Homework assignment 6, due Friday, 2/20.

p.78 #1. Prove that convergence of \( \{s_n\} \) implies convergence of \( \{|s_n|\} \). Is the converse true?

*Hint:* For the first part, use an exercise from Chapter 1.

p.78 #7. Prove that the convergence of \( \sum a_n \) implies the convergence of \( \sum \frac{\sqrt{a_n}}{n} \) if \( a_n \geq 0 \).

*Hint:* Observe that \( 0 \leq \sqrt{AB} \leq A + B \) for \( A, B \geq 0 \).

HW6.1. (a) Let \( \{x_n\} \) be a sequence in \( \mathbb{R} \) satisfying \( x_{n+1} - x_n \geq \frac{1}{n} \). Prove that \( x_n \to +\infty \).

(b) Let \( \{y_n\} \) be a sequence in \( \mathbb{R} \) satisfying \( y_{n+1} - y_n \leq -\frac{1}{n \ln n} \) for \( n \geq 2 \). Prove that \( y_n \to -\infty \).


HW6.3. (a) Let \( b \in (0, 1) \) and \( N \in \mathbb{N} \). Compute \( \sum_{n=N}^{\infty} b^n \).

(b) Let \( \{x_n\} \) be a sequence in \( \mathbb{R} \) satisfying \( |x_{n+1} - x_n| \leq b^n \) for \( n \geq N \), where \( b \in (0, 1) \) and \( N \in \mathbb{N} \). Prove that \( \{x_n\} \) converges.

*Hint:* For \( m > n \) we have \( x_m - x_n = \sum_{k=n}^{m-1} (x_{k+1} - x_k) \). You may also use the following fact: Given \( b \in (0, 1) \) and \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( \frac{b^N}{1-b} < \varepsilon \).

HW6.4. Define \( a^+ = \max \{a, 0\} \) and \( a^- = -\min \{a, 0\} = \max \{-a, 0\} \). Note that \( a^+, a^- \geq 0 \) and \( a = a^+ - a^- \).

Let \( \sum_{n=1}^{\infty} a_n \) be a series satisfying

(1) \( a_n \geq -2^{-n} \) for all \( n \geq 1 \).

(2) There exists \( C \in \mathbb{R} \) such that \( \sum_{n=1}^{N} a_n \leq C \) for all \( N \in \mathbb{N} \).

(a) Prove that \( \sum_{n=1}^{\infty} a_n^- \) converges. *Hint:* use (1).

(b) Prove that \( \sum_{n=1}^{\infty} a_n^+ \) converges. *Hint:* \( a_n^+ = a_n + a_n^- \).

(c) Prove that \( \sum_{n=1}^{\infty} |a_n| \) converges (that is, \( \sum_{n=1}^{\infty} a_n \) converges absolutely).