Homework assignment 5, due in class on Wednesday February 11

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(1) Let \( K_n = (0, \frac{1}{n}) \). Each \( K_n \) is bounded (but not closed). We have that the intersection of any finite subcollection of the collection \( \{K_n\}_{n=1}^{\infty} \) is nonempty. Indeed, \( K_{n_1} \cap \cdots \cap K_{n_k} = K_m \neq \emptyset \), where \( m = \min\{n_1, \ldots, n_k\} \). However, \( \bigcap_{n=1}^{\infty} K_n = \emptyset \). To prove this last statement, let \( x \in \bigcap_{n=1}^{\infty} K_n \). Then \( x \in (0,1) = K_1 \). By the archimedean property, there exists \( n_x \in \mathbb{N} \) such that \( \frac{1}{n_x} < x \). This implies \( x \notin K_{n_x} \), which is a contradiction. Therefore \( \bigcap_{n=1}^{\infty} K_n = \emptyset \).

(2) Let \( K_n = [n, \infty) \). Each \( K_n \) is closed (but not bounded). We have that the intersection of any finite subcollection of the collection \( \{K_n\}_{n=1}^{\infty} \) is nonempty. Indeed, \( K_{n_1} \cap \cdots \cap K_{n_k} = K_M \neq \emptyset \), where \( M = \max\{n_1, \ldots, n_k\} \). However, \( \bigcap_{n=1}^{\infty} K_n = \emptyset \). To prove this last statement, let \( x \in \bigcap_{n=1}^{\infty} K_n \). Since \( x \in \mathbb{R} \), by the archimedean property there exists \( n_x \in \mathbb{N} \) such that \( n_x > x \). This implies \( x \notin K_{n_x} \), which is a contradiction. Therefore \( \bigcap_{n=1}^{\infty} K_n = \emptyset \). \( \Box \)

Page 44, #19. Two subsets \( A \) and \( B \) of a metric space \( X \) are said to be separated if both \( A \cap \overline{B} \) and \( \overline{A} \cap B \) are empty.

(a) Let \( A \) and \( B \) be disjoint closed sets in \( X \). Then \( \overline{A} = A \) and \( \overline{B} = B \). Thus
\[
A \cap \overline{B} = A \cap B = \emptyset, \\
\overline{A} \cap B = A \cap B = \emptyset.
\]

Hence \( A \) and \( B \) are separated.

(b) Let \( A \) and \( B \) be disjoint open sets in \( X \). Then \( A \subset B^c \). Since \( B^c \) is closed, we have \( \overline{A} \subset B^c \) by Theorem 2.27(c). Similarly, one proves that \( \overline{B} \subset A^c \). We conclude that
\[
\overline{A} \cap B = \emptyset, \\
A \cap \overline{B} = \emptyset.
\]

Hence \( A \) and \( B \) are separated.

(c) We have \( A = \{ q \in X : d(q, p) < \delta \} = N_{\delta}(p) \) and \( B = \{ q \in X : d(q, p) > \delta \} \). Then \( A \) and \( B \) are disjoint open sets in \( X \). That \( B \) is open was proved in earlier notes. Hence we may apply (b) to conclude that \( A \) and \( B \) are separated.

(d) Choose \( p \in X \) and \( q \in X \) with \( p \neq q \). Then \( r = d(p, q) > 0 \). Let \( \delta \in (0, r) \) and define \( A \) and \( B \) as in part (c). Then \( A \) and \( B \) are separated. \( A \) and \( B \) are also nonempty since \( p \in A \) and \( q \in B \). Since \( X \) is connected, \( A \cup B \neq X \). Thus there exists \( x_\delta \in X \) with \( x_\delta \notin A \) and \( x_\delta \notin B \). This implies that \( d(x_\delta, p) = \delta \). We have proved that for each \( \delta \in (0, r) \) there exists \( x_\delta \in X \) with \( d(x_\delta, p) = \delta \). Moreover, since we choose only one point for each \( \delta \in (0, r) \), we have the property that if \( x_\delta = x_\delta' \), then \( \delta = \delta' \). This implies that since the set of real numbers \( \delta \in (0, r) \) is uncountable, the set of \( x_\delta \in X \) is uncountable. Hence \( X \) is uncountable. \( \Box \)

HW5.1. By following the proof of Theorem 2.40 in the book, prove that every closed interval \( I = [a, b] \subset \mathbb{R} \), where \( a < b \), is compact. In particular, you may start with the following (or you can start your own way):

Let \( \delta = |b - a| > 0 \). Suppose there exists an open cover \( \{G_\alpha\}_{\alpha \in A} \) of \( I \) which contains no finite subcover. Let \( c = \frac{a+b}{2} \), which is the midpoint of the \( I \). We have \( I = [a, c] \cup [c, b] \). Then, for \( I_1 = [a, c] \) or \( I_1 = [c, b] \), no finite subcollection of \( \{G_\alpha\}_{\alpha \in A} \) covers \( I_1 \). Write \( I_1 = [a_1, b_1] \), where \( |b_1 - a_1| = \frac{\delta}{2} \). Let \( c_1 = \frac{a_1 + b_1}{2} \). Then, for \( I_2 = [a_1, c_1] \) or \( I_2 = [c_1, b_1] \), no finite subcollection of \( \{G_\alpha\}_{\alpha \in A} \) covers \( I_2 = [a_2, b_2] \).

Since \( c_1 \) is the midpoint of \( I_1 \), we have \( |b_2 - a_2| = \frac{\delta}{2} \). Continuing in this way, we obtain a nested sequence of closed intervals \( \{I_n\}_{n=1}^{\infty} \), where \( I_n = [a_n, b_n] \) with \( |b_n - a_n| = 2^{-n} \delta \) and the following properties:

1. \( I \supset I_1 \supset I_2 \supset I_3 \supset \cdots \);
2. for each \( n \in \mathbb{N} \) we have that \( I_n \) is not covered by any finite subcollection of \( \{G_\alpha\}_{\alpha \in A} \).

Since each interval is compact, by Theorem 2.39 there exists \( x^* \in \bigcap_{n=1}^{\infty} I_n \). On the other hand, \( x^* \in G_\beta \) for some \( \beta \in A \). Since \( G_\beta \) is open, there exists \( r > 0 \) such that \( (x^* - r, x^* + r) \subset G_\beta \). Now choose \( n \in \mathbb{N} \) so that \( 2^{-n}\delta < r \) (this is equivalent to \( n > \frac{\ln \delta}{\ln 2} - \frac{\ln r}{\ln 2} \)). Since \( x^* \in I_n = [a_n, b_n] \) and since \( |b_n - a_n| = 2^{-n}\delta \), we
We conclude that \( I_n \subset G_\beta \). This contradicts property (2). We conclude that any open cover \( \{G_\alpha\}_{\alpha \in A} \) of \( I \) must contain a finite subcover. \( \square \)

**HW5.2.** Prove that if \( X \) is the disjoint union of two open sets, then \( X \) is not connected.

**Hint.** Read: http://www.math.ucsd.edu/~benchow/140A-connected.pdf

Let \( X = A \cup B \), where \( A \) and \( B \) are open and \( A \cap B = \emptyset \). Then \( A = \overline{B}^c \) and \( B = \overline{A}^c \), so that \( A \) and \( B \) are closed. Thus \( \emptyset \neq A \cap \overline{B} = \overline{B} \cap A = \overline{A} \cap B \), so that \( A \) and \( B \) are separated. We conclude that \( X \) is not connected. \( \square \)

**HW5.3.** Definition. We say that a subset \( K \subset X \) is **sequentially compact** if every sequence \( \{x_i\}_{i=1}^\infty \) of points in \( K \) has a subsequence that converges to a point \( x_\infty \) in \( K \).

Prove the following (see **Theorem 3.6(a)**, but write the proof in your own words):

If \( K \) is a compact set in a metric space \( X \), then \( K \) is sequentially compact.

Let \( \{x_i\}_{i=1}^\infty \) be a sequence of points in \( K \). Let \( E = \{x_i : i \in \mathbb{N}\} \) be the range of \( \{x_i\}_{i=1}^\infty \).

**Case 1.** \( E \) is finite. This implies there exists \( x_\infty \in E \) and \( i_1 < i_2 < \cdots < i_n \) such that \( x_{i_k} = x_\infty \) for \( k \in \mathbb{N} \). Clearly, \( x_{i_k} \to x_\infty \).

**Case 2.** \( E \) is infinite. Since \( K \) is compact, by Theorem 2.37 there exists a limit point \( x_\infty \in K \). Since \( x_\infty \) is a limit point of the set \( \{x_i : i \in \mathbb{N}\} \), we have the following. There exist \( i_1 \in \mathbb{N} \) such that \( x_{i_1} \in N_1(x) \).

Having chosen \( i_1, \ldots, i_{k-1} \), we can choose \( i_k > i_{k-1} \) such that \( x_{i_k} \in N_{i_k}(x) \). Because of this, \( x_{i_k} \to x_\infty \).

We have proved that every sequence \( \{x_i\}_{i=1}^\infty \) of points in \( K \) has a subsequence that converges to a point \( x_\infty \) in \( K \). \( \square \)

Remark: The proof uses **Theorem 2.37**. If \( E \) is an infinite subset of a compact set \( K \), then \( E \) has a limit point in \( K \).

**HW5.4.** Definition. A set \( T \subset X \) is **totally bounded** if for any \( \varepsilon > 0 \) there exists a finite number of points \( x_1, \ldots, x_k \) such that \( T \subset \bigcup_{i=1}^k N_\varepsilon(x_i) \).

Prove that: Every sequentially compact subset of a metric space is totally bounded.

**Hint.** See p. 27 (Lemma 11) of:

http://www.econ.brown.edu/fac/Mark_Dean/Maths_RA5_10.pdf

Let \( K \subset X \) be a sequentially compact subset. Suppose that \( K \) is not totally bounded. Then there exist \( \varepsilon > 0 \) such that there do NOT exist a finite number of points \( x_1, \ldots, x_k \) such that \( K \subset \bigcup_{i=1}^k N_\varepsilon(x_i) \). Let \( x_1 \in K \). Having chosen \( x_1, \ldots, x_{k-1} \), since \( K \not\subset \bigcup_{i=1}^{k-1} N_\varepsilon(x_i) \), there exists a point \( x_k \in K - \bigcup_{i=1}^{k-1} N_\varepsilon(x_i) \).

We have constructed an infinite sequence of distinct points \( \{x_i\}_{i=1}^\infty \) in \( K \) with this property. In particular, for each \( k, \ell \in \mathbb{N} \) with \( k > \ell \) we have \( d(x_k, x_\ell) \geq \varepsilon \). Indeed, this is true because \( x_k \in K - \bigcup_{i=1}^{k-1} N_\varepsilon(x_i) \) implies that \( x_k \notin N_\varepsilon(x_\ell) \) since \( \ell \leq k - 1 \). Any subsequence of \( \{x_k\}_{k=1}^\infty \) also has this property (\( d(x_k, x_j) \geq \varepsilon \) for all \( k < j \)), which implies that any subsequence is not Cauchy and hence does not converge (any convergent sequence is Cauchy by **Theorem 3.11(a)**). \( \square \)