Chapter 8

#8. The formula is trivially true for \( n = 0 \) and for \( n = 1 \). Suppose that

\[ |\sin nx| \leq n |\sin x| \]

for some \( n \geq 1 \). Then

\[
|\sin (n + 1)x| = |\sin nx \cos x + \sin x \cos nx| \\
\leq |\sin nx| |\cos x| + |\sin x| |\cos nx| \\
\leq |\sin nx| + |\sin x| \quad \text{since } |\cos x| \leq 1 \text{ and } |\cos nx| \leq 1 \\
\leq n |\sin x| + |\sin x| \quad \text{by induction hypothesis} \\
= (n + 1) |\sin x| .
\]

Therefore the result for all \( n \geq 1 \) is true by induction.

#13. By Parseval’s theorem,

\[
\frac{1}{2\pi} \int_0^{2\pi} |x|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 ,
\]

where

\[ c_n = \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \, dx .\]

The left-side is

\[ \frac{1}{2\pi} \int_0^{2\pi} |x|^2 \, dx = \frac{4}{3}\pi^2 . \]

The Fourier coefficients are computed by integration by parts: \( c_0 = \pi \) and for \( n \neq 0 \),

\[
2\pi c_n = \int_0^{2\pi} x e^{-inx} \, dx \\
= \frac{1}{-in} x e^{-inx} \bigg|_0^{2\pi} - \int_0^{2\pi} \frac{1}{-in} e^{-inx} \, dx \\
= \frac{1}{-in} 2\pi - 0 \\
= \frac{2\pi}{-in} .
\]

Hence \( |c_n| = \frac{1}{n} \). Thus

\[ \frac{4}{3}\pi^2 = \pi^2 + \sum_{n \neq 0} \frac{1}{n^2} = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \]

and therefore

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6}\pi^2 . \]
Remark: Cooke’s answer uses the Fourier series expansion in terms of sines and cosines. He does this also for #14. You may use either method. His method is shorter since it makes better use of the symmetries.

#14. \(f(x) = (\pi - |x|)^2\) on \([-\pi, \pi]\). The Fourier coefficients of \(f\) are

\[
2\pi c_n = \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx} \, dx = \int_{0}^{\pi} (\pi - x)^2 e^{-inx} \, dx + \int_{-\pi}^{0} (\pi + x)^2 e^{-inx} \, dx.
\]

Changing variables by \(y = -x\)

\[
\int_{-\pi}^{0} (\pi + x)^2 e^{-inx} \, dx = \int_{0}^{\pi} (\pi - y)^2 e^{iny} (-y) = \int_{0}^{\pi} (\pi - x)^2 e^{inx} \, dx.
\]

So

\[
2\pi c_n = \int_{0}^{\pi} (\pi - x)^2 (e^{inx} + e^{-inx}) \, dx
= 2 \int_{0}^{\pi} (\pi - x)^2 \cos (nx) \, dx
= 2 \int_{0}^{\pi} (\pi^2 - 2\pi x + x^2) \cos (nx) \, dx.
\]

Taking \(n = 0\),

\[
c_0 = \frac{1}{\pi} \int_{0}^{\pi} (\pi^2 - 2\pi x + x^2) \, dx = \frac{1}{3} \pi^2.
\]

Now

\[
\int_{0}^{\pi} \cos (nx) \, dx = 0
\]

\[
\int_{0}^{\pi} x \cos (nx) \, dx = \frac{1}{n} x \sin (nx) \bigg|_{0}^{\pi} - \int_{0}^{\pi} \frac{1}{n} \sin (nx) \, dx
= -\frac{1}{n} \int_{0}^{\pi} \sin (nx) \, dx
= \frac{1}{n^2} \cos (nx) \bigg|_{0}^{\pi}
= \frac{1}{n^2} ((-1)^n - 1)
\]

\[
\int_{0}^{\pi} x^2 \cos (nx) \, dx = \frac{1}{n} x^2 \sin (nx) \bigg|_{0}^{\pi} - \int_{0}^{\pi} 2x \frac{1}{n} \sin (nx) \, dx
= -\frac{2}{n} \int_{0}^{\pi} x \sin (nx) \, dx
= -\frac{2}{n} \left( -\frac{1}{n} x \cos (nx) \bigg|_{0}^{\pi} + \int_{0}^{\pi} \frac{1}{n} \cos (nx) \, dx \right)
= \frac{2}{n^2} \pi (-1)^n
\]
Hence
\[
c_n = \frac{1}{\pi} \int_0^\pi \left( \pi^2 - 2\pi x + x^2 \right) \cos(nx) \, dx
\]
\[
= \frac{1}{\pi} \left( -2\pi \frac{1}{n^2}((-1)^n - 1) + \frac{2}{n^2}\pi (-1)^n \right)
\]
\[
= \frac{2}{n^2}.
\]

Therefore
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}
\]
\[
= \frac{1}{3} \pi^2 + \sum_{n \neq 0} \frac{2}{n^2} e^{inx}
\]
\[
= \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{2}{n^2} (e^{inx} + e^{-inx})
\]
\[
= \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx).
\]

By Theorem 8.14, since \( f \) is Lipschitz continuous and \( 2\pi \)-periodic,
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}
\]
says
\[
(\pi - |x|)^2 = \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx).
\]

Taking \( x = 0 \) we obtain
\[
\pi^2 = \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2},
\]
so that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \pi^2.
\]

By Parseval’s theorem,
\[
\frac{1}{5} \pi^4 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 \, dx
\]
\[
= \sum_{n=-\infty}^{\infty} |c_n|^2
\]
\[
= \frac{1}{9} \pi^4 + 2 \sum_{n=1}^{\infty} \frac{4}{n^4}.
\]
So
\[\sum_{n=1}^{\infty} \frac{4}{n^4} = \frac{2}{45} \pi^4,\]
so that
\[\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90} \pi^4.\]

**#22.** Let
\[f (x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n.\]

Let
\[a_n = \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n.\]

We have
\[\limsup_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \limsup_{n \to \infty} \left| \frac{\alpha - n}{n + 1} \right| = |x|.\]

Hence the series converges for \(|x| < 1\). Moreover, by Theorem we can differentiate the series term by term to obtain
\[f' (x) = \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{(n-1)!} x^{n-1}.\]

Hence
\[(1 + x) f' (x) = \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{(n-1)!} x^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{(n-1)!} x^n\]
\[= \alpha + \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1) (\alpha - n)}{n!} x^n + \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1) n}{n!} x^n\]
\[= \alpha + \alpha \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n = \alpha f (x).\]

Since \(f (0) = 1\), we obtain
\[\int_0^x \frac{f' (x)}{f (x)} \, dx = \int_0^x \frac{\alpha}{1 + x} \, dx\]
so that
\[\ln f (x) = \ln f (x) - \ln f (0) = \alpha \ln (1 + x)\]
and we conclude
\[(1 + x)^\alpha = f (x) = 1 + \sum_{n=1}^{\infty} \frac{\alpha (\alpha - 1) \cdots (\alpha - n + 1)}{n!} x^n.\]
Finally, by replacing $x$ by $-x$ and replacing $\alpha$ by $-\alpha$, we have

$$(1 - x)^{-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{-\alpha(-\alpha-1)\cdots(-\alpha-n+1)}{n!} (-1)^n x^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} x^n.$$ 

On the other hand, the functional equation $\Gamma(x+1) = x\Gamma(x)$ implies

$$\Gamma(x+n) = (x+n-1)\cdots(x+1) x \Gamma(x),$$

so that

$$\Gamma(n+\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1) \Gamma(\alpha).$$

We conclude that

$$(1 - x)^{-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n.$$ 

**Remark.** Note that if $\alpha = m$ is a positive integer, then

$$(1 + x)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n$$

$$= 1 + \sum_{n=1}^{m} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n$$

$$= \sum_{n=0}^{m} \binom{m}{n} x^n,$$

which is a special case of the binomial formula.