Math 140B  Chapter 6. Section 1: Definition and Existence of the Integral

Riemann Integration

A partition $P$ of $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$, where

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n = b.$$  

Let

$$\Delta x_i = x_i - x_{i-1} \quad \text{for } i = 1, 2, \ldots, n.$$  

Note that $\sum_{i=1}^n \Delta x_i = b - a$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Given a partition $P$, define ($M$ for max and $m$ for min)

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

Note that $M_i \geq m_i$ for each $i$. Define ($U$ for upper and $L$ for lower)

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i \quad L(P, f) = \sum_{i=1}^n m_i \Delta x_i.$$  

Note that $U(P, f) \geq L(P, f)$ for each $P$ and $f$.

Define the upper and lower Riemann integrals of $f$ over $[a, b]$ by

$$\int_a^b f(x) \, dx = \inf_P U(P, f) \quad \int_{-a}^b f(x) \, dx = \sup_P L(P, f),$$

respectively, where inf and sup are over all partitions $P$ of $[a, b]$. Note that $\int_a^b f(x) \, dx \geq \int_{-a}^b f(x) \, dx$.

Remark. Since $f$ is bounded, there exist $M$ and $m$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. It is easy to see that each $U(P, f) \leq M(b - a)$ and each $L(P, f) \geq m(b - a)$. So

$$M(b - a) \geq \int_a^b f(x) \, dx \quad \text{and} \quad \int_{-a}^b f(x) \, dx \geq m(b - a).$$

Definition 6.1. We say that $f$ is Riemann integrable on $[a, b]$ and we write $f \in \mathcal{R}$ if

$$\int_a^b f(x) \, dx = \int_{-a}^b f(x) \, dx$$

and we call this $\int_a^b f(x) \, dx$ or $\int_{-a}^b f(x) \, dx$.

A more general version of Riemann Integration
We can integrate a bounded function $f : [a, b] \rightarrow \mathbb{R}$ against something more general than $dx$. Let $\alpha : [a, b] \rightarrow \mathbb{R}$ be an increasing function. Note that $\alpha (a) \leq \alpha (x) \leq \alpha (b)$ for all $x \in [a, b]$, so that $\alpha$ is bounded. Let $P$ be a partition of $[a, b]$. Let

$$\Delta \alpha_i = \alpha (x_i) - \alpha (x_{i-1}).$$

Note that $\Delta \alpha_i \geq 0$ for each $i$ and $\sum_{i=1}^{n} \Delta \alpha_i = \alpha (b) - \alpha (a)$.

**Definition 6.2.** Define

$$U (P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i, \quad L (P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i.$$

Note that $U (P, f, \alpha) \geq L (P, f, \alpha)$ for each $P$, $f$, and $\alpha$.

Define the **upper and lower Riemann–Stieltjes integrals of $f$ with respect to $\alpha$ over $[a, b]$** by

$$\int_{a}^{b} f \, d\alpha = \inf_P U (P, f, \alpha), \quad \int_{a}^{b} f \, d\alpha = \sup_P L (P, f, \alpha).$$

respectively. Note that $\int_{a}^{b} f \, d\alpha \geq \int_{a}^{b} f \, d\alpha$.

We say that $f$ is **Riemann–Stieltjes integrable with respect to $\alpha$ on $[a, b]$** and we write $f \in \mathcal{R}(\alpha)$ if

$$\int_{a}^{b} f \, d\alpha = \int_{a}^{b} f \, d\alpha$$

and we call this $\int_{a}^{b} f \, d\alpha$ or $\int_{a}^{b} \alpha (x) \, d\alpha (x)$.

**Remark.** Definition 6.1 is the special case $\alpha (x) = x$ of Definition 6.2.

**Refinements are Better Sums**

A key idea to understand $\inf_P U (P, f, \alpha)$ is:

**Definition 6.3.** We say that $P^*$ is a **refinement of $P$** if $P \subset P^*$ as sets.

Given two partitions $P_1$ and $P_2$, their **common refinement** is $P_1 \cup P_2$, where the points are in strictly increasing order.

**Example.** $[a, b] = [0, 4]$ and $P_1 = \{0, 1, 2, 4\}$ and $P_2 = \{0, 2, 3, 4\}$. Then $P_1 \cup P_2 = \{0, 1, 2, 3, 4\}$.

**Theorem 6.4.** Let $P^*$ be a refinement of $P$. Then

(1) \quad $L (P^*, f, \alpha) \geq L (P, f, \alpha)$

(2) \quad $U (P^*, f, \alpha) \leq U (P, f, \alpha)$.

An inequality that might seem obvious, but isn’t because of the problem of comparing two different partitions, is obtained by using the common refinement:

**Theorem 6.5.**

$$\int_{a}^{b} f \, d\alpha \geq \int_{a}^{b} f \, d\alpha.$$
and in particular,
\[\int_a^b f\,dx \geq \int_{-a}^{-b} f\,dx.\]

**Proof.** Let \(P_1\) and \(P_2\) be partitions of \([a, b]\). Then by Theorem 6.4,
\[L(P_1, f, \alpha) \leq L(P_1 \cup P_2, f, \alpha) \leq U(P_1 \cup P_2, f, \alpha) \leq U(P_2, f, \alpha).\]
Fixing \(P_2\) and taking the sup over all \(P_1\), we have
\[\int_{-a}^{-b} f\,dx = \sup_{P_1} \left( L(P_1, f, \alpha) \right) \leq U(P_2, f, \alpha).\]
Next we take the inf over all \(P_2\) to get
\[\int_{-a}^{-b} f\,dx \leq \inf_{P_2} \left( U(P_2, f, \alpha) \right) = \int_{-a}^{-b} f\,dx. \square\]

**Cauchy Criterion for Integrability**

A useful criterion for integrability is:

**Theorem 6.6.** \(f \in \mathcal{R}(\alpha)\) on \([a, b]\) if and only if for every \(\varepsilon > 0\) there exists a partition \(P\) such that
\[U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.\]

**Proof.** (\(\Leftarrow\)) Let \(P\) be a partition. Then
\[0 \leq \int_{-a}^{-b} f\,dx - \int_{-a}^{-b} f\,dx \leq U(P, f, \alpha) - L(P, f, \alpha).\]
So if for every \(\varepsilon > 0\) there exists \(P\) such that the right-side can be made less than \(\varepsilon\), we conclude that \(\int_{-a}^{-b} f\,dx = \int_{-a}^{-b} f\,dx\), that is \(f \in \mathcal{R}(\alpha)\) on \([a, b]\).

(\(\Rightarrow\)) Suppose \(f \in \mathcal{R}(\alpha)\) on \([a, b]\). Let \(\varepsilon > 0\). Since \(f \in \mathcal{R}(\alpha)\) and by the definitions of inf and sup, there exist partitions \(P_1\) and \(P_2\) such that
\[L(P_1, f, \alpha) > \int_{-a}^{-b} f\,dx - \frac{\varepsilon}{2} = \int_{-a}^{-b} f\,dx - \frac{\varepsilon}{2}\]
and
\[U(P_2, f, \alpha) < \int_{-a}^{-b} f\,dx + \frac{\varepsilon}{2} = \int_{-a}^{-b} f\,dx + \frac{\varepsilon}{2}.\]
Hence
\[0 \leq U(P_1 \cup P_2, f, \alpha) - L(P_1 \cup P_2, f, \alpha) \leq U(P_2, f, \alpha) - L(P_1, f, \alpha) \leq \int_{-a}^{-b} f\,dx + \frac{\varepsilon}{2} - \left( \int_{-a}^{-b} f\,dx - \frac{\varepsilon}{2} \right) = \varepsilon. \square\]