Math 140B  Chapter 8.
Exponential Function

Define the **exponential function** \( \exp : \mathbb{C} \to \mathbb{C} \) by

\[
\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

Note that \( e^0 = 1 \). We also have

**Lemma 0.**

\[
\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{h \to 0} \sum_{n=1}^{\infty} \frac{h^n}{n!} = 1,
\]

where \( h \) is a complex number.

**Proof.** We compute that

\[
\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{h \to 0} \frac{\sum_{n=0}^{\infty} \frac{h^n}{n!} - 1}{h} = \lim_{h \to 0} \sum_{n=1}^{\infty} \frac{h^n}{n!} = \lim_{h \to 0} \left( 1 + \sum_{n=1}^{\infty} \frac{h^n}{(n+1)!} \right) = \lim_{h \to 0} \left( 1 + h \sum_{n=1}^{\infty} \frac{h^{n-1}}{(n+1)!} \right) = 1. \quad \square
\]

Let \( a_n = \frac{z^n}{n!} \). Then

\[
|a_{n+1}| = \left| \frac{z^{n+1}}{(n+1)!} \right| = \frac{|z|}{n + 1}.
\]

Thus

\[
\limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup_{n \to \infty} \frac{|z|}{n + 1} = 0
\]

for all \( z \in \mathbb{C} \). Hence, by the ratio test, the series

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}
\]

converges for all \( z \in \mathbb{C} \).

We take as a fact the property that (one can use Theorem 3.50 to prove it):

**Lemma 1.**

\[
e^{z+w} = e^z e^w \quad \text{for } z, w \in \mathbb{C}.
\]
As a consequence, we have
\[ e^z e^{-z} = e^{z+(-z)} = e^0 = 1. \]

Another consequence is: Define the complex derivative by
\[ \frac{d}{dz} e^z = \lim_{h \to 0} \frac{e^{z+h} - e^z}{h}, \]
where \( h \) is complex number. We have (using Lemma 0)
\[ \frac{d}{dz} e^z = \lim_{h \to 0} \frac{e^z e^h - e^z}{h} = \lim_{h \to 0} e^z \frac{e^h - 1}{h} = e^z. \]

As a special case, this implies Theorem 8.6(a),(b).

Define
\[
\cos z = \frac{1}{2} \left( e^{iz} + e^{-iz} \right), \\
\sin z = \frac{1}{2i} \left( e^{iz} - e^{-iz} \right).
\]

Note that \( \cos 0 = 1 \) and \( \sin 0 = 0 \).

When \( z \) is real, these are the usual cosine and sine functions. We shall prove this below.

An easy calculation shows
\[ e^{iz} = \cos z + i \sin z. \]

**Lemma 2.**
\[ e^z = e^\overline{z} \]

**Exercise.** Using the definition of the exponential function, prove Lemma 2.

Let \( x \) be real. Then \( \overline{ix} = x \), so that \( e^{ix} = e^{\overline{ix}} = e^{-ix} \). Thus
\[ \cos x = \frac{1}{2} \left( e^{ix} + e^{-ix} \right) = \frac{1}{2} \left( e^{ix} + e^{\overline{ix}} \right) = \text{Re} \left( e^{ix} \right). \]

We also have
\[ \sin x = \frac{1}{2i} \left( e^{ix} - e^{-ix} \right) = \frac{1}{2i} \left( e^{ix} - e^{\overline{ix}} \right) = \text{Im} \left( e^{ix} \right). \]

That is, if \( x \) is real, then both \( \cos x \) and \( \sin x \) are real.

If \( x \) is real, then
\[ |e^{ix}|^2 = e^{ix} e^{\overline{ix}} = e^{ix} e^{-ix} = e^{ix+(-ix)} = e^0 = 1. \]

Let \( x \) be real. Since \( \lim_{h \to 0} \frac{e^{ix+h} - e^x}{h} = e^z \), where \( h \) is complex, we have
\[
\frac{d}{dx} \cos x = \frac{1}{2} \left( \frac{d}{dx} e^{ix} + \frac{d}{dx} e^{-ix} \right) \\
= \frac{1}{2} \left( \lim_{\Delta x \to 0} \frac{e^{ix+i\Delta x} - e^{ix}}{\Delta x} + \lim_{\Delta x \to 0} \frac{e^{-ix-i\Delta x} - e^{-ix}}{\Delta x} \right) \\
= \frac{1}{2} \left( i \lim_{\Delta x \to 0} \frac{e^{ix+i\Delta x} - e^{ix}}{i\Delta x} - i \lim_{\Delta x \to 0} \frac{e^{-ix-i\Delta x} - e^{-ix}}{-i\Delta x} \right) \\
= \frac{1}{2} \left( i e^{ix} - i e^{-ix} \right) \\
= -\frac{1}{2i} (e^{ix} - e^{-ix}) \\n= -\sin x.
\]
Similarly, we can prove that 

\[ \frac{d}{dx} \sin x = \cos x. \]

We define \( \pi \) to be such that \( \frac{\pi}{2} \) is the smallest positive number such that \( \cos \frac{\pi}{2} = 0 \). We’ll take it as a fact that such a number exists. (This is proved at the top of page 183.) Since \( |e^{i\pi}| = 1 \) and

\[ e^{i\frac{\pi}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i \sin \frac{\pi}{2}, \]

we obtain \( \sin \frac{\pi}{2} = \pm 1 \). Since \( \cos x > 0 \) for \( 0 < x < \frac{\pi}{2} \), we have \( \sin x \) is increasing on \( (0, \pi/2) \). Since \( \sin 0 = 0 \), we have \( \sin \frac{\pi}{2} > 0 \), so that \( \sin \frac{\pi}{2} = 1 \). Hence \( 0 < \sin x < 1 \) for \( 0 < x < \frac{\pi}{2} \). Since \( \frac{d}{dx} \cos x = -\sin x < 0 \) for \( 0 < x < \frac{\pi}{2} \), we also have \( 0 < \cos x < 1 \) for \( 0 < x < \frac{\pi}{2} \).

**Lemma 3.**

\[ e^{\frac{\pi}{2}i} = i, \]
\[ e^{\pi i} = -1, \]
\[ e^{2\pi i} = 1. \]

**Proof.** Firstly, from the above, we have

\[ e^{\frac{\pi}{2}i} = \sin \frac{\pi}{2} = i. \]

Secondly, this implies

\[ e^{\pi i} = e^{\frac{\pi}{2}i + \frac{\pi}{2}i} = e^{\frac{\pi}{2}i} \cdot e^{\frac{\pi}{2}i} = i \cdot i = -1, \]

which then implies

\[ e^{2\pi i} = e^{\pi i + \pi i} = e^{\pi i} \cdot e^{\pi i} = (-1)(-1) = 1. \]