#2. This problem uses the results of #1.

**Remark about #1 (which is assigned for the next HW #6).**

We say that \( \{f_n\}_{n \geq 1} \) is a **sequence of bounded functions** on a set \( E \) if for each \( n \geq 1 \) there exists \( M_n \) such that \( |f_n(x)| \leq M_n \) for all \( x \in E \).

We say that \( \{f_n\}_{n \geq 1} \) is **uniformly bounded** if there exists \( M \) such that \( |f_n(x)| \leq M \) for all \( n \geq 1 \) and all \( x \in E \).

**Solution to #2.** (a) Suppose that \( f_n \to f \) uniformly on \( E \) and \( g_n \to g \) uniformly on \( E \). Let \( \varepsilon > 0 \). Then there exists \( N_1, N_2 \in \mathbb{N} \) such that

\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for } x \in E \text{ and } n \geq N_1
\]

and

\[
|g_n(x) - g(x)| < \frac{\varepsilon}{2} \quad \text{for } x \in E \text{ and } n \geq N_2.
\]

Let \( N = \max\{N_1, N_2\} \). Then for \( x \in E \) and \( n \geq N \) we have

\[
|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)|
\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
= \varepsilon.
\]

(b) By Exercise #1, both sequences \( \{f_n\} \) and \( \{g_n\} \) are uniformly bounded. So there exists \( M \) such that \( |f_n(x)| \leq M \) and \( |g_n(x)| \leq M \) for all \( x \in E \) and \( n \geq 1 \). This implies \( |f(x)| \leq M \) and \( |g(x)| \leq M \) for all \( x \in E \). Then there exists \( N \in \mathbb{N} \) such that

\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2M} \quad \text{for } x \in E \text{ and } n \geq N
\]

and

\[
|g_n(x) - g(x)| < \frac{\varepsilon}{2M} \quad \text{for } x \in E \text{ and } n \geq N.
\]

For \( x \in E \) and \( n \geq N \) we have

\[
|(f_ng_n)(x) - (fg)(x)| = |f_n(x)g_n(x) - f(x)g(x)|
= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)|
\leq |(|f_n(x) - f(x)|g_n(x)| + |f(x)|(g_n(x) - g(x))|)
\leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)|
\leq M |f_n(x) - f(x)| + M |g_n(x) - g(x)|
\leq M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M}
= \varepsilon.
\]

#3. On \( \mathbb{R} \) let \( f_n(x) = x \) and \( g_n(x) = \frac{1}{n} \). Then \( f_n \to f \) and \( g_n \to g \) uniformly, where \( f(x) = x \) and \( g(x) = 0 \). Then \( (f_ng_n)(x) = \frac{x}{n} \). Of course, \( f_ng_n \to 0 \). But this convergence is easily seen not to be uniform. E.g., let \( x_n = n \). Then \( (f_ng_n)(x_n) = 1 \).
#4.

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}. \]

(a) For \( x = 0 \) the series diverges since it is \( \sum_{n=1}^{\infty} 1 \).

If \( x = -\frac{1}{n^2} \) for some \( n \geq 1 \), then the series is undefined since one of its terms is undefined.

Otherwise, this series converges absolutely. Indeed, for \( n \) large, the terms of the series \( \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \) behave like the terms of the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), which converges.

(b) The series is defined and converges on the following intervals: \((0, \infty)\), \((-\infty, -1)\), and \((-\frac{1}{n^2}, -\frac{1}{(n+1)^2})\) for \( n \geq 1 \).

On the intervals \((-\infty, -1)\), and \((-\frac{1}{n^2}, -\frac{1}{(n+1)^2})\) for \( n \geq 1 \), the series converges uniformly.

The series does not converge uniformly on \((0, \infty)\). But, for any \( \delta > 0 \) we have that the series converges uniformly on the interval \((\delta, \infty)\). (One just has to stay away from 0.)

(c) The series is continuous wherever it converges, since for each \( x \) at which it converges there exists an \( \varepsilon > 0 \) such that it converges uniformly on \([x - \varepsilon, x + \varepsilon]\) and we can apply Theorem 7.12.

(d) However, the function is not bounded on \((0, \infty)\) since \( \lim_{x \to 0^+} f(x) = \infty \). (Again, one just has to stay away from 0.)

#7.

\[ f_n(x) = \frac{x}{1 + nx^2} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad n \geq 1. \]

(a) If \( |x| \geq \frac{1}{\sqrt{n}} \), then

\[ |f_n(x)| = \frac{|x|}{1 + nx^2} \leq \frac{|x|}{n |x|} = \frac{1}{n} \leq \frac{1}{\sqrt{n}}. \]

(b) If \( |x| \leq \frac{1}{\sqrt{n}} \), then

\[ |f_n(x)| = \frac{|x|}{1 + nx^2} \leq |x| \leq \frac{1}{\sqrt{n}}. \]

From this it is easy to see that \( f_n \to 0 \) uniformly. (Cooke uses the Schwarz inequality to get \( 1 + nx^2 \geq 2\sqrt{n} |x| \), so that

\[ |f_n(x)| = \frac{|x|}{1 + nx^2} \leq \frac{|x|}{2\sqrt{n} |x|} \leq \frac{1}{2\sqrt{n}}. \]

Now

\[ f'_n(x) = \frac{1}{1 + nx^2} - \frac{2nx^2}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad n \geq 1. \]

If \( x \neq 0 \), then

\[ |f'_n(x)| = \frac{|1 - nx^2|}{(1 + nx^2)^2} \leq \frac{1}{1 + nx^2}, \]

so that \( f'_n(x) \to 0 \) as \( n \to \infty \). On the other hand,

\[ f'_n(0) = 1 \quad \text{for} \quad n \geq 1. \]
#8. If
\[
I(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 & \text{if } x > 0,
\end{cases}
\]
if \( \{x_n\} \) is a sequence of distinct points of \((a,b)\), and if \( \sum |c_n| \) converges, prove that the series
\[
f(x) = \sum_{n=1}^{\infty} c_n I(x-x_n), \quad \text{for } a \leq x \leq b,
\]
converges uniformly, and that \( f \) is continuous for every \( x \neq x_n \).

Solution. Let \( x \in [a,b] \). We have \( I(x-x_n) = 1 \) if \( x > x_n \) and \( I(x-x_n) = 0 \) if \( x \leq x_n \). Let
\[
A_x = \{ n \in \mathbb{N} : x_n < x \}.
\]
Define \( b_n = c_n \) if \( n \in A_x \) and \( b_n = 0 \) if \( n \notin A_x \). Then
\[
f(x) = \sum_{n=1}^{\infty} c_n I(x-x_n) = \sum_{n=1}^{\infty} b_n
\]
converges uniformly since \( \sum |c_n| \) converges. Indeed, we can see this from the fact that for any \( N \in \mathbb{N} \),
\[
\sum_{n=N}^{\infty} |c_n I(x-x_n)| = \sum_{n=N}^{\infty} |b_n| \leq \sum_{n=N}^{\infty} |c_n|.
\]
Now recall:

**Theorem 7.11.** Let \( \{f_n\} \) be a sequence of functions on a set \( E \) in a metric space with \( f_n \to f \)
uniformly on \( E \). Let \( x \) be a limit point of \( E \). If \( \lim_{t \to x} f_n(t) = A_n \) exists for all \( n \), then \( \{A_n\} \)
converges and \( \lim_{t \to x} f(t) = \lim_{n \to \infty} A_n \).

Define \( f_n(x) = \sum_{k=1}^{n} c_k I(x-x_k) \). We have just proved that \( f_n \to f \) uniformly on \([a,b]\). Let \( x \in [a,b] \) such that \( x \neq x_n \) for each \( n \). Then each \( f_n \) is continuous at \( x \).
\(^1\) Therefore \( \lim_{t \to x} f_n(t) = A_n = f_n(x) \) exists for all \( n \). By Theorem 7.11 we conclude that
\[
\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n = \lim_{n \to \infty} f_n(x) = f(x).
\]
That is, \( f \) is continuous at \( x \).

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\(^1\) This is not true if \( x = x_n \) for some \( n \), in which case \( f_k \) is not continuous at \( x \) for each \( k \geq n \).