Solution to #26 on p. 119.

1. We have $|f'(x)| \leq A|f(x)|$ on $[a, b]$.

2. Let $x_0 \doteq \min\{a + \frac{1}{2A}, b\}$. Then $x_0 - a \leq \frac{1}{2A}$, so
$$A(x_0 - a) \leq \frac{1}{2}.$$  

3. Define
$$M_0 = \sup_{a \leq x \leq x_0} |f(x)| \quad \text{and} \quad M_1 = \sup_{a \leq x \leq x_0} |f'(x)|.$$  

4. Claim. $M_0 = 0$. The rest of the steps up to the penultimate step work toward proving this.

5. Since $|f'(x)| \leq A|f(x)|$ on $[a, b]$ and since $x_0 \leq b$, we have
$$M_1 = \sup_{a \leq x \leq x_0} |f'(x)| \leq \sup_{a \leq x \leq x_0} (A|f(x)|) = AM_0.$$  

6. Let $a \leq x \leq x_0$. Since $f(a) = 0$, by the MVT and by $M_1 = \sup_{a \leq x \leq x_0} |f'(x)|$, we have
$$|f(x)| = |f(a) + f'(c)(x - a)| \quad \text{for some } a < c < x$$
$$= |f'(c)|(x - a)$$
$$\leq M_1(x - a)$$
$$\leq M_1(x_0 - a) \quad \text{(because } x \leq x_0)$$
$$\leq A(x_0 - a)M_0 \quad \text{(since } M_1 \leq AM_0 \text{ by Step 5)}$$
$$\leq \frac{1}{2}M_0 \quad \text{(since } A(x_0 - a) \leq \frac{1}{2} \text{ by Step 2)}.$$  

7. Since $|f(x)| \leq \frac{1}{2}M_0$ for all $a \leq x \leq x_0$, we have
$$M_0 = \sup_{a \leq x \leq x_0} |f(x)| \leq \frac{1}{2}M_0.$$  

8. Since $M_0 \geq 0$, this implies $M_0 = 0$ (proving the claim), that is, $f(x) = 0$ for $a \leq x \leq x_0$.

9. Assume that $x_0 \neq b$. We can apply the same argument as above with $a$ replaced by $x_0$ to prove that
$$f(x) = 0 \quad \text{for } x_0 \leq x \leq x_1,$$
where $x_1 \doteq \min\{x_0 + \frac{1}{2A}, b\} = \min\{a + 2 \cdot \frac{1}{2A}, b\}$. Hence we get a finite sequence $x_0 < x_1 < x_2 < \cdots < x_k = b$ and conclude that
$$f(x) = 0 \quad \text{for } a \leq x \leq b. \quad \Box$$

**Remark.** We have
$$\frac{1}{2A} = x_1 - x_0 = x_2 - x_1 = \cdots = x_{k-1} - x_{k-2} \geq x_k - x_{k-1}.$$