Weierstrass theorem

Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then \( f \) is uniformly continuous. It is easy to see that for any \( \varepsilon > 0 \) there exists a piecewise linear function \( g \) (i.e., the graph of \( g \) is a polygonal line) such that \( |g(x) - f(x)| < \varepsilon \) for all \( x \in [a, b] \).

**Proof:** Let \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there exists \( \delta > 0 \) such that \( |x - y| \leq \delta \) implies \( |f(x) - f(y)| < \varepsilon \). Let \( P = \{x_0, \ldots, x_n\} \) be a partition \([a, b]\) such that \( \Delta x_i < \delta \) for all \( 1 \leq i \leq n \). Let \( g \) be the unique function with \( g(x_i) = f(x_i) \) for all \( i \) and \( g \) is linear on each \([x_{i-1}, x_i]\). Then \( |g(x) - f(x)| \leq |M_i - m_i| < \varepsilon \) for all \( x \in [x_{i-1}, x_i] \).

Bernoulli’s inequality:

\[
(1 + h)^n \geq 1 + nh
\]

for any real \( h \geq -1 \) and integer \( n \geq 1 \).

Define

\[
q_n(x) = (1 - x^n)^{2^n}.
\]

Note that \( q_n(0) = 1 \) and \( q_n(1) = 0 \). Each function \( q_n \) is a decreasing.

**Lemma.**

\[
\lim_{n \to \infty} q_n(x) = \begin{cases} 
1 & \text{if } x \in [0, \frac{1}{2}), \\
0 & \text{if } x \in (\frac{1}{2}, 1]. 
\end{cases}
\]

**Proof.** (1) Let \( x \in [0, \frac{1}{2}) \). By Bernoulli’s inequality, since \(-x^n \geq -1\),

\[
1 \geq q_n(x) = (1 - x^n)^{2^n} \geq 1 - 2^n x^n = 1 - (2x)^n.
\]

Since \( x \in [0, \frac{1}{2}) \), \( \lim_{n \to \infty} (1 - (2x)^n) = 1 \). Hence \( \lim_{n \to \infty} q_n(x) = 1 \).

(2) Let \( x \in (\frac{1}{2}, 1) \). By Bernoulli’s inequality,

\[
\frac{1}{q_n(x)} = \left(\frac{1}{1 - x^n}\right)^{2^n} = \left(1 + \frac{x^n}{1 - x^n}\right)^{2^n} \geq 1 + 2^n \frac{x^n}{1 - x^n} = 1 + \frac{(2x)^n}{1 - x^n}.
\]

Since \( 2x > 1 \), we have \( \lim_{n \to \infty} \left(1 + \frac{(2x)^n}{1 - x^n}\right) = \infty \). Hence \( \lim_{n \to \infty} q_n(x) = 0 \).

Note that

\[
\lim_{n \to \infty} q_n\left(\frac{1}{2}\right) = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right)^{2^n} = \frac{1}{e}.
\]
Define
\[ p_n(x) = \left( 1 - \left( \frac{1 - x}{2} \right)^n \right)^2. \]

Note that \( p_n(x) = q_n\left( \frac{1 - x}{2} \right) \).

Define
\[ r_n(x) = x \left( 2p_n(x) - 1 \right). \]

**Exercise:** Show that \( r_n(x) \) converges uniformly to \(|x|\) on \([-1, 1]\).

Consider the function \( x \mapsto |x| \). We can scale and translate this to get
\[ x \mapsto a |x - b| + c \quad \text{on} \quad [-d, d], \]
for any \( a, b, c \in \mathbb{R} \) and \( d > 0 \). Clearly, by the Exercise, we can approximate this function uniformly by polynomials.

Let \( g \) be a piecewise linear function on \([a, b]\). Let \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) be a partition such that \( g \) is linear on each interval \([x_{i-1}, x_i]\).

**Exercise.** Show that we can approximate \( g \) uniformly by polynomials.