Math 31BH  HW5, due Friday Feb 19 at the beginning of class

HW5, #1 (see equation 3.1.23 on p. 299 in §3.1). Consider the parametrization

\[ \gamma : U = (0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}^2 \to \gamma(U) = M \subset S^2 \subset \mathbb{R}^3 \]

defined by

\[ \gamma \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{array} \right). \]

Here, \( S^2 \) denotes the unit sphere in \( \mathbb{R}^3 \) centered at the origin.

(a) Describe the set \( S^2 - M \), that is, the complement of the image \( \gamma(U) \) in \( S^2 \).

(b) Compute \( D_1 \gamma \left( \begin{array}{c} u \\ v \end{array} \right) \), \( D_2 \gamma \left( \begin{array}{c} u \\ v \end{array} \right) \), and their cross product \( D_1 \gamma \left( \begin{array}{c} u \\ v \end{array} \right) \times D_2 \gamma \left( \begin{array}{c} u \\ v \end{array} \right) \). (See Definition 1.4.17 on p. 77 for the cross product.)

(c) What is the relation of this cross product with the vector \( \gamma \left( \begin{array}{c} u \\ v \end{array} \right) \)?

Answer to (a). To see the relation to polar coordinates on \( \mathbb{R}^2 \) and spherical coordinates on \( \mathbb{R}^3 \), let \( z = \sin v \), \( r = \cos v \), \( x = r \cos u \), \( y = r \sin u \). Then

\[ \gamma \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} r \cos u \\ r \sin u \\ z \end{array} \right) = \left( \begin{array}{c} x \\ y \\ z \end{array} \right). \]

We have \( r^2 + z^2 = \cos^2 v + \sin^2 v = 1 \) and \( x^2 + y^2 = r^2 \cos^2 u + r^2 \sin^2 u = r^2 \), so \( x^2 + y^2 + z^2 = 1 \). This confirms that \( \gamma \left( \begin{array}{c} u \\ v \end{array} \right) \in S^2 \), the unit 2-sphere. Now \( u \in (0, 2\pi) \) is the longitude and \( v \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) is the latitude.

Let \( C = S^2 - \gamma(U) \), the complement of the image \( \gamma(U) \) in \( S^2 \). The north pole \( \text{NP} = (0, 0, 1) \) has latitude \( \frac{\pi}{2} \), whereas the south pole \( \text{SP} = (0, 0, -1) \) has latitude \( -\frac{\pi}{2} \). So we see that \( \text{NP} \in C \) and \( \text{SP} \in C \).
Remark: For the point \( \gamma \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \), the latitude \( v \) measures the angle of the vector \( \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \) to the \( xy \)-plane.

The projection of \( \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \) to the \( xy \)-plane is \( \left( \begin{array}{c} x \\ y \\ 0 \end{array} \right) \). The longitude \( u \) measures the angle of the vector \( \left( \begin{array}{c} x \\ y \\ 0 \end{array} \right) \) to the positive \( x \)-axis (where \( y = 0 \) and \( x > 0 \)) in the \( xy \)-plane. Since we allow this angle to range in \((0, 2\pi)\), the only part of \( S^2 \) missing from the image \( \gamma(U) \) is the intersection of \( S^2 \) to the half-plane where \( z \leq 0, y = 0, \) and \( x \geq 0 \). Thus \( C \) is a semi-circle. One can parametrize this set \( C \) by

\[ v \mapsto \left( \begin{array}{c} \cos v \\ 0 \\ \sin v \end{array} \right), \quad -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}. \]

**Answer to (b).** We compute

\[
D_1\gamma \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} -\sin u \cos v \\ \cos u \cos v \\ 0 \end{array} \right), \quad D_2\gamma \left( \begin{array}{c} u \\ v \end{array} \right) = \left( \begin{array}{c} -\cos u \sin v \\ -\sin u \sin v \\ \cos v \end{array} \right),
\]

and

\[
D_1\gamma \left( \begin{array}{c} u \\ v \end{array} \right) \times D_2\gamma \left( \begin{array}{c} u \\ v \end{array} \right) = \cos v \left( \begin{array}{c} \cos u \cos v \\ \sin u \cos v \\ \sin v \end{array} \right) = \cos v \gamma \left( \begin{array}{c} u \\ v \end{array} \right).
\]

This answers part (c) also.

**HW5, #2.** Let \( r \) and \( s \) be positive numbers. Define \( F : \mathbb{R}^3 \to \mathbb{R} \) by \( F(x_1, x_2, x_3) = \frac{x_1^2}{r^2} + \frac{x_2^2}{s^2} + x_3^2 - 1. \)

(a) Show that \( M = F^{-1}(0) \) is a 2-dimensional manifold, i.e., a surface.

(b) Given \( c = (c_1, c_2, c_3) \), find the equation for the tangent space \( T_c M = \ker [DF_c] \).

(c) Write down a normal vector to \( T_c M \), call it \( \vec{N}_1 \), that your answer to (b) gives.

**Answer to (a).** Let \( x = (x_1, x_2, x_3) \). We compute

\[
DF_x = \left[ \begin{array}{ccc} D_1 F_x & D_2 F_x & D_3 F_x \end{array} \right] = \left[ \begin{array}{ccc} \frac{2x_1}{r^2} & \frac{2x_2}{s^2} & 2 \end{array} \right].
\]

Since \( x \in F^{-1}(0) \) says \( \frac{x_1^2}{r^2} + \frac{x_2^2}{s^2} + x_3^2 = 1 \), we have that at least one of \( x_1, x_2, x_3 \) is nonzero. Hence \( DF_x \) is surjective. By the implicit function theorem, \( F^{-1}(0) \) is a 2-dimensional manifold.

**Answer to (b).** \( DF_c = \left[ \begin{array}{cc} \frac{2c_1}{r^2} & \frac{2c_2}{s^2} \\ \frac{2c_1}{r^2} & 2c_3 \end{array} \right] \), so \( v = (v_1, v_2, v_3) \in \ker [DF_c] \) if and only if \( v \cdot (DF_c)^\top = 0 \), i.e.,

\[
\frac{2c_1}{r^2} v_1 + \frac{2c_2}{s^2} v_2 + 2c_3 v_3 = 0.
\]

**Answer to (c).** \( \vec{N}_1 = (DF_c)^\top = \left( \begin{array}{c} \frac{2c_2}{s^2} \\ 2c_3 \end{array} \right) \).
HW5, #3. Let $r$ and $s$ be positive numbers and let $U \subset \mathbb{R}^2$ be the set of $(x_1, x_2)$ such that $\frac{x_1^2}{r^2} + \frac{x_2^2}{s^2} < 1$. Define $f : U \to \mathbb{R}$ by $f(x_1, x_2) = \sqrt{1 - \frac{x_1^2}{r^2} - \frac{x_2^2}{s^2}}$.

(a) Let $\gamma : U \to \mathbb{R}^3$ be the graphical parametrization defined by $\gamma(x_1, x_2) = (x_1, x_2, f(x_1, x_2))$. Compute $D_1\gamma(u), D_2\gamma(u)$, and their cross product $D_1\gamma(u) \times D_2\gamma(u)$.

(b) Given $a = (c_1, c_2)$, find the equation for the tangent space $T_{(a, f(a))} M$.

(c) From your answer in (b) gives a normal vector, call it $\vec{N}_2$ and write down what it is. Show that $\vec{N}_2$ is a multiple of $\vec{N}_1$.

**Answer to (a).** We compute

$$D_1\gamma(u) = \begin{pmatrix} 1 \\ 0 \\ \frac{-c_2}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \end{pmatrix}, \quad D_2\gamma(u) = \begin{pmatrix} 0 \\ 1 \\ \frac{-c_1}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \end{pmatrix},$$

and

$$D_1\gamma(u) \times D_2\gamma(u) = \begin{pmatrix} \frac{c_1}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \\ \frac{c_2}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \\ \frac{1}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \end{pmatrix}.$$

**Answer to (b).** Taking $(u, v) = (c_1, c_2)$, we define

$$\vec{N}_2 = D_1\gamma(c_1, c_2) \times D_2\gamma(c_1, c_2) = \begin{pmatrix} \frac{c_1}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \\ \frac{c_2}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \\ \frac{1}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \end{pmatrix} = \frac{1}{\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \begin{pmatrix} \frac{2c_1}{s^2} \\ \frac{2c_2}{r^2} \\ 2c_3 \end{pmatrix}.$$

Observe that $f(c_1, c_2) = \sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}} = c_3$, so since $\vec{N}_1 = \begin{pmatrix} \frac{2c_1}{s^2} \\ \frac{2c_2}{r^2} \\ 2c_3 \end{pmatrix}$, we have

$$\vec{N}_2 = \frac{1}{2\sqrt{1 - \frac{c_1^2}{r^2} - \frac{c_2^2}{s^2}}} \vec{N}_1.$$

This also answers part (c).

HW5, #4. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a $C^1$ function. Define a parametrization $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}^n$ of the graph $M = \Gamma(f)$ by $\gamma(x) = (x, f(x))$, where $x = (x_1, \ldots, x_{n-1})$.

(a) Show that $D_1\gamma, \ldots, D_{n-1}\gamma$ is a basis for $T_{\gamma(x)} M$.

(b) Recall from Formula 1.7.22 on p. 128 that the gradient of $f$ is defined by $\nabla f_x = \sqrt{f_x} = \begin{pmatrix} D_1 f_x \\ \vdots \\ D_{n-1} f_x \end{pmatrix}$. Show that the vector $[\begin{pmatrix} -\nabla f_x \\ 1 \end{pmatrix}]^\top$ is perpendicular to $T_{\gamma(x)} M$. 

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Answer to (a). We have $T_{\gamma(x)}M = \text{img}(D_{\gamma_x})$. Since $\{e_1, \ldots, e_{n-1}\}$ is a basis for $\mathbb{R}^{n-1}$ and since $D_{\gamma_x}$ is an isomorphism onto its image (this is easy to see), we have that $\{D_1\gamma_x, \ldots, D_{n-1}\gamma_x\} = \{D_{\gamma_x}(e_1), \ldots, D_{\gamma_x}(e_{n-1})\}$ is a basis for $T_{\gamma(x)}M$.

Answer to (b). We calculate for $1 \leq i \leq n-1$ that

$$D_i \gamma_x = e_i + D_i f_x e_n,$$

so that

$$\begin{bmatrix} -\nabla f_x \\ 1 \end{bmatrix} \cdot D_i \gamma_x = \left( -\nabla f_x + e_n \right) \cdot (e_i + D_i f_x e_n)$$

$$= -\nabla f_x \cdot e_i + e_n \cdot D_i f_x e_n$$

$$= -D_i f_x + D_i f_x$$

$$= 0.$$

To shorten things, here we used a slight abuse of notation of embedding $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ by mapping $v$ to $(v, 0)$ so that, e.g., $-\nabla f_x$ maps to $(-\nabla f_x, 0)$.

HW5, #5. Do Exercise 3.3.2.

Answer to (a).

$$4 + 3x_2 + 4x_1 x_3^2 + 2x_1 x_2 x_3^2 + x_1^2 x_3^2 + 2x_1 x_3^2 + 3x_1^5.$$

Answer to (b).

$$a_{(0,1,0)} = 2,$$

$$a_{(1,1,0)} = 1,$$

$$a_{(1,1,1)} = -1,$$

$$a_{(2,0,0)} = 1,$$

$$a_{(0,2,1)} = 5.$$

So it is $2x^{(0,1,0)} + x^{(1,1,0)} - x^{(1,1,1)} + x^{(2,0,0)} + 5x^{(0,2,1)}$.

Part (c) is similar.

HW5, #6. Do Exercise 3.5.4.

Answer to (a).

$$x^2 + 4xy + 4y^2 = (x + 2y)^2$$

has signature $(1,0)$ and is degenerate.

Answer to (b).

$$x^2 + 2xy + 2y^2 + 2yz + z^2 = (x + y)^2 + (y + z)^2$$

has signature $(2,0)$ and is degenerate.

Answer to (c).

$$= 2x^2 + 4xy + 2xz - 2xw + 2y^2 + z^2 + w^2 - 2yw$$

$$= 2 \left( x + y + \frac{1}{2}z - \frac{1}{2}w \right)^2 + \frac{1}{2}w^2 + wz + \frac{1}{2}z^2 - 2yz$$

$$= 2 \left( x + y + \frac{1}{2}z - \frac{1}{2}w \right)^2 + \frac{1}{2} \left( z + w - 2y \right)^2 + 2wy - 2y^2$$

$$= 2 \left( x + y + \frac{1}{2}z - \frac{1}{2}w \right)^2 + \frac{1}{2} \left( z + w - 2y \right)^2 - 2 \left( y - \frac{1}{2}w \right)^2 + \frac{1}{2}w^2$$
The matrix corresponding to these 4 linear functions is:

\[
\begin{pmatrix}
1 & 1 & \frac{1}{2} & -\frac{1}{2} \\
0 & -2 & 1 & 1 \\
0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

whose determinant is $-\frac{1}{2}$, so the 4 linear functions are linearly independent. The quadratic form is nondegenerate.

**HW5, #7.** Do Exercise 3.5.16.

**Answer.** (Did this mostly in class.) $S^{n-1}$ is bounded because for every $x \in S^{n-1}$ we have $|x| = 1$. So, for example, $S^{n-1} \subset B_2(0)$, the open ball of radius 2 centered at the origin.

$S^{n-1}$ is closed because its complement is open: Let $x \notin S^{n-1}$. Then $|x| \neq 1$ and $B_\varepsilon(0) \cap S^{n-1} = \emptyset$ for $\varepsilon = | |x| - 1| > 0$.

By Theorem 1.6.9, since $S^{n-1}$ is compact and since $Q$ is continuous, there exists $x_0 \in S^{n-1}$ such that $Q(x) \geq Q(x_0)$ for all $x \in S^{n-1}$. Let $C \equiv Q(x_0)$. Since $Q$ is positive definite, $C > 0$. We have

$$Q(x) \geq C \quad \text{for all } x \in S^{n-1}.$$

Next, observe that $Q(cx) = c^2 Q(x)$ for any $c \in \mathbb{R}$. In particular, if $x \neq 0$, then $|x| \neq 0$, so that

$$Q(x) = Q(|x| \frac{x}{|x|}) = |x|^2 Q\left(\frac{x}{|x|}\right).$$

We conclude that if $x \neq 0$, then

$$Q(x) = |x|^2 Q\left(\frac{x}{|x|}\right) \geq C |x|^2$$

since $\frac{x}{|x|} \in S^{n-1}$. On the other hand, if $x = 0$, then $Q(x) \geq C |x|^2$ is trivially true.