HW 4: Feb 12. Chapter 7 Exercises #1, 2, 7, 8, 9, 11.

#1. Definitions for #1:

We say that \( \{f_n\}_{n \geq 1} \) is a sequence of bounded functions on a set \( E \) if for each \( n \geq 1 \) there exists \( M_n \) such that \( |f_n(x)| \leq M_n \) for all \( x \in E \).

We say that \( \{f_n\}_{n \geq 1} \) is uniformly bounded if there exists \( M \) such that \( |f_n(x)| \leq M \) for all \( n \geq 1 \) and all \( x \in E \).

Example 1. Let \( E \) be any nonempty set. Define \( f_n(x) = n \) for all \( x \in E \). Then \( \{f_n\}_{n \geq 1} \) is a sequence of bounded functions on \( E \). But \( \{f_n\}_{n \geq 1} \) is not uniformly bounded.

Example 2. Let \( E = [0, 1] \). Define \( f_n : [0, 1] \rightarrow \mathbb{R} \) by

\[
f_n(x) = \begin{cases} 
0 & \text{if } \frac{2}{n} \leq x \leq 1, \\
n^2x & \text{if } 0 \leq x \leq \frac{1}{n}, \\
2n - n^2x & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n}.
\end{cases}
\]

This is a sequence of bounded functions converging to the zero function, where the sequence is not uniformly bounded.

#1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution. Suppose \( \{f_n\} \) is a sequence of bounded functions on a set \( E \) uniformly converging to \( f \).

By hypothesis, for each \( n \geq 1 \) there exists \( M_n \) such that \( |f_n(x)| \leq M_n \) for all \( x \in E \).

Corresponding to taking \( \varepsilon = 1 \), by the definition of uniformly convergent, there exists \( N \in \mathbb{N} \) such that

\[
|f_n(x) - f(x)| < 1 \quad \text{for all } x \in E \text{ and } n \geq N.
\]

(i) \( f \) is bounded. We have

\[
|f(x)| \leq |f_N(x)| + |f_N(x) - f(x)| \leq M_N + 1.
\]

(ii) \( \{f_n\}_{n \geq 1} \) is uniformly bounded. For \( n \geq N \) we have

\[
|f_n(x)| \leq |f(x)| + |f_n(x) - f(x)| \leq M_N + 2.
\]

Hence, for all \( n \geq 1 \) we have

\[
|f_n(x)| \leq M = \max \{M_1, \ldots, M_{N-1}, M_N + 2\}.
\]

Remark: A similar proof is given in Cooke’s solution manual, where he avoids proving that \( f \) is bounded, but rather uses the Cauchy criterion for uniform convergence.
#2. (a) Suppose that \( f_n \to f \) uniformly on \( E \) and \( g_n \to g \) uniformly on \( E \). Let \( \varepsilon > 0 \). Then there exists \( N_1, N_2 \in \mathbb{N} \) such that
\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for} \ x \in E \text{ and } n \geq N_1
\]
and
\[
|g_n(x) - g(x)| < \frac{\varepsilon}{2} \quad \text{for} \ x \in E \text{ and } n \geq N_2.
\]
Let \( N = \max \{N_1, N_2\} \). Then for \( x \in E \) and \( n \geq N \) we have
\[
|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)|
\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
= \varepsilon.
\]
(b) By Exercise \#1, both sequences \( \{f_n\} \) and \( \{g_n\} \) are uniformly bounded. So there exists \( M \) such that \( |f_n(x)| \leq M \) and \( |g_n(x)| \leq M \) for all \( x \in E \) and \( n \geq 1 \). This implies \( |f(x)| \leq M \) and \( |g(x)| \leq M \) for all \( x \in E \). Then there exists \( N \in \mathbb{N} \) such that
\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2M} \quad \text{for} \ x \in E \text{ and } n \geq N
\]
and
\[
|g_n(x) - g(x)| < \frac{\varepsilon}{2M} \quad \text{for} \ x \in E \text{ and } n \geq N.
\]
For \( x \in E \) and \( n \geq N \) we have
\[
|(f_n g_n)(x) - (f g)(x)| = |f_n(x) g_n(x) - f(x) g(x)|
\leq |f_n(x)| |g_n(x) - g(x)| + |g_n(x) - g(x)| |f_n(x) - f(x)| + |f_n(x)| |g_n(x) - g(x)|
\leq M |f_n(x)| |g_n(x) - g(x)| + M |g_n(x) - g(x)|
\leq M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M}
= \varepsilon.
\]

#7.

\[
f_n(x) = \frac{x}{1 + nx^2} \quad \text{for } x \in \mathbb{R} \text{ and } n \geq 1.
\]

(a) If \( |x| \geq \frac{1}{\sqrt{n}} \), then
\[
|f_n(x)| = \frac{|x|}{1 + nx^2} \leq \frac{|x|}{nx^2} = \frac{1}{n|x|} \leq \frac{1}{\sqrt{n}}.
\]

(b) If \( |x| \leq \frac{1}{\sqrt{n}} \), then
\[
|f_n(x)| = \frac{|x|}{1 + nx^2} \leq |x| \leq \frac{1}{\sqrt{n}}.
\]
From this it is easy to see that \( f_n \to 0 \) uniformly. (Cooke uses the Schwarz inequality to get \( 1 + nx^2 \geq 2\sqrt{n} |x| \), so that

\[
|f_n(x)| = \frac{|x|}{1 + nx^2} \leq \frac{|x|}{2\sqrt{n} |x|} \leq \frac{1}{2\sqrt{n}}.
\]

Now

\[
f_n'(x) = \frac{1}{1 + nx^2} - \frac{2nx^2}{(1 + nx^2)^2} = \frac{1 - nx^2}{(1 + nx^2)^2}
\]

for \( x \in \mathbb{R} \) and \( n \geq 1 \).

If \( x \neq 0 \), then

\[
|f_n'(x)| = \frac{|1 - nx^2|}{(1 + nx^2)^2} \leq \frac{1}{1 + nx^2},
\]

so that \( f_n'(x) \to 0 \) as \( n \to \infty \). On the other hand,

\[
f_n'(0) = 1 \quad \text{for all } n \geq 1.
\]

\#8. If

\[
I(x) = \begin{cases} 
0 & \text{if } x \leq 0, \\
1 & \text{if } x > 0,
\end{cases}
\]

if \( \{x_n\} \) is a sequence of distinct points of \( (a, b) \), and if \( \sum |c_n| \) converges, prove that the series

\[
f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n), \quad \text{for } a \leq x \leq b,
\]

converges uniformly, and that \( f \) is continuous for every \( x \neq x_n \).

**Solution.** Let \( x \in [a, b] \). We have \( I(x - x_n) = 1 \) if \( x > x_n \) and \( I(x - x_n) = 0 \) if \( x \leq x_n \). Let

\[
A_x \doteq \{n \in \mathbb{N} : x_n < x\}.
\]

Define \( b_n = c_n \) if \( n \in A_x \) and \( b_n = 0 \) if \( n \notin A_x \). Then

\[
f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) = \sum_{n=1}^{\infty} b_n
\]

converges uniformly since \( \sum |c_n| \) converges. Indeed, we can see this from the fact that for any \( N \in \mathbb{N} \),

\[
\sum_{n=N}^{\infty} |c_n I(x - x_n)| = \sum_{n=N}^{\infty} |b_n| \leq \sum_{n=N}^{\infty} |c_n|.
\]

Now recall:

**Theorem 7.11.** Let \( \{f_n\} \) be a sequence of functions on a set \( E \) in a metric space with \( f_n \to f \) uniformly on \( E \). Let \( x \) be a limit point of \( E \). If \( \lim_{t \to x} f_n(t) = A_n \) exists for all \( n \), then \( \{A_n\} \) converges and \( \lim_{t \to x} f(t) = \lim_{n \to \infty} A_n \).

Define \( f_n(x) = \sum_{k=1}^{n} c_k I(x - x_k) \). We have just proved that \( f_n \to f \) uniformly on \( [a, b] \). Let \( x \in [a, b] \) such that \( x \neq x_n \) for each \( n \). Then each \( f_n \) is continuous at \( x \).\(^1\) Therefore \( \lim_{t \to x} f_n(t) = A_n = f_n(x) \) exists for all \( n \). By Theorem 7.11 we conclude that

\[
\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n = \lim_{n \to \infty} f_n(x) = f(x).
\]

\(^1\)This is not true if \( x = x_n \) for some \( n \), in which case \( f_k \) is not continuous at \( x \) for each \( k \geq n \).
That is, \( f \) is continuous at \( x \).

#9. Let \( \{ f_n \} \) be a sequence of continuous functions which converges uniformly to a function \( f \) on a set \( E \). Prove that

\[
\lim_{n \to \infty} f_n(x_n) = f(x)
\]

for every sequence of points \( x_n \in E \) such that \( x_n \to x \), and \( x \in E \). Is the converse of this true?

**Solution.** (a) Let \( \varepsilon > 0 \). Since \( \{ f_n \} \) converges uniformly to \( f \), there exists \( N_1 \in \mathbb{N} \) such that

\[
|f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \text{for } x \in E \text{ and } n \geq N_1.
\]

By Theorem 7.12, \( f \) is continuous on \( E \). So there exists \( \delta > 0 \) such that

\[
|f(y) - f(x)| < \frac{\varepsilon}{2} \quad \text{for } y \in N_\delta(x).
\]

Since \( x_n \to x \), there exists \( N_2 \in \mathbb{N} \) such that if \( n \geq N_2 \), then \( x_n \in N_\delta(x) \).

Let \( N = \max \{ N_1, N_2 \} \). If \( n \geq N \), then

\[
|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This proves \( \lim_{n \to \infty} f_n(x_n) = f(x) \).

(b) The converse is not true. Actually, we need to formulate what the converse means. To this end, we rephrase the original statement:

Let \( \{ f_n \} \) be a sequence of continuous functions which converges to a function \( f \) on a set \( E \).

If the convergence is uniform, then \( \lim_{n \to \infty} f_n(x_n) = f(x) \) for every sequence of points \( x_n \in E \) such that \( x_n \to x \), and \( x \in E \).

So the converse is:

If \( \lim_{n \to \infty} f_n(x_n) = f(x) \) for every sequence of points \( x_n \in E \) such that \( x_n \to x \) and \( x \in E \), then the convergence is uniform.

Simply define \( f_n : \mathbb{R} \to \mathbb{R} \) by

\[
f_n(x) = \begin{cases} 
0 & \text{if } x \leq n, \\
\frac{x}{x-n} & \text{if } x \geq n.
\end{cases}
\]

Then \( \{ f_n \} \) is a sequence of continuous functions which converges to the zero function \( f(x) \equiv 0 \). Moreover, for any \( x \in \mathbb{R} \) and sequence \( x_n \in \mathbb{R} \) with \( x_n \to x \), we have \( \lim_{n \to \infty} f_n(x_n) = 0 = f(x) \).
#11. Suppose \( \{f_n\}, \{g_n\} \) are defined on \( E \), and

(a) \( \sum f_n \) has uniformly bounded partial sums;
(b) \( g_n \to 0 \) uniformly on \( E \);
(c) \( g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots \) for every \( x \in E \).

Prove \( \sum f_n g_n \) converges uniformly on \( E \). Hint: Compare with Theorem 3.42.

**Solution.** Basically, we follow the proof of Theorem 3.42 exactly, where the constants are now independent of \( x \). Let \( s_n(x) = \sum_{k=1}^{n} f_k(x) \) be the \( n \)th partial sum. By (a), there exists \( M \in \mathbb{R} \) such that \( |s_n(x)| \leq M \) for all \( x \in E \) and \( n \geq 1 \). Let \( \varepsilon > 0 \). By (b), there exists \( N \in \mathbb{N} \) such that \( g_N \leq \frac{\varepsilon}{2M} \).

For any \( q \geq p \geq N \) and \( x \in E \), we have

\[
\left| \sum_{n=p}^{q} f_n(x) \, g_n(x) \right| = \left| \sum_{n=p}^{q-1} s_n(x) \left( g_n(x) - g_{n+1}(x) \right) + s_q(x) \, g_q(x) - s_{p-1}(x) \, g_p(x) \right|
\leq M \left| \sum_{n=p}^{q-1} \left( g_n(x) - g_{n+1}(x) \right) + g_q(x) + g_p(x) \right|
= 2M g_p(x) \leq 2M g_N(x) \leq \varepsilon.
\]

Now uniform convergence of the series \( \sum_{n=1}^{\infty} f_n(x) \, g_n(x) \) follows from the Cauchy Criterion for Uniform Convergence (Theorem 7.8).