Section 4.1: #1, 4(a), 9, 12, 13.

#1. (a) False: $f(x) = |x|$ at $x_0 = 0$.
(b) True: Proposition 4.5.
(c) False: $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \notin \mathbb{Q}. \end{cases}$

#4(a).

$$\lim_{x \to 1} \frac{\sqrt{x+1} - \sqrt{2}}{x-1} = \lim_{x \to 1} \frac{\sqrt{x+1} - \sqrt{2} \sqrt{x+1} + \sqrt{2}}{x-1 \sqrt{x+1} + \sqrt{2}} = \lim_{x \to 1} \frac{1}{\sqrt{x+1} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

#9. Since $-x^2 \leq f(x) \leq x^2$, we have $f(0) = 0$ and

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x-0} = \lim_{x \to 0} \frac{f(x)}{x}.$$ 

Now $-x^2 \leq f(x) \leq x^2$ implies $-|x| \leq |f(x)| \leq |x|$ for $x \neq 0$. Since $\lim_{x \to 0} |x| = 0$, by the squeeze theorem from calculus for limits, we conclude that $\lim_{x \to 0} \frac{f(x)}{x} = 0$.

#12. Let $x_0 \in \mathbb{R}$. We have

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x-x_0} = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x-x_0}.$$ 

Since $x > x_0$ implies $f(x) - f(x_0) \geq 0$ and $x-x_0 > 0$, we have $\frac{f(x) - f(x_0)}{x-x_0} \geq 0$, so that $f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x-x_0} \geq 0$.

#13. By the sequential compactness theorem and since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$, which we also call $\{x_n\}$, which converges to some number $x_0$. Since $x_n \to x_0$, $f(x_n) = 0$, and $f$ is continuous, we have $f(x_0) = \lim_{n \to \infty} f(x_n) = 0$. It is possible that there exists $m \geq 1$ such that $x_m = x_0$. But in this case, $x_n \neq x_0$ for $n \neq m$. In particular, $x_n \neq x_0$ for $n \geq m + 1$. So we can take the limit:

$$f'(x_0) = \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = \lim_{n \to \infty} \frac{0 - 0}{x_n - x_0} = 0.$$
Section 4.2: #3, 6, 8, 9.

#3. \( f : (0, \infty) \to \mathbb{R} \) is defined by \( f(x) = 1/x^2 \). By the intermediate value theorem and that \( \lim_{x \to 0^+} f(x) = +\infty \) and \( \lim_{x \to \infty} f(x) = 0 \), one can prove that \( f([0, \infty)) = (0, \infty) \). \( f'(x) = -2/x^3 < 0 \), so \( f \) is one-to-one. Hence the inverse \( f^{-1} : (0, \infty) \to (0, \infty) \) exists. The rest of the problem consists of routine calculations.

#6. 

\[
g'(x_0) = \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(cx) - f(cx_0)}{x - x_0} \\
= c \lim_{x \to x_0} \frac{f(cx) - f(cx_0)}{cx - cx_0} = cf'(cx_0).
\]

#8. This is similar to Exercise #12 in Section 4.1.

#9. Apply the definition of derivative and the fact that \( f \) is odd and differentiable to compute \( f'(-x) \).

Or we can just apply the chain rule: Let \( g(x) = -x \). Then

\[
(f \circ g)(x) = f(-x) = -f(x).
\]

Hence

\[
-f'(x) = (f \circ g)'(x) = f'(g(x)) g'(x) = f'(-x)(-1).
\]

That is, \( f'(-x) = f'(x) \).