Math 140B  Chapter 7.
Sections 1 and 2: Discussion of the Main Problem and Uniform Convergence

Definitions

For the study of partial differential equations, it is important to consider sequences of functions, find out when there are converging subsequences, understand in what sense the functions convergence, and to discover properties of the limits.

**Definition 7.1.** Let $E$ be a set. We say that a sequence of functions $f_n : E \to \mathbb{C}$ converges pointwise to a function $f : E \to \mathbb{C}$ if for every $x \in E$, $f_n(x)$ converges to $f(x)$. That is, for every $x \in E$ and $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| \leq \varepsilon$.

**Remark 1.** Here $N$ depends on $x$ and $\varepsilon$.

**Definition 7.7.** Let $E$ be a set. We say that a sequence of functions $f_n : E \to \mathbb{C}$ converges uniformly to a function $f : E \to \mathbb{C}$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in E$.

**Remark 2.** Here $N$ only depends on $\varepsilon$ and not on $x$. In Definition 7.7, the conclusion is equivalent to $\sup_{x \in E} |f_n(x) - f(x)| \leq \varepsilon$.

**Remark 3.** Suppose that $f_n \to f$. Then the convergence if not uniform if and only if there exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $n \geq N$ such that $|f_n(x) - f(x)| > \varepsilon$ for some $x \in E$.

**Examples**

Consider the following real valued functions on the metric space $[0, 1] \subset \mathbb{R}$.

**Example 1:** Define the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 - \frac{1}{n}, \\ n(x - 1) + 1 & \text{if } 1 - \frac{1}{n} \leq x \leq 1. \end{cases}$$

We have $\lim_{n \to \infty} f_n(x) = f(x)$, where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$
Note that, although the functions $f_n$ are each continuous, $f$ has a (jump) discontinuity at $x = 1$. Their graphs look like:

Although each $f_n$ is continuous, their limit $f$ is not continuous.

By Theorem 7.12, proved below, the convergence of $f_n$ to $f$ cannot be uniform. Indeed, let $\varepsilon = \frac{1}{3}$. For each $n$ there exists $t_n \in [0, 1]$ such that $f(t_n) = \frac{1}{2}$ (take $t_n = 1 - \frac{1}{2n}$). Since $f(t) = 0$ or $f(t) = 1$ for all $t \in [0, 1]$, we have for all $n \in \mathbb{N}$,

$$|f_n(t_n) - f(t_n)| = \frac{1}{2} > \varepsilon.$$ 

By Remark 3, this implies the convergence of $f_n \to f$ is not uniform.

**Example 2:** On the other hand, even if the convergence is not uniform, the limit could still be continuous. Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$ defined by the following diagram:

A limit function $f = \lim_{n \to \infty} f_n$ is continuous if and only if

$$\lim_{x \to t} f(x) = f(t),$$

that is,

$$\lim_{x \to t} \left( \lim_{n \to \infty} f_n(x) \right) = \lim_{n \to \infty} \left( \lim_{x \to t} f_n(x) \right).$$

That is, whether we can switch the order of the limits for a sequence of continuous functions.
(Discrete) Example 7.2. Define the sequence of functions $s_m : \mathbb{N} \to \mathbb{R}$ by $s_m(n) = \frac{m}{m+n}$ for $m \in \mathbb{N}$. We have

$$\lim_{m \to \infty} s_m(n) = 1.$$  

That is, the functions $s_m$ converge to the constant function 1 as $m \to \infty$. (But not uniformly, for $n$ large it takes longer for $s_m(n)$ to approach 1.) Then, of course,

$$\lim_{n \to \infty} \left( \lim_{m \to \infty} s_m(n) \right) = 1.$$  

On the other hand, if we reverse the limits, we get the following:

$$\lim_{m \to \infty} s_m(n) = 0,$$

so

$$\lim_{n \to \infty} \left( \lim_{m \to \infty} s_m(n) \right) = 1,$$

which is a different answer.

Example 7.3 (Series of functions). Consider the sequence of continuous functions

$$f_n(x) = \frac{x^2}{(1+x^2)^n}$$  

for $x \in \mathbb{R}$ and $n \geq 0$.

Define the series

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}. $$

We have $f(0) = 0$ and if $x \neq 0$, then

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=0}^{\infty} \left( \frac{1}{1+x^2} \right)^n = x^2 \frac{1}{1 - \frac{1}{1+x^2}} = 1 + x^2$$

(since $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$). That is,

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is discontinuous at $x = 0$.

Example 7.5 (limiting to zero but derivatives getting bigger). Let

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}$$  

for $x \in \mathbb{R}$ and $n \geq 1$.

Clearly,

$$f(x) \doteq \lim_{n \to \infty} f_n(x) = 0,$$

where the convergence is uniform. But, the derivatives are

$$f'_n(x) = \sqrt{n} \cos nx$$  

for $x \in \mathbb{R}$ and $n \geq 1$.

In particular, $f'_n(0) = \sqrt{n}$, so that

$$\lim_{n \to \infty} f'_n(0) = \infty \neq 0 = f'(0).$$
Theorems

The **Cauchy Criterion for Uniform Convergence**:

**Theorem 7.8.** A sequence of functions \( f_n : E \to \mathbb{R} \) converges uniformly on \( E \) if and only if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( m, n \geq N \) and \( x \in E \), then

\[
|f_n(x) - f_m(x)| \leq \varepsilon.
\]

**Proof.** (\( \Rightarrow \)) Suppose \( \{f_n\} \) converges uniformly on \( E \) to \( f \). Let \( \varepsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that if \( k \geq N \) and \( x \in E \), then

\[
|f_k(x) - f(x)| \leq \frac{\varepsilon}{2}.
\]

Hence, if \( m, n \geq N \) and \( x \in E \), then

\[
|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

(\( \Leftarrow \)) Suppose we have the property that: for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( m, n \geq N \) and \( x \in E \), then

\[
|f_n(x) - f_m(x)| \leq \varepsilon.
\]

Fixing \( x \), since \( \{f_n(x)\} \) is a Cauchy sequence of real numbers, by Theorem 3.11(c), \( f_n(x) \) converges as \( n \to \infty \) to a real number, which we call \( f(x) \). We need to show that \( f_n \to f \) uniformly. Let \( \varepsilon > 0 \). Then there exists \( N \) such that if \( m, n \geq N \) and \( x \in E \), then

\[
|f_n(x) - f_m(x)| \leq \varepsilon.
\]

Taking the limit as \( m \to \infty \), we obtain that if \( n \geq N \) and \( x \in E \), then

\[
|f_n(x) - f(x)| \leq \varepsilon. \quad \square
\]

**Criterion for uniform convergence** are given by Theorems 7.9 and 7.10.
Theorems on uniform convergence and continuity

Recall the following facts.

**Theorem 1.** Let \( f : E \to Y \), where \( E \) is a subset of a metric space \( Y \).

1. If \( x \in E \) is an isolated point, then \( f \) is continuous at \( x \).

2. If \( x \in E \) is a limit point of \( E \), then \( f \) is continuous at \( x \) if and only if \( \lim_{t \to x} f(t) = f(x) \).

We have the following important result:

**Theorem 7.12.** Let \( E \) be a subset of a metric space \( X \) and let \( f_n : E \to \mathbb{C} \) be a sequence of continuous functions. If \( f_n \to f \) uniformly on \( E \), then \( f \) is continuous on \( E \).

It follows from:

**Theorem 7.11.** Let \( \{f_n\} \) be a sequence of continuous functions on a set \( E \) in a metric space with \( f_n \to f \) uniformly on \( E \). Let \( x \) be a limit point of \( E \). If \( \lim_{t \to x} f_n(t) = A_n \) exists for all \( n \), then \( \{A_n\} \) converges and \( \lim_{t \to x} f(t) = \lim_{n \to \infty} A_n \).

**Proof of T7.12 assuming T7.11 is true.** By Theorem 1, we only need to consider the case where \( x \in E \) is a limit point of \( E \). Since each \( f_n \) is continuous and \( x \in E \), we have \( \lim_{t \to x} f_n(t) = f_n(x) \). By Theorem 7.11 we conclude that the sequence of numbers \( \{f_n(x)\} \) converges and that \( \lim_{t \to x} f(t) = \lim_{n \to \infty} f_n(x) = f(x) \), where the last equality is because \( f_n \to f \). \( \square \)

**Proof of T7.11.**

**Step 1.** \( \{A_n\} \) is a Cauchy sequence, and hence converges to some number \( A \). Let \( \varepsilon > 0 \). Since \( f_n \to f \) uniformly, there exists \( N \in \mathbb{N} \) such that \( |f_n(t) - f(t)| \leq \frac{\varepsilon}{2} \) for all \( t \in E \) and \( n \geq N \). Hence, for all \( t \in E \) and \( n, m \geq N \) we have

\[
|f_n(t) - f_m(t)| = |f_n(t) - f(t) + f(t) - f_m(t)| \\ 
\leq |f_n(t) - f(t)| + |f_m(t) - f(t)| \\ 
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Hence

\[
\varepsilon \geq \lim_{t \to x} |f_n(t) - f_m(t)| = \left| \lim_{t \to x} f_n(t) - \lim_{t \to x} f_m(t) \right| = |A_n - A_m|
\]

where the first equality is by a limit law. This proves that \( \{A_n\} \) is a Cauchy sequence and hence converges. Let \( A = \lim_{n \to \infty} A_n \). This proves step 1.

**Step 2.** Again let \( \varepsilon > 0 \).

(i) Since \( A_n \to A \), there exists \( N_1 \in \mathbb{N} \) such that

\[
|A_n - A| \leq \frac{\varepsilon}{3} \quad \text{for all } n \geq N_1.
\]

(ii) Since \( f_n \to f \) uniformly by hypothesis, there exists \( N_2 \in \mathbb{N} \) such that

\[
|f_n(t) - f(t)| \leq \frac{\varepsilon}{3} \quad \text{for all } t \in E \text{ and } n \geq N_2.
\]

(iii) Let \( N = \max\{N_1, N_2\} \). Then

\[
|A_N - A| \leq \frac{\varepsilon}{3} \quad \text{and} \quad |f_N(t) - f(t)| \leq \frac{\varepsilon}{3} \quad \text{for all } t \in E.
\]

(iv) Since \( \lim_{t \to x} f_N(t) = A_N \) by hypothesis, there exists \( \delta > 0 \) such that if \( t \in N_\delta(x) \) with \( t \neq x \), then

\[
|f_N(t) - A_N| \leq \frac{\varepsilon}{3}.
\]

We conclude from the triangle inequality that \( t \in N_\delta(x) \) with \( t \neq x \), then

\[
|f(t) - A| \leq |f_N(t) - f(t)| + |f_N(t) - A_N| + |A_N - A| \leq \varepsilon.
\]

We have proved that \( \lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n \). \( \square \)