Lemma 6.001. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ be increasing. Suppose that there exists a number $S$ such that for any $\varepsilon > 0$ there exists partitions $P_U$ and $P_L$ such that

$$ U (P_U, f, \alpha) \leq S + \varepsilon \quad \text{and} \quad L (P_L, f, \alpha) \geq S - \varepsilon. $$

Then $f \in \mathcal{R}(\alpha)$ and

$$ \int_a^b f \, d\alpha = S. $$

Idea of the proof (fill in details). Then

$$ \int_a^b f \, d\alpha = \inf_{P} \sum_{i=1}^{n} f(x_i) \Delta x_i, $$

which imply $\int_a^b f \, d\alpha = \int_a^b f \, d\alpha = S$. \qed

Lemma 6.002. If $P_\delta$ is some family of partitions, then

$$ \inf_{P} U (P, f, \alpha) \leq \lim_{\delta \to 0} U (P_\delta, f, \alpha) $$

whenever the limit on the right exists. Similarly,

$$ \sup_{P} L (P, f, \alpha) \geq \lim_{\delta \to 0} L (P_\delta, f, \alpha) $$

whenever the limit on the right exists.

Proof. For each $\delta$ we have

$$ U (P_\delta, f, \alpha) \geq \inf_{P} U (P, f, \alpha). $$

Noting that the right side is just a number, we get

$$ \lim_{\delta \to 0} U (P_\delta, f, \alpha) \geq \inf_{P} U (P, f, \alpha). $$

Similarly, we get the same result for $U$ replaced by $L$. \qed

Definition 6.003. Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, a “particular Riemann sum” of $f$ is

$$ \sum_{i=1}^{n} f (s_i) \Delta x_i, \quad \text{where} \ s_i \in [x_{i-1}, x_i]. $$

Then $$(\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}))$$

$$ L (P, f, \alpha) \leq \sum_{i=1}^{n} f (s_i) \Delta \alpha_i \leq U (P, f, \alpha). \quad (1) $$
Theorem 6.7(b). Suppose \( U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon \), where \( P = \{x_0, \ldots, x_n\} \). Then for any \( s_i, t_i \in [x_{i-1}, x_i] \) we have
\[
\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta \alpha_i < \varepsilon.
\]

Proof. Since \( m_i \leq f(s_i), f(t_i) \leq M_i \), we have
\[
|f(s_i) - f(t_i)| \leq M_i - m_i.
\]
Hence
\[
\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta \alpha_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i = U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon. \quad \square
\]

Definition 6.14. The unit step function (Heaviside function) is
\[
I(x) = \begin{cases} 
0 & x \leq 0, \\
1 & x > 0.
\end{cases}
\]

Theorem 6.15. Suppose \( 0 \in (a, b) \), \( f: [a, b] \to \mathbb{R} \) is bounded, \( f \) is continuous at \( 0 \). Then \( f \in \mathcal{R}(I) \) and
\[
\int_a^b f \, dI = f(0).
\]

Proof. Let \( \delta \in (0, b) \). Define the partition
\[
P_\delta = \{x_0, x_1, x_2, x_3\} = \{a, 0, \delta, b\}.
\]

Then
\[
\Delta I_1 = I(0) - I(a) = 0, \\
\Delta I_2 = I(\delta) - I(0) = 1, \\
\Delta I_3 = I(b) - I(\delta) = 0.
\]

Therefore
\[
U(P_\delta, f, I) = \sum_{i=1}^{3} M_i(\delta) \Delta I_i = M_2(\delta) \triangleq \sup_{x \in [0, \delta]} f(x).
\]

Similarly,
\[
L(P_\delta, f, I) = \sum_{i=1}^{3} m_i(\delta) \Delta I_i = m_2(\delta) \triangleq \inf_{x \in [0, \delta]} f(x).
\]

Since \( f \) is (right) continuous at \( 0 \), we have
\[
\lim_{\delta \to 0} M_2(\delta) = f(0) = \lim_{\delta \to 0} m_2(\delta).
\]

We conclude that (using Lemma 6.01 for the inequalities)
\[
\int_a^b f \, dI = \inf_{P} U(P, f, I) \leq \lim_{\delta \to 0} U(P_\delta, f, I) = f(0)
\]
and
\[
\int_{a}^{b} f \, dI = \sup_{P} L(P, f, I) \geq \lim_{\delta \to 0} L(P_{\delta}, f, I) = f(0).
\]
This implies that \( f \in \mathcal{R}(I) \) and
\[
\int_{a}^{b} f \, dI = f(0). \quad \square
\]

**Theorem 6.17.** Suppose \( \alpha \) is increasing and \( \alpha' \in \mathcal{R} \) on \([a, b]\) and that \( f : [a, b] \to \mathbb{R} \) is bounded. Then
\[
f \in \mathcal{R}(\alpha) \text{ if and only if } f \alpha' \in \mathcal{R}.
\]
In this case,
\[
\int_{a}^{b} f \, d\alpha = \int_{a}^{b} f(x) \alpha'(x) \, dx.
\]

**Proof.** (The idea is to use the MVT to relate \( \alpha' \) to \( \Delta \alpha \).) Let \( \varepsilon > 0 \).

1. Since \( \alpha' \in \mathcal{R} \), there exists a partition \( P = \{x_0, \ldots, x_n\} \) of \([a, b]\) such that
\[
U(P, \alpha') - L(P, \alpha') < \varepsilon. \quad (2)
\]

2. By the MVT, for each \( i \) there exists \( t_i \in (x_{i-1}, x_i) \) such that
\[
\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i) \Delta x_i.
\]

3. By (2) and Theorem 6.7(b), for any \( s_i \in [x_{i-1}, x_i] \) we have
\[
\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon.
\]

4. So
\[
\left| \sum_{i=1}^{n} f(s_i) \Delta \alpha_i - \sum_{i=1}^{n} f(s_i) \alpha'(s_i) \Delta x_i \right| = \left| \sum_{i=1}^{n} f(s_i) (\alpha'(t_i) - \alpha'(s_i)) \Delta x_i \right|
\leq \sum_{i=1}^{n} |f(s_i)| |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i
\leq M \sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i
\leq M \varepsilon.
\]
where \( M \doteq \sup_{x \in [a, b]} |f(x)| \). So (compare with (1))
\[
\sum_{i=1}^{n} f(s_i) \Delta \alpha_i \leq \sum_{i=1}^{n} f(s_i) \alpha'(s_i) \Delta x_i + M \varepsilon
\leq \sum_{i=1}^{n} M^* \Delta x_i + M \varepsilon
\leq U(P, f \alpha') + M \varepsilon,
\]
where $M_i^* \triangleq \sup_{x \in [x_{i-1}, x_i]} f (x) \alpha' (x)$. Since $s_i \in [x_{i-1}, x_i]$ are arbitrary, we obtain
\begin{equation}
U (P, f, \alpha) \leq U (P, f \alpha') + M \varepsilon. \tag{3}
\end{equation}

Similarly, we argue that (reversing the sums)
\[
\sum_{i=1}^{n} f (s_i) \alpha' (s_i) \Delta x_i \leq \sum_{i=1}^{n} f (s_i) \Delta \alpha_i + M \varepsilon \\
\leq \sum_{i=1}^{n} M_i \Delta \alpha_i + M \varepsilon \\
\leq U (P, f, \alpha) + M \varepsilon.
\]

Since $s_i \in [x_{i-1}, x_i]$ are arbitrary, we obtain
\begin{equation}
U (P, f') \leq U (P, f, \alpha) + M \varepsilon. \tag{4}
\end{equation}

By (3) and (4), we conclude that
\[
|U (P, f \alpha') - U (P, f, \alpha)| \leq M \varepsilon.
\]

The completely analogous argument, with $U$ replaced by $L$ (and sup replaced by inf), yields
\[
|L (P, f \alpha') - L (P, f, \alpha)| \leq M \varepsilon.
\]

Since the above two inequalities remain true for any refinement of $P$, we conclude that
\[
\left| \int_{a}^{b} f \, d\alpha - \int_{a}^{b} f (x) \alpha' (x) \, dx \right| \leq M \varepsilon
\]
and similarly,
\[
\left| \int_{-a}^{b} f \, d\alpha - \int_{-a}^{b} f (x) \alpha' (x) \, dx \right| \leq M \varepsilon.
\]

Since $\varepsilon > 0$ is arbitrary, we conclude the theorem. □

**Remark:** The proof had slight jumps in logic toward the end. Fill these in.

**Theorem 6.19.** Let $\varphi : [A, B] \to [a, b]$ be a strictly increasing, continuous, onto function. If $\alpha : [a, b] \to \mathbb{R}$ is increasing and $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $f \circ \varphi \in \mathcal{R}(\alpha \circ \varphi)$ on $[A, B]$ and
\[
\int_{A}^{B} f \circ \varphi \, d(\alpha \circ \varphi) = \int_{a}^{b} f \, d\alpha.
\]

**Proof.** (Idea: correspondence.) Any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ corresponds to a partition $Q = \{y_0, \ldots, y_n\}$ of $[A, B]$ by $x_i = \varphi (y_i)$ for each $i$, that is, $y_i = \varphi^{-1} (x_i)$. It is easy to see that under this correspondence,
\[
U (Q, f \circ \varphi, \alpha \circ \varphi) = U (P, f, \alpha), \\
L (Q, f \circ \varphi, \alpha \circ \varphi) = L (P, f, \alpha).
\]

The theorem follows from taking the inf of the $U$’s and the sup of the $L$’s. □