Let \((X,d)\) be a metric space.

**Definition 2.31.** An open cover of a subset \(E\) is a collection \(\{G_\alpha\}_{\alpha \in A}\) of open sets such that \(E \subseteq \bigcup_{\alpha \in A} G_\alpha\).

That is, we have a bunch of open sets whose union contains the set \(E\).

**Definition 2.32.** A subset \(K\) is compact if for each open cover of \(K\) there exists a finite subcover.

We elaborate on the last definition. By a subcover of \(\{G_\alpha\}_{\alpha \in A}\) we mean a subcollection which still covers \(K\). That is, a subcover is a collection \(\{G_\alpha\}_{\alpha \in B}\) where \(B \subseteq A\) and \(K \subseteq \bigcup_{\alpha \in B} G_\alpha\).

By a finite subcover of \(\{G_\alpha\}_{\alpha \in A}\) we mean a finite subcollection which still covers \(K\). That is, \(B\) is finite. In this case we may write \(B = \{\alpha_1, \ldots, \alpha_k\}\) for some \(\alpha_1, \ldots, \alpha_k \in A\), where \(k \in \mathbb{N}\). Thus \(K \subseteq G_{\alpha_1} \cup \cdots \cup G_{\alpha_k}\).

So a set \(K\) being compact means that for any bunch of open sets whose union contains the set \(E\), we can always find finitely many of these sets whose union contains \(K\).

Some main theorems and ideas of their proofs

**Theorem 2.34.** Compact subsets of metric spaces are closed.

*Idea of the proof:* Use compactness to prove the complement is open, i.e., any point in the complement has a neighborhood contained in the complement.

**Theorem 2.35.** Closed subsets of compact sets are compact.

*Idea of the proof:* For any cover of the closed subset, append its complement and then use that it is contained in a compact set.

**Theorem 2.36.** If \(\{K_\alpha\}\) is a collection of compact subsets of a metric space \(X\) such that the intersection of every finite subcollection of \(\{K_\alpha\}\) is nonempty, then \(\bigcap_{\alpha \in A} K_\alpha\) is nonempty.

*Idea of the proof:* Obtain a contradiction to their intersection being empty, which is equivalent to the union of their complements being the whole metric space.

**Proposition.** Compact subsets of metric spaces are bounded.

*Idea of the proof:* Choose any point \(p \in X\). The cover \(\{N_n(p)\}_{n=1}^\infty\) has a finite subcover.

See: http://www.math.ucsd.edu/~benchow/140Notes09b.pdf

Useful ways to think of some concepts

- If \(x\) is not a limit point of a set \(E\), then there exists \(r > 0\) such that \(N_r(x) \cap E \subseteq \{x\}\). And vice-versa.
- If \(A \subseteq B^c\), then \(A \cap B = \emptyset\). And vice-versa.
- \(x \in E^c\) is equivalent to \(E \subseteq X - \{x\}\).
- \(\bigcap_{n=1}^\infty N_{1/n}(x) = \{x\}\). And thus \(\bigcup_{n=1}^\infty (N_{1/n}(x))^c = X - \{x\}\).

An idea related to a HW problem

A set \(E\) is dense in \(X\) if every point is a limit point of \(E\) or a point of \(E\) (or both). That is, \(\bar{E} = X\).

\(\mathbb{Q}\) is dense in \(\mathbb{R}\). In fact, \(\mathbb{Q}' = \mathbb{R}\), which implies \(\mathbb{Q} = \mathbb{R}\).

Let \(x = (x_1, \ldots, x_k)\) and \(y = (y_1, \ldots, y_k)\) be points in \(\mathbb{R}^k\). Then

\[|x - y|^2 = \sum_{i=1}^k |x_i - y_i|^2.\]

Suppose \(|x_i - y_i| \leq \frac{r}{\sqrt{k}}\) for each \(i = 1, \ldots, k\). What does this imply about \(|x - y|\)?