Math 31BH  HW2, due Friday January 22 at the beginning of class

HW2, #1.
(1) What is the closure of \( B_r(\vec{x}) \subset \mathbb{R}^n \)?
(2) What is the closure of \( X = \{ (x,0) \mid x > 0 \} \subset \mathbb{R}^2 \)?
(3) What is the interior of \( Y = \{ (x,y) \mid x \text{ and } y \text{ are rational numbers} \} \)?

HW2, #2. Let \( A \subset \mathbb{R}^n \) and let \( \overline{A} \) denote the closure of \( A \). Let \( x \in \mathbb{R}^n \) and \( r > 0 \). Prove that if \( B_r(x) \cap A = \emptyset \), then \( B_r(x) \cap \overline{A} = \emptyset \).

HW2, #3. Prove that if \( A \subset \mathbb{R}^n \) is closed, then \( \overline{A} = A \).

HW2, #4. Using #3, show that if \( A \subset \mathbb{R}^n \) is a closed set and \( \vec{x}_m \) is a sequence of points in \( A \) such that \( \vec{x}_m \to \vec{x}_0 \) for some \( \vec{x}_0 \in \mathbb{R}^n \), then \( \vec{x}_0 \in A \).

HW2, #5. Suppose that \( A \subset \mathbb{R}^n \) has the property that for any sequence \( \vec{x}_m \) of points in \( A \) with \( \vec{x}_m \to \vec{x}_0 \) for some \( \vec{x}_0 \in \mathbb{R}^n \), \( \vec{x}_0 \in A \). Prove by contradiction that if \( \vec{x} \in A^c \), then there exists \( r > 0 \) such that \( B_r(\vec{x}) \subset A^c \).

HW2, #6. Let \( X \subset \mathbb{R}^n \) and let \( f : X \to \mathbb{R}^m \). Suppose that \( f \) is not continuous at \( \vec{x}_0 \in X \). Prove that there exists a sequence \( \vec{x}_m \in X \) with \( \vec{x}_m \to \vec{x}_0 \) such that \( \lim_{m \to \infty} f(\vec{x}_m) \neq f(\vec{x}_0) \), i.e., either \( \lim_{m \to \infty} f(\vec{x}_m) \) does not exist, or if it exists, it is not equal to \( f(\vec{x}_0) \).

HW2, #7. Consider the sequence \( \{(−1)^m\}_{m \geq 1} \). Give a necessary and sufficient condition for a subsequence \( \{(−1)^{i(m)}\}_{m \geq 1} \) to converge.

HW2, #8. Let \( C \subset \mathbb{R}^n \) be a compact set and let \( f : C \to \mathbb{R} \) be a continuous function. Suppose that for each positive integer \( m \) there exists \( \vec{x}_m \in C \) such that \( |f(\vec{x}_m)| \geq m \).
(1) Prove that there exists a subsequence \( \vec{x}_{i(m)} \) which converges to some point \( \vec{x}_0 \in C \).
(2) Prove that there exists \( M \) such that \( |f(\vec{x}_{i(m)}) − f(\vec{x}_0)| < 1 \) for all \( m \geq M \).
(3) Derive a contradiction.

HW2, #9. Let \( \{a_m\}_{m \geq 1} \) and \( \{b_m\}_{m \geq 1} \) be sequences of real numbers such that \( |a_m − b_m| \to 0 \) as \( m \to \infty \). Prove that \( \{a_m\}_{m \geq 1} \) converges if and only if \( \{b_m\}_{m \geq 1} \) converges.