Integration of Vector-Valued Functions

Theorem 6.25. If \( f : [a, b] \to \mathbb{R}^k \) and if \( f \in \mathcal{R}(\alpha) \), then \( |f| \in \mathcal{R}(\alpha) \) and

\[
\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha.
\]

**Proof.** By definition, since \( f = (f_1, \ldots, f_k) \in \mathcal{R}(\alpha) \), we have each \( f_i \in \mathcal{R}(\alpha) \). Hence \( f_i^2 \in \mathcal{R}(\alpha) \) (by Theorem 6.11), so that

\[
|f|^2 = \sum_{i=1}^k f_i^2 \in \mathcal{R}(\alpha).
\]

Since \( \sqrt{ \cdot } : [0, \infty) \to [0, \infty) \) is continuous and \( |f|^2 \in \mathcal{R}(\alpha) \), by Theorem 6.11, we have \( |f| = \sqrt{|f|^2} \in \mathcal{R}(\alpha) \).

Next, let \( y = \int_a^b f \, d\alpha \), that is, \( y = (y_1, \ldots, y_k) \), where \( y_i = \int_a^b f_i \, d\alpha \). We have

\[
\left| \int_a^b f \, d\alpha \right|^2 = y \cdot \int_a^b f \, d\alpha = \int_a^b y \cdot f \, d\alpha \leq \int_a^b |y||f| \, d\alpha = \left( \int_a^b f \, d\alpha \right) \left( \int_a^b |f| \, d\alpha \right),
\]

where we used the Schwarz inequality. If \( \left| \int_a^b f \, d\alpha \right| = 0 \), then we are done. Otherwise, we are also done since we get

\[
\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha. \quad \square
\]

Rectifiable Curves

**Definition.** A curve in \( \mathbb{R}^k \) is a continuous mapping \( \gamma : [a, b] \to \mathbb{R}^k \).

Given a partition \( P = \{x_0, \ldots, x_n\} \) of \([a, b]\), define

\[
\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|.
\]

Define the length of \( \gamma \) to be

\[
\Lambda(\gamma) = \sup_P \Lambda(P, \gamma).
\]

If the length of \( \gamma \) is finite, then we say that \( \gamma \) is rectifiable.

**Theorem 6.27.** If \( \gamma' \) is continuous on \([a, b]\), then \( \gamma \) is rectifiable and

\[
\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt \equiv L(\gamma).
\]

**Proof.** \((\leq)\) Let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\). Since \( \gamma' \) is continuous, by Theorem 6.24 (i.e., Theorem 6.21 applied to each component), we have

\[
\gamma(x_i) - \gamma(x_{i-1}) = \int_{x_{i-1}}^{x_i} \gamma'(t) \, dt.
\]
So, by Theorem 6.25,

\[ |\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, dt \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, dt \]

This implies

\[ \Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, dt = \int_{a}^{b} |\gamma'(t)| \, dt = L(\gamma). \]

Since this is true for each partition \( P \), we have

\[ \Lambda(\gamma) = \sup_{P} \Lambda(P, \gamma) \leq \int_{a}^{b} |\gamma'(t)| \, dt. \]

(≥) Let \( \varepsilon > 0 \). Since \( \gamma' \) is continuous on \([a, b]\) and \([a, b]\) is compact, \( \gamma' \) is uniformly continuous on \([a, b]\). So there exists \( \delta > 0 \) such that

\[ |\gamma'(s) - \gamma'(t)| < \varepsilon \quad \text{if} \quad |s - t| \leq \delta. \]

Let \( P \) be a partition with \( \Delta x_i < \delta \) for all \( i \). If \( t \in [x_{i-1}, x_i] \), then

\[ |\gamma'(x_i) - \gamma'(t)| < \varepsilon, \]

so that

\[ |\gamma'(t)| \leq |\gamma'(x_i)| + \varepsilon. \]

Thus

\[ \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, dt \leq \int_{x_{i-1}}^{x_i} (|\gamma'(x_i)| + \varepsilon) \, dt = |\gamma'(x_i)| \Delta x_i + \varepsilon \Delta x_i. \]

On the other hand,

\[ |\gamma'(x_i)| \Delta x_i = \left| \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t) + \gamma'(t)) \, dt \right| \]

\[ \leq \left| \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t)) \, dt \right| + \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, dt \right| \]

\[ \leq \int_{x_{i-1}}^{x_i} |\gamma'(x_i) - \gamma'(t)| \, dt + |\gamma(x_i) - \gamma(x_{i-1})| \]

\[ \leq \varepsilon \Delta x_i + |\gamma(x_i) - \gamma(x_{i-1})|. \]

Combining the above two inequalities implies

\[ \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, dt \leq |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon \Delta x_i. \]

Then summing over \( i \) yields

\[ \int_{a}^{b} |\gamma'(t)| \, dt \leq \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| + \sum_{i=1}^{n} 2\varepsilon \Delta x_i \]

\[ = \Lambda(P, \gamma) + 2\varepsilon (b - a) \]

\[ \leq \Lambda(\gamma) + 2\varepsilon (b - a). \]

Since \( \varepsilon > 0 \) is arbitrary, we conclude that

\[ \Lambda(\gamma) \geq \int_{a}^{b} |\gamma'(t)| \, dt. \quad \square \]