Math 140B    HW1, due Wednesday Apr 8 at the end of class

HW1-#1. (a) Prove the following fact. If \( \lim_{x \to 0} h(x) = 0 \) and \( |k(x)| \leq 1 \), then \( \lim_{x \to 0} h(x) k(x) = 0 \).

(b) Prove that if \( \lim_{x \to 0} h(x) = 0 \), then \( \lim_{x \to 0} |h(x)| = 0 \).

HW1-#2. Prove that if \( \lim_{x \to 0} h(x) \) exists and is nonzero, then \( \lim_{x \to 0} h(x) \cos \frac{1}{x} \) does not exist.

HW1-#3. (Compare with #1 on p. 114.) Let \( f \) be defined for all real \( x \), and suppose that

\[
|f(x) - f(y)| \leq |x - y|^{1+\alpha}
\]

for all real \( x \) and \( y \), where \( \alpha > 0 \). Prove that \( f \) is constant.

HW1-#4. (a) Let \( g : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying \( g'(x) > 0 \) for all \( x \neq 0 \). Prove that \( g \) is one-to-one.

(b) (Compare with #3 on p. 114.) Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying \( |f'(x)| \leq Mx^2 \), where \( M \) is a positive constant. Prove for \( \varepsilon \in \mathbb{R} \) sufficiently small (how small depends on \( M \)) that the function

\[
f_\varepsilon(x) = x^3 + \varepsilon f(x)
\]

is one-to-one.

HW1-#5. (#7 on p. 114.) Let \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \). Let \( x \in \mathbb{R} \). Suppose \( f'(x) \), \( g'(x) \) exist, \( g'(x) \neq 0 \), and \( f(x) = g(x) = 0 \). Prove that

\[
\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.
\]

HW1-#6. (Part of #8 on pp. 114–115.) Suppose \( f' \) is continuous on \([a, b]\) and \( \varepsilon > 0 \). Prove that there exists \( \delta > 0 \) such that

\[
\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon
\]

whenever \( 0 < |t - x| < \delta \), \( a \leq x \leq b \), \( a \leq t \leq b \).