Theorem 6.20. Let \( f \in \mathcal{R} \) on \([a, b]\). For \( a \leq x \leq b \), put
\[
F(x) = \int_a^x f(t) \, dt.
\]
Then \( F \) is continuous on \([a, b]\); furthermore, if \( f \) is continuous at a point \( x_0 \) of \([a, b]\), then \( F \) is differentiable at \( x_0 \), and
\[
F'(x_0) = f(x_0).
\]

Proof. (1) Since \( f \in \mathcal{R} \), by definition \( f \) is bounded, so there is an \( M < \infty \) such that \( |f(t)| \leq M \) for \( t \in [a, b] \). Hence, for any \( a \leq x < y \leq b \) we have
\[
|F(y) - F(x)| = \left| \int_a^y f(t) \, dt - \int_a^x f(t) \, dt \right|
\leq \int_x^y |f(t)| \, dt
\leq M |y - x|.
\]
This implies that \( F \) is Lipschitz continuous (by definition), and in particular, \( F \) is continuous.

(2) Suppose that \( f \) is continuous at \( x_0 \). We have
\[
\frac{F(x_0 + h) - F(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0 + h} f(t) \, dt,
\]
so that
\[
\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| = \frac{1}{h} \left| \int_{x_0}^{x_0 + h} (f(t) - f(x_0)) \, dt \right| \rightarrow 0
\]
as \( h \to 0 \) since \( f \) is continuous at \( x_0 \). \( \square \)

Theorem 6.21. The Fundamental Theorem of Calculus. If \( f \in \mathcal{R} \) on \([a, b]\) and if there is a differentiable function \( F \) on \([a, b]\) such that \( F' = f \), then
\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

Proof. Let \( \varepsilon > 0 \). Since \( f \in \mathcal{R} \), by Theorem 6.6 there exists a partition \( P = \{x_0, x_1, \ldots, x_n\} \) such that
\[
U(P, f) - L(P, f) < \varepsilon.
\]
By the Mean Value Theorem 5.10, there exist \( c_i \in (x_{i-1}, x_i) \) such that
\[
F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i) \Delta x_i.
\]
Therefore
\[
F(b) - F(a) = \sum_{i=1}^n (F(x_i) - F(x_{i-1}))
= \sum_{i=1}^n f(c_i) \Delta x_i \in [L(P, f), U(P, f)].
\]
Since

\[ L(P, f) \leq \int_a^b f(x) \, dx \leq U(P, f), \]

we conclude that

\[ |F(b) - F(a) - \int_a^b f(x) \, dx| \leq \varepsilon. \]

The theorem follows since \( \varepsilon > 0 \) is arbitrary. \( \square \)

**Theorem 6.22. Integration by Parts.** Suppose \( F \) and \( G \) are differentiable functions on \([a, b]\), \( F' \in \mathcal{R} \) and \( G' \in \mathcal{R} \). Then

\[ \int_a^b F(x) G'(x) \, dx = F(b) G(b) - F(a) G(a) - \int_a^b F'(x) G(x) \, dx. \]

**Proof.** Let \( H = FG \). Then \( H' \in \mathcal{R} \) by Theorem 6.13. Theorem 6.21 implies

\[ F(b) G(b) - F(a) G(a) = \int_a^b (FG)'(x) \, dx = \int_a^b (F'(x) G(x) + F(x) G'(x)) \, dx. \]