1.35 Theorem If \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) are complex numbers, then
\[
\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2
\]

Define the vectors \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \). Define the Hermitian inner product
\[
\langle a, b \rangle = \sum_{j=1}^{n} a_j \bar{b}_j.
\]

For properties of this, see:
http://mathworld.wolfram.com/HermitianInnerProduct.html

Define the corresponding norm by
\[
|a|^2 = \langle a, a \rangle = \sum_{j=1}^{n} a_j \bar{a}_j = \sum_{j=1}^{n} |a_j|^2.
\]

Then the theorem says that
\[
|\langle a, b \rangle|^2 \leq |a|^2 |b|^2.
\]

Exercise 1 Show that:
1. If \( \lambda \in \mathbb{C} \), then
\[
\langle \lambda a, b \rangle = \lambda \langle a, b \rangle,
\]
\[
\langle a, \lambda b \rangle = \overline{\lambda} \langle a, b \rangle.
\]
2. \( \langle b, a \rangle = \langle a, b \rangle \).

Proof of the theorem. If \( b = 0 \), the theorem is easy to prove. So assume that \( b \neq 0 \), so that \( |b|^2 \neq 0 \). Let \( \lambda \in \mathbb{C} \) to be chosen later. We have
\[
0 \leq |a - \lambda b|^2 = \langle a - \lambda b, a - \lambda b \rangle.
\]

Expanding this, we get
\[
0 \leq \langle a, a \rangle + \langle \lambda b, \lambda b \rangle - \langle a, \lambda b \rangle - \langle \lambda b, a \rangle
\]
\[
= |a|^2 + |\lambda|^2 |b|^2 - \overline{\lambda} \langle a, b \rangle - \lambda \langle a, b \rangle.
\]

The vector projection of \( a \) onto \( b \) is: \( \frac{\langle a, b \rangle}{|b|^2} b \). So let \( \lambda = \frac{\langle a, b \rangle}{|b|^2} \).

To simplify notation, let \( A = |a|^2, B = |b|^2, \) and \( C = \langle a, b \rangle \). We then have \( \lambda = \frac{C}{B} \) and (1) implies
\[
0 \leq A + \left| \frac{C}{B} \right|^2 B - \frac{C}{B} \frac{C}{B} - \frac{C}{B} \frac{C}{B} = A - \frac{|C|^2}{B}.
\]

Hence \( |C|^2 \leq AB \).
Let $S$ be an ordered set. Let $E \subseteq S$ with $E$ nonempty and bounded above. We say that $\beta$ is an upper bound for $E$ if $\beta \geq x$ for each $x \in E$.

We say that $\alpha = \sup E$ is the least upper bound of $E$ if $\alpha$ is an upper bound for $E$ and each upper bound $\beta$ of $E$ satisfies $\beta \geq \alpha$.

Similarly, given a set $F$ which is nonempty and bounded below, we say that $\omega = \inf F$ is the greatest lower bound of $F$ if $\omega$ is a lower bound for $F$ and each lower bound $\eta$ of $F$ satisfies $\eta \leq \omega$.

1.11 Theorem Let $S$ be an ordered set with the least upper bound property. Let $B \subseteq S$ be nonempty and bounded below. Define $L$ to be the set of all lower bounds of $B$. Then

$$\sup L = \inf B$$

Proof. Step 1. $\sup L$ exists.

Since $B$ is bounded from below, $L$ is nonempty. By definition, if $x \in L$, then $x \leq x$ for each $y \in B$. So each $y \in B$ is an upper bound for $L$; in particular, $L$ is bounded above. By the least upper bound property, $\sup L$ exists.

Step 2. $\sup L = \inf B$.

Let $y \in B$. Since $y$ is an upper bound for $L$, $y \geq \sup L$. So $\sup L$ is a lower bound for $B$. On the other hand, if $x$ is a lower bound for $B$, then $x \in L$, so that $x \leq \sup L$. Hence $\sup L = \inf B$
A simple application of vectors: http://en.wikipedia.org/wiki/Parabola

Let the point \((0, p)\) be the **focus**. Let the line \(y = -p\) be the **directrix**. Consider the set of all points equidistant from the focus and the directrix. We get the equation

\[|(x, y) - (0, p)|^2 = \text{dist}^2((x, y), \{y = -p\}).\]

This is

\[x^2 + (y - p)^2 = (y + p)^2,\]

which simplifies to \(x^2 = 4yp\), or equivalently, the **parabola**

\[y = \frac{1}{4p}x^2.\]

**Reflective property:** Consider the parabola \(y = x^2\), which has focus \(F = (0, \frac{1}{4})\) and directrix \(y = -\frac{1}{4}\).

Let \(B = (\frac{x}{2}, 0)\), \(C = (x, -\frac{1}{4})\) which is on the directrix, \(D = (x, 0)\), and \(E = (x, x^2)\).

By calculus, the slope of the tangent line to the parabola passing through \(E\) is equal to \(\frac{d}{dx}x^2 = 2x\). On the other hand, since

\[\frac{DE}{BD} = \frac{x^2}{2} = 2x,\]

we conclude that \(B\) lies on the tangent line. Clearly \(B\) bisects \(FC\). Also,

\[EF = \sqrt{x^2 + \left(x^2 - \frac{1}{4}\right)^2} = \sqrt{\left(x^2 + \frac{1}{4}\right)^2} = x^2 + \frac{1}{4} = EC.\]

So the triangles \(\triangle EBF\) and \(\triangle EBC\) are congruent. We conclude that the angles labelled \(\alpha\) are all equal.