#1. Suppose \( \alpha \) increases on \([a, b]\), \(a \leq x_0 \leq b\), \( \alpha \) is continuous at \(x_0\), \( f(x_0) = 1 \), and \( f(x) = 0 \) if \( x \neq x_0 \). Prove that \( f \in \mathcal{R}(\alpha) \) and that \( \int f \, d\alpha = 0 \).

**Commentary.** One can start the proof as follows. Let \( \varepsilon > 0 \). Since \( \alpha \) is continuous at \( x_0 \) there exists \( \delta > 0 \) such that

\[
\text{If } |t - x_0| \leq \delta, \text{ then } |\alpha(t) - \alpha(x_0)| < \frac{\varepsilon}{2}.
\] (1)

Choose a partition \( P = \{t_0 \leq t_1 \leq t_2 \leq t_3\} \) such that ...

**Some things you may want to think about:**

1. Certainly \( t_0 = a \) and \( t_3 = b \). What properties should \( t_1 \) and \( t_2 \) satisfy? Its length should be less than what? Where’s \( x_0 \)? What about when \( x_0 \) is \( a \) or \( b \)?

2. It’s easy to show that \( L(P, f, \alpha) = 0 \). Why?

3. Show that \( U(P, f, \alpha) < \varepsilon \).

4. Why are we now done?

#2. Suppose \( f \geq 0 \), \( f \) is continuous on \([a, b]\), and \( \int_a^b f(x) \, dx = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [a, b] \). (Compare this with Exercise 1.)

**Commentary.** One can start the proof as follows. Suppose there exists \( x_0 \in [a, b] \) such that \( f(x_0) > 0 \). Since \( f \) is continuous, there exists \( \delta \in (0, \frac{b-a}{2}) \) such that if \( |t - x_0| \leq \delta \), then \( f(t) > \frac{1}{2} f(x_0) > 0 \). Choose a partition \( P = \{t_0 \leq t_1 \leq t_2 \leq t_3\} \) such that \( t_0 = a \), \( t_3 = b \), and \( [t_1, t_2] = [a, b] \cap [x_0 - \delta, x_0 + \delta] \).

**Some things you may want to think about:**

1. Why is \( t_2 > t_1 \)?

2. What is \( L(P, f) \)?

3. Why do we have a contradiction?

#4. Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
0 & \text{if } x \text{ is irrational}, \\
1 & \text{if } x \text{ is rational}.
\end{cases}
\]

Prove that \( f \notin \mathcal{R} \) on \([a, b]\) for any \( a < b \).

**Commentary.** Should be easy.

#5. Let \( f : [a, b] \to \mathbb{R} \) be a bounded function with \( f^2 \in \mathcal{R} \) on \([a, b]\), where \( a < b \).

(a) Does it follow that \( f \in \mathcal{R} \)?

(b) Does the answer change if we assume that \( f^3 \in \mathcal{R} \)?

**Commentary.** Cooke does a good job explaining this.
#7. Commentary. (a) One can start by: Let $c \in (0, 1]$. By Theorem 6.12(c) we have $f \in R$ on $[0, c]$ and
\[ \int_0^1 f(x) \, dx - \int_c^1 f(x) \, dx = \int_0^c f(x) \, dx. \]
So it suffices to show that $\lim_{c \to 0} \int_0^c f(x) \, dx = 0$.

(b) One can start by (this is a bit general): Let $\{a_n\}$ be a strictly decreasing sequence with $a_1 = 1$ and $a_n \to 0$ as $n \to \infty$. Define $f : (0, 1] \to \mathbb{R}$ by $f(x) = (-1)^n \frac{1}{n(a_n-a_{n+1})}$ for $x \in (a_{n+1}, a_n)$. Then $\int_{a_{n+1}}^{a_n} f(x) \, dx = (-1)^n \frac{1}{n}$ and $\int_{a_{n+1}}^{a_n} |f(x)| \, dx = \frac{1}{n}$.

Remark: If we choose say $a_n = \frac{1}{n}$, then
\[ \frac{1}{n(a_n-a_{n+1})} = \frac{1}{n\left(\frac{1}{n}-\frac{1}{n+1}\right)} = n + 1. \]
Compare with Cooke’s solution.

#8. Commentary. Loosely speaking, the infinite series can be both an upper sum and a lower sum for $\int_1^\infty f(x) \, dx$. 
