrt{2}} u \, du = \frac{\pi}{2} u^2 \bigg|_{1}^{\sqrt{2}} = \frac{\pi}{2} (2 - 1) = \frac{\pi}{2}. \]

9. Under the hypothesis that \( y = e^{kx} \) we have \( \frac{dy}{dx} = ke^{kx} \) and \( \frac{d^2y}{dx^2} = k^2e^{kx} \). Plugging this into the equation \( \frac{d^2y}{dx^2} = 4y \), we get

\[ k^2e^{kx} = 4e^{kx}. \]

Since \( e^{kx} > 0 \) (for all \( x \)), this is equivalent to \( k^2 = 4 \), that is \( k = \pm 2 \).

10. Solve the initial value problem:

\[ \frac{dy}{dt} = -t(y - 2), \quad \text{with } y(0) = 5. \]

Separating variables yields

\[ \int \frac{dy}{y - 2} = \int -tdt, \]

which gives (note \( y - 2 > 0 \))

\[ \ln (y - 2) = -\frac{t^2}{2} + C. \]

The initial condition \( y(0) = 5 \) implies \( \ln 3 = C \).

So

\[ \ln (y - 2) = -\frac{t^2}{2} + \ln 3 \]

and exponentiating:

\[ y - 2 = e^{\ln(y - 2)} = e^{-\frac{t^2}{2} + \ln 3} = e^{\ln 3}e^{-\frac{t^2}{2}} = 3e^{-\frac{t^2}{2}}. \]

We conclude that

\[ y = 3e^{-\frac{t^2}{2}} + 2. \]

If you are unsure of your answer, you can check the initial condition:

\[ y(0) = 3e^{0^2} + 2 = 3e^0 + 2 = 5 \]

and the equation:

\[ \frac{dy}{dt} = -3te^{-\frac{t^2}{2}} = -t \left(3e^{-\frac{t^2}{2}}\right) = -t(y - 2). \]