Math 140B  HW6, due Friday March 4 at the beginning of class

HW6, #1. (a) Show that for $z, w \in \mathbb{C}$,

$$\sin(z + w) = \sin z \cos w + \cos z \sin w, \quad \cos(z + w) = \cos z \cos w - \sin z \sin w.$$  

(b) Prove the complex derivative formulas:

$$\frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z.$$  

HW6, #2. Do Chapter 8, Exercise #1 on p. 196.

**Hint:** Prove this by induction. For $n \geq 1$ let $S(n)$ be the following statement.

(i) There exists a polynomial $P_n(x)$ such that $f^{(n)}(x) = P_n\left(\frac{1}{x}\right)e^{-1/x^2}$ for $x \neq 0$.

(ii) $f^{(n)}(0) = 0$.

(iii) You may need $\lim_{x \to \infty} x^m e^{-x} = 0$, which you can prove by applying l'Hospital’s rule.

HW6, #3. Do Chapter 8, Exercise #6 on p. 197.

**Hints:** (a) What is $f(0)$? From the hypotheses, it is easy to see that $\lim_{h \to 0} \frac{f(h) - 1}{h}$ exists. (Why?) Show that $f$ satisfies a simple first order ODE.

(b) By defining $g(x) = \ln(f(x))$, we have $g(x + y) = g(x) + g(y)$ and $g(0) = 0$. Show that $g(x) = g(1) x$ for all $x \in \mathbb{Q}$.

HW6, #4. Let $P(z) = a_n z^n + \cdots + a_0$ be a polynomial, where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$.

(a) Prove that for any $M \in \mathbb{R}$ there exists $R_0 < \infty$ such that if $|z| > R_0$, then $|P(z)| \geq M$.

(b) Let $\mu = \inf_{z \in \mathbb{C}} |P(z)|$. Prove that there exists $z_0 \in \mathbb{C}$ such that $|P(z_0)| = \mu$.

HW6, #5. (a) Let $Q(z) = 1 + b_k z^k + \cdots + b_n z^n$, where $k \geq 1$, $b_k, \ldots, b_n \in \mathbb{C}$ with with $b_k \neq 0$. Prove that there exists $z \in \mathbb{C}$ such that $|Q(z)| < 1$.

(b) Let $P(z) = a_n z^n + \cdots + a_0$ be a polynomial, where $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Let $z_0 \in \mathbb{C}$ be as in #4(b), so that $|P(z_0)| = \mu = \inf_{z \in \mathbb{C}} |P(z)|$. Suppose that $\mu > 0$ and define $Q(z) = P(z + z_0) / P(z_0)$. Obtain a contradiction.

HW6, #6. Let $P(z) = a_n z^n + \cdots + a_0$ be a polynomial, where $a_0, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Prove that $P(z)$ is the product of $a_n$ times linear factors $z - r$, where $r \in \mathbb{R}$, and quadratic factors $z^2 + bz + c$, where $b, c \in \mathbb{R}$, with complex conjugate roots $z = \lambda \pm i\mu$, where $\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$. 
