Math 140B  HW1, due Wednesday Apr 8 at the end of class

HW1-#1. (a) Prove the following fact. If \( \lim_{x\to0} h(x) = 0 \) and \( |k(x)| \leq 1 \), then
\[
\lim_{x\to0} h(x)k(x) = 0.
\]

(b) Prove that if \( \lim_{x\to0} h(x) = 0 \), then \( \lim_{x\to0} |h(x)| = 0 \).

Solution to #1. (a) Let \( \epsilon > 0 \). Since \( \lim_{x\to0} h(x) = 0 \) there exist \( \delta > 0 \) such that if \( 0 < |x - 0| < \delta \), then \( |h(x) - 0| < \epsilon \). Hence, if \( 0 < |x - 0| < \delta \), then using \( |k(x)| \leq 1 \) we obtain
\[
|h(x)k(x) - 0| = |h(x)| |k(x)| \leq |h(x)| < \epsilon
\]
This proves \( \lim_{x\to0} h(x)k(x) = 0 \). \( \square \)

(b) Define \( k(x) = \text{sign}(h(x)) \), which is 1, 0, -1 depending on whether \( h(x) \) is positive, zero, negative, respectively. Then \( |k(x)| \leq 1 \) and \( h(x)k(x) = |h(x)| \) so we may apply part (a). \( \square \)

Remark. This is a version of the squeeze theorem.

HW1-#2. Prove that if \( \lim_{x\to0} h(x) \) exists and is nonzero, then \( \lim_{x\to0} h(x) \cos \frac{1}{x} \) does not exist.

Solution to #2. Let \( L = \lim_{x\to0} h(x) \).

(1) Let \( a_n = \frac{1}{2\pi n} \), so that \( a_n \to 0 \). Since \( \cos(2\pi n) = 1 \), we have
\[
\lim_{n\to\infty} h(a_n) \cos \frac{1}{a_n} = \lim_{n\to\infty} h(a_n) = L.
\]

(2) Let \( b_n = \frac{1}{(2n+1)\pi} \), so that \( b_n \to 0 \). Since \( \cos((2n+1)\pi) = -1 \), we have
\[
\lim_{n\to\infty} h(b_n) \cos \frac{1}{b_n} = -\lim_{n\to\infty} h(b_n) = -L.
\]
Since \( L \neq 0 \), we conclude that \( \lim_{x\to0} h(x) \cos \frac{1}{x} \) does not exist. \( \square \)

Remark 1. One can choose other sequences such as \( c_n = \frac{1}{2\pi n + \frac{\pi}{2}} \). We just need two subsequential limits to be different.

Remark 2. The following statements are true (you may have used them implicitly in solving #2):

Lemma. Suppose \( L = \lim_{x\to a} f(x) \) exists. Then for any sequence \( \{a_n\} \) such that \( a_n \to a \) we have \( \lim_{n\to\infty} f(a_n) = L \).

Corollary. Suppose there exist \( a_n \to a \) and \( a'_n \to a \) such that \( \lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(a'_n) \). Then \( \lim_{x\to a} f(x) \) does not exist.

HW1-#3. (Compare with #1 on p. 114.) Let \( f \) be defined for all real \( x \), and suppose that
\[
|f(x) - f(y)| \leq |x - y|^{1+\alpha}
\]
for all real \( x \) and \( y \), where \( \alpha > 0 \). Prove that \( f \) is constant.

Solution to #3. We have
\[
\left| \frac{f(x) - f(y)}{x - y} \right| = \frac{|f(x) - f(y)|}{|x - y|} \leq |x - y|^\alpha
\]
for \( x \neq y \). Since \( \lim_{x \to y} |x - y|^\alpha = 0 \) (from \( \alpha > 0 \)), by the squeeze theorem we conclude that

\[
f'(y) = \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0.
\]

Since this true for all \( y \in \mathbb{R} \), we conclude from Theorem 5.11(b) that \( f \) is constant. \( \square \)

**Remark.** To prove that \( \lim_{x \to y} |x - y|^\alpha = 0 \), just observe the following. Given \( \epsilon > 0 \), let \( \delta = \epsilon^{1/\alpha} \). If \( |x - y| < \delta \), then \( |x - y|^\alpha - 0 = |x - y|^\alpha < \delta^\alpha = \epsilon \).

**HW1-#4.** (a) Let \( g : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying \( g'(x) > 0 \) for all \( x \neq 0 \). Prove that \( g \) is one-to-one.

(b) (Compare with #3 on p. 114.) Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying \( |f'(x)| \leq Mx^2 \), where \( M \) is a positive constant. Prove for \( \epsilon \in \mathbb{R} \) sufficiently small (how small depends on \( M \)) that the function

\[
f_\epsilon(x) = x^3 + \epsilon f(x)
\]

is one-to-one.

**Solution to #4.** (a) The proof is similar to that for Theorem 5.11(a), except we have \( > \) (except at 0) instead of \( \geq \). Let \( x_1, x_2 \in \mathbb{R} \) with \( x_1 < x_2 \). By Theorem 5.10, there exists \( c \in (x_1, x_2) \) such that

\[
g(x_2) - g(x_1) = (x_2 - x_1) g'(c).
\]

**Case 1.** \( 0 \notin (x_1, x_2) \). Then \( g'(c) > 0 \), which implies \( g(x_2) - g(x_1) > 0 \).

**Case 2.** \( 0 \in (x_1, x_2) \). We apply Theorem 5.10 on the intervals \([x_1, 0]\) and \([0, x_2]\) to conclude that:

(i) there exists \( c_1 \in (x_1, 0) \) such that

\[
g(0) - g(x_1) = (0 - x_1) g'(c_1) > 0.
\]

(ii) there exists \( c_2 \in (0, x_2) \) such that

\[
g(x_2) - g(0) = (x_2 - 0) g'(c_2) = 0.
\]

Thus \( g(x_2) - g(x_1) = g(x_2) - g(0) + g(0) - g(x_1) > 0 \).

Hence, not only is \( g \) one-to-one, it is strictly increasing. \( \square \)

(b) Since

\[
f_\epsilon'(x) = 3x^2 + \epsilon f'(x)
\]

and \( |f'(x)| \leq Mx^2 \) (for all \( x \) and where \( M > 0 \)), we have

\[
f_\epsilon'(x) \geq 3x^2 - \epsilon Mx^2.
\]

Choose \( \epsilon \in (0, \frac{3}{M}) \). Then \( f_\epsilon'(x) \geq ax^2 \), where \( a = 3 - \epsilon M > 0 \). So \( f_\epsilon'(x) > 0 \) for \( x \neq 0 \). We may now apply part (a) to conclude that \( f_\epsilon \) is one-to-one. \( \square \)

**HW1-#5.** (#7 on p. 114.) Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \). Let \( x \in \mathbb{R} \). Suppose \( f'(x) \), \( g'(x) \) exist, \( g'(x) \neq 0 \), and \( f(x) = g(x) = 0 \). Prove that

\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.
\]
Solution to #5. First of all, we can’t simply say that by l’Hospital’s rule:
\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f'(t)}{g'(t)} = \frac{f'(x)}{g'(x)}
\]
because we don’t know that \(f'\) and \(g'\) are continuous at \(x\).

Since \(f(x) = g(x) = 0\) (justifying the first equality below), we have
\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)} = \frac{\lim_{t \to x} \frac{f(t) - f(x)}{t - x}}{\lim_{t \to x} \frac{g(t) - g(x)}{t - x}} = \frac{f'(x)}{g'(x)}
\]
where the second equality is a limit law (true since both top and bottom limits exist) and the third equality is by definition of derivative and since it is defined by \(g'(x) \neq 0\). □

HW1-#6. (Part of #8 on pp. 114–115.) Suppose \(f'\) is continuous on \([a, b]\) and \(\varepsilon > 0\). Prove that there exists \(\delta > 0\) such that
\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon
\]
whenever \(0 < |t - x| < \delta, a \leq x \leq b, a \leq t \leq b\).

Solution to #6. Let \(\varepsilon > 0\). Since \(f'\) is continuous on \([a, b]\) and since \([a, b]\) is compact, by Theorem 4.19 \(f'\) is uniformly continuous. Then there exists \(\delta > 0\) such that
\[
|f'(y) - f'(x)| < \varepsilon \quad \text{whenever} \quad |y - x| < \delta.
\]
Let \(a \leq x, t \leq b\). Then there exists \(c\) between \(x\) and \(t\) such that
\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)|.
\]
Hence, if \(0 < |t - x| < \delta\), then \(|c - x| < \delta\), which implies \(|f'(c) - f'(x)| < \varepsilon\). □