Section 3.5. $\varepsilon$-$\delta$ criterion for convergence

These are notes for the $\varepsilon$-$\delta$ definitions of continuity and uniform continuity. We prove that they are equivalent to the previous 'sequential limit' definitions of continuity and uniform continuity: Theorems 3.20 and 3.22 in §3.5.

Continuity

§3.1 Definition (A). $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if for any sequence $\{x_n\}$ in $D$ with $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

§3.5 Definition (B). $f : D \to \mathbb{R}$ is continuous at $x_0 \in D$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in D \text{ and } |x-x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Theorem 3.20. These two definitions (A) and (B) of continuity are equivalent.

Proof. ((A) $\Rightarrow$ (B), via the contrapositive) Suppose $f$ does not satisfy (B). Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n \in D$ with $|x_n-x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$. Then $x_n \to x_0$, whereas $f(x_n) \not\to f(x_0)$. Hence $f$ does not satisfy (A).

((B) $\Rightarrow$ (A)). Suppose $f$ satisfies (B). Let $\{x_n\}$ in $D$, with $x_n \to x_0$. Let $\varepsilon > 0$. By (B), there exists $\delta > 0$ such that

$$x \in D \text{ and } |x-x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that $|x_n-x_0| < \delta$ for all $n \geq N$. Hence $n \geq N$ implies $|f(x_n) - f(x_0)| < \varepsilon$. Thus $f(x_n) \to f(x_0)$. This proves that $f$ satisfies (A). □
Uniform Continuity

§3.4 Definition (C). \( f : D \to \mathbb{R} \) is uniformly continuous if for any sequences \( \{u_n\} \) and \( \{v_n\} \) in \( D \) with \( u_n - v_n \to 0 \), we have \( f(u_n) - f(v_n) \to 0 \).

§3.5 Definition (D). \( f : D \to \mathbb{R} \) is uniformly continuous if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
 u, v \in D \text{ and } |u - v| < \delta \quad \text{implies} \quad |f(u) - f(v)| < \varepsilon .
\]

Theorem 3.22. These two definitions (C) and (D) of uniform continuity are equivalent.

Proof. ((C) \( \Rightarrow \) (D), via the contrapositive) Suppose \( f \) does not satisfy (D). Then there exists \( \varepsilon > 0 \) such that for each \( n \in \mathbb{N} \) there exists \( u_n, v_n \in D \) with \( |u_n - v_n| < \frac{1}{n} \) and \( |f(u_n) - f(v_n)| \geq \varepsilon \). Then \( u_n - v_n \to 0 \), whereas \( f(u_n) - f(v_n) \not\to 0 \). Hence \( f \) does not satisfy (C).

((D) \( \Rightarrow \) (C)). Suppose \( f \) satisfies (D). Let \( \{u_n\} \) and \( \{v_n\} \) in \( D \) with \( u_n - v_n \to 0 \). Let \( \varepsilon > 0 \). By (D), there exists \( \delta > 0 \) such that

\[
 u, v \in D \text{ and } |u - v| < \delta \quad \text{implies} \quad |f(u) - f(v)| < \varepsilon .
\]

Since \( u_n - v_n \to 0 \), there exists \( N \in \mathbb{N} \) such that \( |u_n - v_n| < \delta \) for all \( n \geq N \). Hence \( n \geq N \) implies \( |f(u_n) - f(v_n)| < \varepsilon \). Thus \( f(u_n) - f(v_n) \to 0 \). This proves that \( f \) satisfies (C). \( \Box \)
Example. Show directly that the squaring function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ satisfies the $\varepsilon$-$\delta$ definition of continuity.

Solution. The nonlinearity of the function complicates things a bit, requiring a backward proof as the most motivated.

Let $x_0 \in \mathbb{R}$. We shall show that $f$ is continuous at $x_0$. Let $\varepsilon > 0$. The inequality

$$\varepsilon > |f(x) - f(x_0)| = |x^2 - x_0^2|$$

is equivalent to

$$\varepsilon > |x - x_0||x + x_0|.$$

Let $\delta > 0$ (we need find $\delta$ that works for the given $\varepsilon$) and suppose $|x - x_0| < \delta$. Since

$$|x + x_0| = |x - x_0 + 2x_0| \leq |x - x_0| + 2|x_0| < \delta + 2|x_0|,$$

we have

$$|x - x_0||x + x_0| < \delta (\delta + 2|x_0|).$$

Claim. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $\delta (\delta + 2|x_0|) \leq \varepsilon$.

Let $\delta > 0$ be as in the claim. If $|x - x_0| < \delta$, then $|x - x_0||x + x_0| < \delta (\delta + 2|x_0|) \leq \varepsilon$, that is,

$$|f(x) - f(x_0)| < \varepsilon.$$

Proof of the claim.

Case 1. If $x_0 = 0$, then choose $\delta = \sqrt{\varepsilon}$. If $|x - x_0| = |x| < \delta$, then

$$|f(x) - f(x_0)| = |x - x_0||x + x_0| = x^2 < \delta^2 = \varepsilon.$$

Case 2. If $x_0 \neq 0$, then choose $\delta = \min\{|x_0|, \frac{\varepsilon}{3|x_0|}\}$. Then $\delta (\delta + 2|x_0|) \leq \frac{\varepsilon}{3|x_0|} (|x_0| + 2|x_0|) = \varepsilon$. Hence

$$|f(x) - f(x_0)| = |x - x_0||x + x_0| < \delta (\delta + 2|x_0|) = \varepsilon.$$